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An Approach for Constructing Some $2_{\mathrm{III}}^{m-(m-k)}$ Designs with the Maximum Number of Clear Two-Factor Interactions *

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Abstract

This paper provides an approach for constructing $2_{\mathrm{III}}^{m-(m-k)}$ designs containing the maximum number of clear two-factor interactions. The designs obtained contain more clear two-factor interactions than those obtained Tang et al. (2002) for some m and k. Moreover, the designs constructed are shown to have concise grid representations.

Keywords: Clear effect, minimum aberration, resolution.

AMS Subject Classification: 62K15.

§1. Introduction

In this paper, a $2^{m-(m-k)}$ design stands for a two-level fractional factorial design with m factors and 2^k runs. In the defining relation of a $2^{m-(m-k)}$ design, the numbers $1, 2, \cdots, m$ attached to the factors are called letters, a product of any subset of the letters is called a word, and the number of letters in a word is called its wordlength. Associated with every $2^{m-(m-k)}$ design is a set of $m-k$ words $W_1, W_2, \cdots, W_{m-k}$ called generators. The set of distinct words formed by all possible products involving the $m-k$ generators gives the defining relation. Let d be a $2^{m-(m-k)}$ design and $A_i(d)$ be the number of words of length i in its defining relation. $A(d) = (A_3(d), A_4(d), \cdots, A_m(d))$ is called the wordlength pattern of d. With this notation, the resolution of d is the smallest i with positive $A_i(d)$ in $A(d)$, and a $2_{\mathrm{III}}^{m-(m-k)}$ design represents a $2^{m-(m-k)}$ design of resolution III.

Fractional factorial designs with factors at two-levels are widely used in experimental investigations. Particularly, regular $2^{m-(m-k)}$ designs with resolution III or IV are very important in many scientific experiments. Maximum resolution (Box and Hunter, 1961)

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and minimum aberration (Fries and Hunter, 1980) are commonly used criteria for selecting such $2^{m-(m-k)}$ designs. Under the hierarchical assumption of experimental designs, appropriate designs are those of minimum aberration (Fries and Hunter, 1980). Wu and Chen (1992) classified two-factor interactions (or 2FIs for short) into three categories – ineligible, eligible and clear. A 2FI is called clear if it is not aliased with any main effect and any other 2FIs. Clear 2FIs are estimable under the assumption that three-factor and higher-order interactions are negligible. Chen, Sun and Wu (1993) showed that both maximum resolution and minimum aberration do not completely characterize the number of clear 2FIs in a $2_{\mathrm{III}}^{m-(m-k)}$ design. It is useful to know the maximum number of clear 2FIs in a $2_{\mathrm{III}}^{m-(m-k)}$ design. Since it is still open what is the precise maximum number of clear 2FIs in a $2_{\mathrm{III}}^{m-(m-k)}$ design, the designs containing the maximum number of clear 2FIs stand for those having as many clear 2FIs as possible in the following.

Tang et al. (2002) pointed out that a $2_{\mathrm{III}}^{m-(m-k)}$ design containing the maximum number of clear 2FIs can estimate as many 2FIs as possible without making the assumption that the remaining 2FIs are negligible, if we can assume that the magnitude of the main effects is much larger than that of the 2FIs, which is not unreasonable in many applications. The presence of other 2FIs does not affect the estimation of the clear 2FIs, although they can bias the estimates of the main effects. Since the magnitude of the main effects is much larger, this bias will not be substantial. It is valuable to construct a $2_{\mathrm{III}}^{m-(m-k)}$ design containing the maximum number of clear 2FIs.

In a $2^{m-(m-k)}$ design with resolution at least V, all 2FIs are clear. For given k, let M_k be the maximum value of m for a $2^{m-(m-k)}$ design to have resolution at least V. For $k = 4, 5, 6, 7, 8, 9, 10, 11$ and 12, the value of M_k is $5, 6, 8, 11, 17, 23, 32, 41$ and 65, respectively (Draper and Lin, 1990). Chen and Hedayat (1998) proved that there exist $2_{\mathrm{III}}^{m-(m-k)}$ designs containing clear 2FI if and only if $m \leqslant 2^{k-1}$. Thus, we only need to consider m that $M_k < m \leqslant 2^{k-1}$.

Tang et al. (2002) provided an approach to construct $2_{\mathrm{III}}^{m-(m-k)}$ designs. The number $\alpha_l(k,m)$ of clear 2FIs in these designs is

$$\alpha_l(k,m) = \begin{cases} (2^j-1)(m-2^j+1) & \text{if } \ m_j \geqslant m > m_{j+1}, \ \text{ for } \ j = 1, \cdots, J, \\ [m/2](m-[m/2]) & \text{if } \ m \leqslant \min(m_J, m_{J+1}), \end{cases}$$

where $m_j = 2^j + 2^{k-j} - 2$, $m_{J+1} = 2(2^J - 1) + 1$ and $J = [k/2]$, $[x]$ stands for the largest integer not exceeding x. $\alpha_l(k,m)$ is not the optimal except when $m = m_j$. This paper will provide another approach to construct $2_{\mathrm{III}}^{m-(m-k)}$ designs containing the maximum number of clear 2FIs, and show that the designs constructed have more clear 2FIs than those constructed by Tang et al. (2002) for $1 < j \leqslant [k/2]$ and $m_j < m < (m_{j-1} + 2m_j - 1)/3$.

§2. Main Results

For a $2_{\rm III}^{m-(m-k)}$ design, still let $m_j = 2^{k-j} + 2^j - 2$ for $j = 1, \cdots, J\,(= [k/2])$. We have $m_1 > m_2 > \cdots > m_J$. Also let $H_{2^{k-j}} = (\gamma_0, \cdots, \gamma_{2^{k-j}-1})$ and $H_{2^j} = (c_0, \cdots, c_{2^j-1})$ be $2^{k-j} \times 2^{k-j}$ and $2^j \times 2^j$ normalized Hadamard matrices respectively, where $\gamma_0 = 1_{2^{k-j}}$ and $c_0 = 1_{2^j}$. Again let

$$\begin{aligned} E_{l_1} &= \gamma_{l_1} \otimes c_0 \ \text{ for } \ l_1 = 1, \cdots, 2^{k-j} - 1, \\ F_{l_2} &= \gamma_0 \otimes c_{l_2} \ \text{ for } \ l_2 = 1, \cdots, 2^j - 1, \end{aligned} \tag{2.1}$$

where $\gamma_i \otimes c_j$ represents the Kronecker product of γ_i and c_j. The 2FI grid for a $2_{\rm III}^{m_j-(m_j-k)}$ design, denoted by $d_1 = \{E_1 - E_{2^{k-j}-1}, F_1 - F_{2^j-1}\}$, is given by

$$\begin{array}{c|ccccc|} & 1 & F_1 & F_2 & \cdots & F_{2^j-1} \\ \hline 1 & & F_1 & F_2 & \cdots & F_{2^j-1} \\ E_1 & E_1 & E_1F_1 & E_1F_2 & \cdots & E_1F_{2^j-1} \\ E_2 & E_2 & E_2F_1 & E_2F_2 & \cdots & E_2F_{2^j-1} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ E_{2^{k-j}-1} & E_{2^{k-j}-1} & E_{2^{k-j}-1}F_1 & E_{2^{k-j}-1}F_2 & \cdots & E_{2^{k-j}-1}F_{2^j-1} \\ \hline \end{array}\,. \tag{2.2}$$

Each position in the 2FI grid represents an alias set, and the column of an alias set is the Hadamard product of its horizontal and vertical co-ordinates labelled outside the grid. Note that the Hadamard product of two columns is the entry-wise product of the two columns. In the 2FI grid (2.2), if a main effect and 2FIs are aliased with each other, they are in the same alias set. For simplicity, we omit the 2FIs, if they are aliased with some main effect. From the definition of clear effects, 2FI is clear if it is in the alias set without other main effects or 2FIs in the 2FI grid (2.2). Thus, all 2FIs $E_{l_1}F_{l_2}$ for $1 \le l_1 \le 2^{k-j} - 1$ and $1 \le l_2 \le 2^j - 1$ are clear. $E_{l_1}E_{l_2}$ for $1 \le l_1, l_2 \le 2^{k-j} - 1$ are in an alias set containing one of $E_1, \cdots, E_{2^{k-j}-1}$, $F_{l_1}F_{l_2}$ for $1 \le l_1, l_2 \le 2^j - 1$ in an alias set containing of $F_1, \cdots, F_{2^j-1}$, and they are not clear. Thus the number of clear 2FIs in d_1 is $C(d_1) = (2^{k-j} - 1)(2^j - 1) = \alpha_l(k, m_j)$.

Lemma 2.1 If $j = 3$, $k > 6$ or $j \neq 3$, $k > \max(j + 2, 2j - 1)$, the maximum number $N(k, m_j+1)$ of clear 2FIs in a $2_{\text{III}}^{m_j+1-(m_j+1-k)}$ design is

$$N(k, m_j+1) = (2^j-2)(2^{k-j}-2). \tag{2.3}$$

Proof We take $H_{2^{k-j}} = (\gamma_0, \cdots, \gamma_{2^{k-j}-1})$ and $H_{2^j} = (c_0, \cdots, c_{2^j-1})$ as $2^{k-j} \times 2^{k-j}$ and $2^j \times 2^j$ normalized Hadamard matrices respectively, where $\gamma_0 = 1_{2^{k-j}}$ and $c_0 = 1_{2^j}$. Let $G = E_1F_1$, E_{l_1} and F_{l_2} for $l_1 = 1$ up to $2^{k-j}-1$ and $l_2 = 1$ through $2^j - 1$ be defined in (2.1). The 2FI grid for a $2_{\text{III}}^{m-(m-k)}$ design for $m = m_j+1$, denoted by $d_2 = \{E_1 - E_{2^{k-j}-1}, F_1 - F_{2^j-1}, G\}$, is given by

$$\begin{array}{c|ccccc|} & 1 & F_1 & F_2 & \cdots & F_{2^j-1} \\ \hline 1 & & F_1 & F_2 & \cdots & F_{2^j-1} \\ E_1 & E_1 & G & * & \cdots & * \\ E_2 & E_2 & * & E_2F_2 & \cdots & E_2F_{2^j-1} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ E_{2^{k-j}-1} & E_{2^{k-j}-1} & * & E_{2^{k-j}-1}F_2 & \cdots & E_{2^{k-j}-1}F_{2^j-1} \\ \hline \end{array}, \tag{2.4}$$

where alias sets with '$*$' contain two or more 2FIs. Thus, the total number of clear 2FIs in d_2 is $(2^j-2)(2^{k-j}-2)$. This number can be taken as the maximum number of clear 2FIs in a $2_{\text{III}}^{m-(m-k)}$ design.

For $m = m_j + 1$ and $1 < j \leqslant [k/2]$, the $2_{\text{III}}^{m-(m-k)}$ design obtained by Tang et al. (2002) contains $\alpha_l(k,m) = (2^{j-1}-1)(m-2^{j-1}+1)$ clear 2FIs. Thus,

$$\begin{aligned} N(k,m) - \alpha_l(k,m) &= (2^j-2)(2^{k-j}-2) - (2^{j-1}-1)(m-2^{j-1}+1) \\ &= (2^{j-1}-1)(2^{k-j}-2^{j-1}-4). \end{aligned}$$

When $j = 3$, $k > 6$ or $j \neq 3$, $k > \max(j+2, 2j-1)$, we have $N(k, m_j+1) > \alpha_l(k, m_j+1)$. This lemma is valid. □

This lemma and its proof provide an approach for constructing $2_{\text{III}}^{m-(m-k)}$ designs with the maximum number of clear 2FIs. The resulting designs have more clear 2FIs than those obtained by Tang et al. (2002), when the conditions of Lemma 2.1 are satisfied.

Example 1 The maximum number of clear 2FIs in a $2_{\text{III}}^{11-(11-5)}$ design is 12.

We take $H_8 = (\gamma_0, \cdots, \gamma_7)$ and $H_4 = (c_0, \cdots, c_3)$ as 8×8 and 4×4 normalized Hadamard matrices respectively, where $\gamma_0 = 1_8$ and $c_0 = 1_4$. Let $G = E_1F_1$, E_{l_1} and F_{l_2} be defined in (2.1) for $l_1 = 1$ up to 7 and $l_2 = 1$ through 3. The 2FI grid for a $2_{\text{III}}^{11-(11-5)}$

design, denoted by $d_3 = \{E_1 - E_7, F_1 - F_3, G\}$, is given as in

$$\begin{array}{c|cccc|} & 1 & F_1 & F_2 & F_3 \\ \hline 1 & & F_1 & F_2 & F_3 \\ E_1 & E_1 & G & * & * \\ E_2 & E_2 & * & E_2F_2 & E_2F_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ E_7 & E_7 & * & E_7F_2 & E_7F_3 \\ \hline \end{array}. \tag{2.5}$$

Thus, the total number of clear 2FIs in d_3 is 12. The clear 2FIs are $E_2F_2, \cdots, E_2F_7$, $E_3F_2, \cdots, E_3F_7$. d_3 is one of the best $2_{\rm III}^{11-(11-5)}$ designs, as the precise maximum number of clear 2FIs in a $2_{\rm III}^{11-(11-5)}$ design is 12 from Chen, Sun and Wu (1993). However, Tang et al. (2002) provided the number $\alpha_l(5, 11) = (2-1)(11-2+1) = 10$ of clear 2FIs in a $2_{\rm III}^{11-(11-5)}$ design constructed.

Theorem 2.1 For $1 < j \leqslant [k/2]$ and $m_j < m < (m_{j-1} + 2m_j - 1)/3$, the maximum number $N(k, m)$ of clear 2FIs in a $2_{\rm III}^{m-(m-k)}$ design is

$$N(k, m) = (2^j - 2)(2^{k-j+1} + 2^j - 3 - m). \tag{2.6}$$

Proof Let $H_{2^{k-j}} = (\gamma_0, \cdots, \gamma_{2^{k-j}-1})$ and $H_{2^j} = (c_0, \cdots, c_{2^j-1})$ be $2^{k-j} \times 2^{k-j}$ and $2^j \times 2^j$ normalized Hadamard matrices respectively, where $\gamma_0 = 1_{2^{k-j}}$ and $c_0 = 1_{2^j}$. Let $G_{l_1} = E_{l_1}F_1$, and E_{l_2} and F_{l_3} be defined in (2.1) for $l_1 = 1$ up to $m - m_j$, $l_2 = 1$ up to $2^{k-j} - 1$ and $l_3 = 1$ through $2^j - 1$. The 2FI grid for a $2_{\rm III}^{m-(m-k)}$ design, denoted by $d_4 = \{E_1 - E_{2^{k-j}-1}, F_1 - F_{2^j-1}, G_1 - G_{m-m_j}\}$, is

$$\begin{array}{c|ccccc|} & 1 & F_1 & F_2 & \cdots & F_{2^j-1} \\ \hline 1 & & F_1 & F_2 & \cdots & F_{2^j-1} \\ E_1 & E_1 & G_1 & * & \cdots & * \\ E_2 & E_2 & G_2 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ E_{m-m_j} & E_{m-m_j} & G_{m-m_j} & * & \cdots & * \\ E_{m-m_j+1} & E_{m-m_j+1} & * & E_{m-m_j+1}F_2 & \cdots & E_{m-m_j+1}F_{2^j-1} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ E_{2^{k-j}-1} & E_{2^{k-j}-1} & * & E_{2^{k-j}-1}F_2 & \cdots & E_{2^{k-j}-1}F_{2^j-1} \\ \hline \end{array}. \tag{2.7}$$

Thus, the total number $N(k, m)$ of clear 2FIs in d_4 is $N(k, m) = (2^j - 2)(2^{k-j} - 1 - (m - m_j)) = (2^j - 2)(2^{k-j+1} + 2^j - 3 - m)$. Moreover, $\alpha_l(k, m) = (2^{j-1} - 1)(m - 2^{j-1} + 1)$ from

Tang et al. (2002). We have

$$\begin{aligned} N(k,m)-\alpha_l(k,m) &= (2^{j-1}-1)[2(2^{k-j+1}+2^j-3-m)-(m-2^{j-1}+1)] \\ &= (2^{j-1}-1)[m_{j-1}+2m_j-3m-1]. \end{aligned}$$

Therefore, for $1<j\leqslant[k/2]$, when $m_j<m<(m_{j-1}+2m_j-1)/3$, $N(k,m)-\alpha_l(k,m)>0$. This theorem is valid. $\square$

From this theorem, when $1<j\leqslant[k/2]$ and $m_j<m<(m_{j-1}+2m_j-1)/3$, the resulting designs have more clear 2FIs than those obtained by Tang et al. (2002). Let us construct a $2_{\rm III}^{20-(20-6)}$ design by using the approach of Theorem 2.1.

Let $H_{16}=(\gamma_0,\cdots,\gamma_{15})$ and $H_4=(c_0,\cdots,c_3)$ be 16×16 and 4×4 normalized Hadamard matrices respectively, where $\gamma_0=1_{16}$ and $c_0=1_4$. Let $G_1=E_1F_1$, $G_2=E_2F_1$, E_{i_1} and F_{i_2} be defined by (2.1) for $i_1=1$ up to 15 and $i_2=1$ through 3. The 2FI grid for a $2_{\rm III}^{20-(20-14)}$ design, denoted by $d_5=\{E_1-E_{15},F_1-F_3,G_1,G_2\}$, is

$$\begin{array}{c|cccc|}
 & 1 & F_1 & F_2 & F_3 \\ \cline{2-5}
1 & & F_1 & F_2 & F_3 \\
E_1 & E_1 & G_1 & * & * \\
E_2 & E_2 & G_2 & * & * \\
E_3 & E_3 & * & E_3F_2 & E_3F_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
E_{15} & E_{15} & * & E_{15}F_2 & E_{15}F_3 \\ \cline{2-5}
\end{array}\ . \qquad (2.8)$$

The design d_5 contains $N(6,20)=2(15-2)=26$ clear 2FIs. The $2_{\rm III}^{20-(20-6)}$ design obtained by Tang et al. (2002) has $\alpha_l(6,20)=19$ clear 2FIs. Obviously, $N(6,20)>\alpha_l(6,20)$, and the design in (2.8) contains more clear 2FIs.

§3. Summary Remarks

In this paper, we provide an approach to construct $2_{\rm III}^{m-(m-k)}$ designs containing the maximum number of clear 2FIs. The designs constructed contain more clear 2FIs than those obtained Tang et al. (2002) for some m and k. It is still an open problem on the precise maximum number of clear 2FIs in a $2_{\rm III}^{m-(m-k)}$ design. This problem is under investigation now.

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某些包含最多纯净两因子交互效应$2_{\mathrm{III}}^{m-(m-k)}$设计的一种构造方法

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本文给出了构造包含最多纯净两因子交互效应$2_{\mathrm{III}}^{m-(m-k)}$设计的一种方法. 对于某些设计参数$m$和$k$, 验证了所构造的设计包含纯净两因子交互效应的数量多于Tang et al. (2002)所构造的设计. 并且所构造的设计都给出了格子点表示.

关 键 词: 纯净效应, 最小低阶混杂, 分辨度.

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