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Convergence Rate of Wavelet Estimator in Semiparametric Models with Dependent MA(∞) Error Process *

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Abstract

Consider semiparametric regression model $y_i = x_i\beta + g(t_i) + V_i$ $(1 \le i \le n)$, where the known design points (x_i, t_i), the unknown slope parameter β, and the nonparametric component g are non-random, and the correlated errors $V_i = \sum\limits_{j=-\infty}^{\infty} c_j e_{i-j}$ with $\sum\limits_{j=-\infty}^{\infty} |c_j| < \infty$ and e_i are negatively associated random variables. Under appropriate conditions, we study rates of strong convergence for wavelet estimators of β and $g(\cdot)$. The results show that the wavelet estimator of $g(\cdot)$ can attain the optimal convergence rate. Finite sample behavior of the estimator of β is investigated via simulations too.

Keywords: Semiparametric regression model, negatively associated, wavelet estimator, convergence rate.

AMS Subject Classification: 62G05.

§1. Introduction

Consider the following semiparametric regression model:

$$y_i = x_i\beta + g(t_i) + V_i, \qquad 1 \le i \le n, \tag{1.1}$$

where β is an unknown parameter of interest, (x_i, t_i) are nonrandom design points, y_i are the response variables, V_i are random errors, and $g(\cdot)$ is an unknown function defined on the closed interval $[0, 1]$.

The model (1.1) was first introduced by Engle et al. (1986) and has been extensively studied. When the errors V_i are i.i.d. random variables, various estimation methods have been used to obtain estimators of the unknown quantities in (1.1), see e.g. Speckman (1988), Hamilton and Truong (1997), Qian and Chai (1999), Qian et al. (2000) among others. However, the independence assumption for the errors is not always appropriate in applications, especially for sequentially collected economic data, which often exhibit

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evident dependence in the errors. Recently, the semiparametric regression with serially correlated errors has attracted increasing attention by statisticians. For model (1.1) with MA(∞) errors, which are of the form $V_i = \sum_{j=0}^{\infty} c_j \xi_{i-j}$, where ξ_j are i.i.d. random variables, Sun et al. (2002) discussed the law of iterated logarithm for the semiparametric least square estimator of β and strong convergence rates of the nonparametric estimator of $g(\cdot)$, under c_j satisfying $\sum_{j=0}^{\infty} |c_j| < \infty$ and $\sup_{n\geq 1} n \sum_{j=n}^{\infty} |c_j| < \infty$.

It is well known that the wavelet analysis has been used extensively in engineering and technological fields. In order to meet practical demand, since 90's, some authors have considered to use the wavelet methods in statistics. For instance, Antoniadis et al. (1994) and Donoho et al. (1996) estimated regression function and density function by using wavelet technique, respectively.

Related to the wavelet estimation for model (1.1) and assume that the errors are independent random variables, We refer to papers by Qian and Chai (1999), Qian et al. (2000) and Xue (2003). But, up to now, there have been few results related to wavelet estimation for model (1.1) with MA(∞) error process based on negatively associated random variables.

A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be *negatively associated* (NA) if, for every pair of disjoint subsets A and B of $\{1, 2, \cdots, n\}$, we have

$$\mathsf{Cov}\,(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0,$$

whenever f_1 and f_2 are coordinatewise increasing and such that the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA. The definition of NA random variables was introduced by Alam and Saxena (1981) and carefully studied by Joag-Dev and Proschan (1983). Because of its wide applications in multivariate statistical analysis and systems reliability, the notion of NA received considerable attention recently. See Liang and Jing (2005), Liang et al. (2006), Jing and Liang (2004), Shao (2000), Roussas (2000), Chen et al. (2003), and so on.

In this paper, we consider the model (1.1) and assume the following form for $\{V_i\}$:

$$V_i = \sum_{j=-\infty}^{\infty} c_j e_{i-j} \qquad \text{with} \qquad \sum_{j=-\infty}^{\infty} |c_j| < \infty, \tag{1.2}$$

where $\{e_i\}$ are identically distributed, negatively associated random variables with $\mathsf{E}e_i = 0$. We shall investigate the wavelet estimators of β and $g(\cdot)$ in model (1.1) and discuss strong convergence rates of these estimators. Our results show that the wavelet estimator of $g(\cdot)$ can attain the optimal convergence rate.

The paper is organized as follows. We introduce the wavelet estimators of β and $g(\cdot)$ in Section 2. In Section 3, we list some assumptions and remarks. Main result is formulated in Section 4. Some preliminary lemmas, which are used in the proof of the main result, are collected in Section 5. Section 6 gives proof of the main result. A simulation study is presented in section 7.

§2. Estimators

Let ϕ be father wavelet with compact support and unit integral of multiresolution analysis $\{V_m, m \in Z\}$, where Z is integer set. Since $\{\phi(x-k), k \in Z\}$ is an orthogonal family of $L^2(R)$ and V_0 is the subspace spanned, if we denote

$$\phi_{mk}(x) = 2^{m/2}\phi(2^m x - k), \qquad k \in Z,$$

then $\{\phi_{0k}, k \in Z\}$ is an orthogonal basis of V_0, and $\{\phi_{mk}, k \in Z\}$ is an orthogonal basis of V_m. For the more on wavelet see Watler (1994).

By ϕ, we can define the following wavelet kernel:

$$E_m(x,u) = 2^m E_0(2^m x, 2^m u), \qquad E_0(x,u) = \sum_{k\in Z}\phi(x-k)\phi(u-k).$$

We now construct the wavelet estimators of β and g. Assume that β in model (1.1) is given, since $\mathsf{E}V_i = 0$, we have $g(t_i) = \mathsf{E}(y_i - x_i\beta)$ (for $i = 1, \cdots, n$). Hence, a natural estimator of $g(\cdot)$ is

$$g_n(t,\beta) = \sum_{i=1}^n (y_i - x_i\beta)\int_{A_i} E_m(t,s)\mathrm{d}s,$$

where $A_i = [s_{i-1}, s_i]$, $s_0 = 0$, $s_n = 1$, $s_i = (1/2)(t_i + t_{i+1})$, $i = 1, \cdots, n-1$. Hence $t_i \in A_i$ for $1 \le i \le n$. In order to estimate β, we minimize

$$SS(\beta) = \sum_{i=1}^n [y_i - x_i\beta - g_n(t_i,\beta)]^2. \tag{2.1}$$

Define $S_n^2 = \sum_{i=1}^n \widetilde{x}_i^2$, $\widetilde{x}_i = x_i - \sum_{j=1}^n x_j \int_{A_j} E_m(t_i,s)\mathrm{d}s$, $\widetilde{y}_i = y_i - \sum_{j=1}^n y_j \int_{A_j} E_m(t_i,s)\mathrm{d}s$. The minimizer in (2.1) is found to be

$$\widehat{\beta}_n = \sum_{i=1}^n \widetilde{x}_i\widetilde{y}_i / S_n^2. \tag{2.2}$$

So, a plug-in estimator of the nonparametric component $g(\cdot)$, based on $\widehat{\beta}_n$, is given by

$$\widehat{g}_n(t) = \sum_{i=1}^n (y_i - x_i\widehat{\beta}_n)\int_{A_i} E_m(t_i,s)\mathrm{d}s. \tag{2.3}$$

In the sequel, let $C, C_1, \cdots, c, c_1, \cdots$ denote positive constants whose values may vary at each occurrence. $a \ll b$ means $a \le Cb$. For any function $A(\cdot)$ defined on $[0,1]$, let

$$\widetilde{A}(t) = A(t) - \sum_{j=1}^{n} A(t_j) \int_{A_j} E_m(t,s)\mathrm{d}s.$$

Let $(j_1, \cdots, j_n)$ and $(\pi_1, \cdots, \pi_n)$ denote permutations of $(1, \cdots, n)$ and the permutations may vary at different places.

§3. Some Assumptions

In order to list some restrictions for ϕ and g, we give two definitions here.

Definition 3.1 A father wavelet ϕ is said to be q-regular (S_q, $q \in \mathcal{N}$) if for any $l \le q$, and for any integer l and k, one has $|\mathrm{d}^l\phi/\mathrm{d}x^l| \le C_k(1+|x|)^{-k}$, where C_k is a generic constant depending only on k.

Definition 3.2 A function space H^γ ($\gamma \in R$) is said to be Sobolev space with order γ, i.e. if $h \in H^\gamma$ then $\displaystyle\int |\widehat{h}(w)|^2(1+w^2)^\gamma \mathrm{d}w < \infty$, where $\widehat{h}$ is the Fourier transform of h.

Now we list the following some assumptions.

(A1) There exists a function $h(\cdot)$ defined on $[0,1]$ such that $x_i = h(t_i) + u_i$ and

(i) $\lim\limits_{n\to\infty} n^{-1}\sum\limits_{i=1}^{n} u_i^2 = \sigma$ $(0 < \sigma < \infty)$,

(ii) $\limsup\limits_{n\to\infty} \dfrac{1}{\sqrt{n}\log n} \max\limits_{1\le m\le n} \Big|\sum\limits_{i=1}^{m} u_{j_i}\Big| < \infty$ for all permutations $(j_1, \cdots, j_n)$ of $(1, \cdots, n)$;

(A2) $g(\cdot), h(\cdot) \in H^\alpha$, $\alpha > 2/3$;

(A3) $\phi \in S_r$, $r \ge \alpha$. Let $\phi(\cdot)$ satisfy Lipschitz condition of order 1 and $|\widehat{\phi}(\xi) - 1| = O(\xi)$ as $\xi \to 0$, where $\widehat{\phi}$ is the Fourier transformation of ϕ;

(A4) $c_1/n \le \min\limits_{1\le i\le n}(t_i - t_{i-1}) \le \max\limits_{1\le i\le n}(t_i - t_{i-1}) \le c_2/n$;

(A5) $c_3 n^{1/4}(\log n)^{1/4} \le 2^m \le c_4 n^{1/2}(\log n)^{-3/2}$;

(A5*) $c_5 n^{1/3} \le 2^m \le c_6 n^{1/3}$.

Remark 1 (i) Condition (A1) is assumed in Sun et al. (2002) and Härdle et al. (2000) for multivariate setting; Conditions (A2)-(A3) were used by Xue (2003), Qian et al. (2000), Chen et al. (2003) and Liang et al. (2004); Condition (A4) is assumed in Xue (2003) and Shi (1998), and (A4) implies $\max\limits_{1\le i\le n}(s_i - s_{i-1}) = O(n^{-1})$.

(ii) According to Antoniadis et al. (1994), condition (A2) follows that $g(t)$ and $h(t)$ are continuous and differentiable on [0,1], which imply that $g(t)$ and $h(t)$ satisfy Lipschitz condition of order 1 on $[0,1]$.

Remark 2 Let us have a look at some consequences of the above assumptions:

(a) Assumptions (A1), (A2) for $h(\cdot)$, (A3)-(A4), and (A5) or (A5*) imply

$$\frac{1}{n}\sum_{i=1}^n \widetilde{x}_i^2 \to \sigma \qquad \text{and} \qquad S_n^{-2}\sum_{i=1}^n |\widetilde{x}_i| \le C.$$

In fact. From $\widetilde{x}_i = u_i + \widetilde{h}(t_i) - \sum\limits_{j=1}^n u_j \int_{A_j} E_m(t_i,s)\mathrm{d}s$, we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n \widetilde{x}_i^2 &= \frac{1}{n}\sum_{i=1}^n u_i^2 + \frac{1}{n}\sum_{i=1}^n \widetilde{h}^2(t_i) + \frac{1}{n}\sum_{i=1}^n\Big(\sum_{j=1}^n u_j\int_{A_j} E_m(t_i,s)\mathrm{d}s\Big)^2 + \frac{2}{n}\sum_{i=1}^n u_i\widetilde{h}(t_i)\\ &\quad -\frac{2}{n}\sum_{i=1}^n u_i\Big(\sum_{j=1}^n u_j\int_{A_j} E_m(t_i,s)\mathrm{d}s\Big) - \frac{2}{n}\sum_{i=1}^n \widetilde{h}(t_i)\Big(\sum_{j=1}^n u_j\int_{A_j} E_m(t_i,s)\mathrm{d}s\Big)\\ &= L_1 + L_2 + L_3 + 2L_4 - 2L_5 - 2L_6.\end{aligned}$$

Note that $\max\limits_{1\le i\le n}|\widetilde{h}(t_i)| = O(2^{-m})$ from (b) below. In order to verify the conclusion, we now introduce the following *Abel Inequality* (see Härdle, Liang and Gao (2000), p.183):

Let $A_1, A_2, \cdots, A_n$; $B_1, B_2, \cdots, B_n$ $(B_1 \ge B_2 \ge \cdots \ge B_n \ge 0)$ be two sequences of real numbers, and let $S_k = \sum\limits_{i=1}^k A_i$, $M_1 = \min\limits_{1\le k\le n} S_k$ and $M_2 = \max\limits_{1\le k\le n} S_k$. Then

$$B_1 M_1 \le \sum_{k=1}^n A_k B_k \le B_1 M_2. \tag{3.1}$$

Now, let G_k, H_k $(1 \le k \le n)$ be arbitrary real numbers satisfying (without loss of generality) $H_1 \ge H_2 \ge \cdots \ge H_n$. Put $Q_s = H_s - H_n$, $1 \le s \le n$. By an application of (3.1), we have

$$\Big|\sum_{k=1}^n G_k H_k\Big| \le \Big|\sum_{k=1}^n G_k Q_k\Big| + \Big|\sum_{k=1}^n G_k H_n\Big| \le 5\max_{1\le i\le n}|H_i| \max_{1\le m\le n}\Big|\sum_{k=1}^m G_k\Big|. \tag{3.2}$$

So, on applying the assumptions, Lemma 5.4 below and (3.2), it is easy to verify that $L_l \to 0$ $(l = 2,3,4,5,6)$. Hence, we have $(1/n)\sum\limits_{i=1}^n \widetilde{x}_i^2 \to \sigma$ from (A1)(i). Further we get $S_n^{-2}\sum\limits_{i=1}^n |\widetilde{x}_i| \le C$.

(b) Assume that function $A(\cdot)$ satisfies a Lipschitz condition of order 1 on $[0,1]$. From Lemma 2.1 of Qian et al. (2000), under the assumptions (A3)-(A4) we have $\sup\limits_t |\widetilde{A}(t)| = O(2^{-m}) + O(n^{-1})$.

§4. Main Results

Our main results are as follows.

Theorem 4.1 (a) Suppose that (A1)-(A5) hold and $\sup\limits_{n\ge1} n\sqrt{\log n}\sum\limits_{|j|>n}|c_j|<\infty$. If $\mathsf{E}|e_0|^{2+\delta}<\infty$ for some $\delta>0$ and $\max\limits_{1\le i\le n}|u_i|=O(n^{\delta/2(2+\delta)}(\log n)^{1/2})$, then $\widehat{\beta}_n-\beta=O(n^{-1/2}(\log n)^{1/2})$ a.s..

(b) Suppose that (A1)-(A4) and (A5*) hold. Let $\mathsf{E}|e_0|^{2+\delta}<\infty$ for some $\delta>0$. If $\sup\limits_{n\ge1} n^{1/2+\gamma}\sum\limits_{|j|>n}|c_j|<\infty$ for some $\gamma\le1/(2+\delta)$ and $\max\limits_{1\le i\le n}|u_i|=O(n^{(\delta(1-\gamma)+(1-2\gamma))/(2+\delta)}\log n)$, then $\widehat{\beta}_n-\beta=o(n^{-\gamma}\log n)$ a.s..

In particular, if $\delta=1$, $\gamma=1/3$, then, under $\sup\limits_{n\ge1} n^{5/6}\sum\limits_{|j|>n}|c_j|<\infty$, $\max\limits_{1\le i\le n}|u_i|=O(n^{1/3}\log n)$ and $\mathsf{E}|e_0|^3<\infty$, we have $\widehat{\beta}_n-\beta=o(n^{-1/3}\log n)$ a.s..

Theorem 4.2 Suppose that (A1)-(A4) and (A5*) hold, and that $n^{5/6}\sum\limits_{|j|>n}|c_j|<\infty$, $\max\limits_{1\le i\le n}|u_i|=O(n^{1/3}\log n)$. If $\mathsf{E}|e_0|^3<\infty$, then $\sup\limits_{0\le t\le1}|\widehat{g}_n(t)-g(t)|=O(n^{-1/3}\log n)$ a.s..

Remark 3 Since the NA property includes independence, our theorems include the results in the i.i.d. setting as a special case. Theorem 4.2 shows that the estimator of the nonparametric component in model (1.1) attains the optimal strong uniform convergence rate known from nonparametric estimation in corresponding i.i.d. models.

§5. Preliminary Lemmas

In this section, let $\{e_i\}$ be identically distributed, negatively associated random variables with $\mathsf{E}e_i=0$, and let $\{V_i\}$ be defined as in (1.2).

Lemma 5.1 (Jing and Liang (2004)) Let $\mathsf{E}|e_1|^p<\infty$ for some $p>2$. Assume that $\{b_{ni}, 1\le i\le k_n, n\ge1\}$ is an array of real numbers satisfying $\max\limits_{1\le i\le k_n}|b_{ni}|=O(n^{-1/p})$, where $k_n\le Mn$ (M an integer not depending on n) is a sequence of positive integers. If $\sum\limits_{i=1}^{k_n}b_{ni}^2=o((\log n)^{-1})$, then $\sum\limits_{i=1}^{k_n}b_{ni}e_i=o(1)$ a.s.; if $\sum\limits_{i=1}^{k_n}b_{ni}^2=O((\log n)^{-1})$, then $\sum\limits_{i=1}^{k_n}b_{ni}e_i=O(1)$ a.s.

Lemma 5.2 (Liang et al. (2006)) Assume that $\{a_{ni}(t), 1\le i\le n, n\ge1\}$ is a family of real functions defined on $[0,1]$ satisfying

(i) $\sum\limits_{i=1}^{n}|a_{ni}(t)|=O(1)$ and $\max\limits_{1\le i\le n}|a_{ni}(t)|=O(n^{-s})$, for some $s>0$, uniformly in $t\in[0,1]$;

(ii) $\max\limits_{1\le i\le n}|a_{ni}(s)-a_{ni}(t)|\le c|s-t|$, uniformly in $t\in[0,1]$ and $n\ge 1$.

Assume that $\mathsf{E}|e_0|^p<\infty$ for some $p\ge 2$. Then, for the linear process defined in (1.2) with $\sup\limits_{n\ge1} n^{1/2+\alpha}\sum\limits_{|j|>n}|c_j|<\infty$ for $1/p\le\alpha\le\min(s-1/p,1-1/p,s/2)$, we have $\sup\limits_{0\le t\le 1}\Big|\sum\limits_{i=1}^{n}a_{ni}(t)V_i\Big|=O(n^{-\alpha}\log n)$ a.s..

Lemma 5.3 (Liang et al. (2006)) Let $\mathsf{E}|e_0|^p<\infty$ for some $p\ge 2$. If $\sup\limits_{n\ge1} n^{3/2-\eta}\cdot\sum\limits_{|j|>n}|c_j|<\infty$, for some $\eta\ge\max(1/p,1-1/p)$, then $\max\limits_{1\le m\le n}\Big|\sum\limits_{i=1}^{m}V_{j_i}\Big|=O(n^{\eta}\log n)$ a.s.. In particular, if $p=2$, $\eta=1/2$, then, under $\sup\limits_{n\ge1} n\sum\limits_{|j|>n}|c_j|<\infty$, we have $\max\limits_{1\le m\le n}\Big|\sum\limits_{i=1}^{m}V_{j_i}\Big|=O(n^{1/2}\log n)$ a.s.; if $p=2+\delta$ for some $\delta>0$, $\eta=1-\gamma$ for some $\gamma\le 1/(2+\delta)$, then, under $\mathsf{E}|e_0|^{2+\delta}<\infty$ and $\sup\limits_{n\ge1} n^{1/2+\gamma}\sum\limits_{|j|>n}|c_j|<\infty$, we have $\max\limits_{1\le m\le n}\Big|\sum\limits_{i=1}^{m}V_{j_i}\Big|=O(n^{1-\gamma}\log n)$ a.s..

Lemma 5.4 (Xue (2003)) (a) Under (A3) we have

(1) $\sup\limits_{0\le t,s\le 1}|E_m(t,s)|=O(2^m)$; (2) $\sup\limits_{0\le t\le 1}\displaystyle\int_0^1|E_m(t,s)|\mathrm{d}s\le C$.

(b) Under (A3) and (A4) we have

(3) $\max\limits_{1\le i\le n}\sum\limits_{j=1}^{n}\displaystyle\int_{A_j}|E_m(t_i,s)|\mathrm{d}s\le C$; (4) $\max\limits_{1\le j\le n}\sum\limits_{i=1}^{n}\displaystyle\int_{A_j}|E_m(t_i,s)|\mathrm{d}s\le C$.

§6. Proofs of Main Results

Proof of Theorem 4.1 (a) From the definition of $\widehat{\beta}_n$ in (2.2),

$$\begin{aligned}\widehat{\beta}_n-\beta &= S_n^{-2}\Big\{\sum_{i=1}^{n}\widetilde{x}_iV_i-\sum_{i=1}^{n}\widetilde{x}_i\Big(\sum_{j=1}^{n}V_j\int_{A_j}E_m(t_i,s)\mathrm{d}s\Big)+\sum_{i=1}^{n}\widetilde{x}_i\widetilde{g}_i\Big\}\\ &:= S_n^{-2}\{I_{1n}-I_{2n}+I_{3n}\}\end{aligned}$$

and $I_{1n}=\sum\limits_{i=1}^{n}u_iV_i+\sum\limits_{i=1}^{n}\widetilde{h}_iV_i-\sum\limits_{i=1}^{n}V_i\Big(\sum\limits_{j=1}^{n}u_j\displaystyle\int_{A_j}E_m(t_i,s)\mathrm{d}s\Big)=:I_{11n}+I_{12n}-I_{13n}$, where $\widetilde{g}_i=\widetilde{g}(t_i)$, $\widetilde{h}_i=\widetilde{h}(t_i)$, and $\widetilde{x}_i=\widetilde{h}_i+u_i-\sum\limits_{j=1}^{n}u_j\displaystyle\int_{A_j}E_m(t_i,s)\mathrm{d}s$. Hence, from Remark 2 (a), it suffices to show that

$$I_{11n}+I_{12n}-I_{13n}-I_{2n}+I_{3n}=O(\sqrt{n\log n}),\qquad \text{a.s..}$$

It is easy to see that

$$\begin{aligned}I_{2n} &= \sum_{i=1}^{n}u_i\Big(\sum_{j=1}^{n}V_j\int_{A_j}E_m(t_i,s)\mathrm{d}s\Big)+\sum_{i=1}^{n}\widetilde{h}_i\Big(\sum_{j=1}^{n}V_j\int_{A_j}E_m(t_i,s)\mathrm{d}s\Big)\\ &\quad+\sum_{i=1}^{n}\Big(\sum_{p=1}^{n}u_p\int_{A_p}E_m(t_i,s)\mathrm{d}s\Big)\Big(\sum_{j=1}^{n}V_j\int_{A_j}E_m(t_i,s)\mathrm{d}s\Big).\end{aligned}\tag{6.1}$$

On applying (3.2), (A1), (A5), and according to Remark 2, Lemma 5.4 and Lemma 5.3, we have

$$
\begin{aligned}
&\Big|\sum_{i=1}^{n} u_i\Big(\sum_{j=1}^{n} V_j\int_{A_j} E_m(t_i,s)\mathrm{d}s\Big)\Big| \\
\ll\ & \max_{1\le i,j\le n}\Big|\int_{A_j} E_m(t_i,s)\mathrm{d}s\Big| \max_{1\le m\le n}\Big|\sum_{i=1}^{m} V_{\pi_i}\Big|\cdot\max_{1\le k\le n}\Big|\sum_{i=1}^{k} u_{j_i}\Big| \\
=\ & O\Big(\frac{2^m}{n}\Big)\cdot O(n^{1/2}\log n)\cdot O(n^{1/2}\log n)=O(\sqrt{n\log n}), \qquad \text{a.s.,} && (6.2)\\
&\Big|\sum_{i=1}^{n} \widetilde{h}_i\Big(\sum_{j=1}^{n} V_j\int_{A_j} E_m(t_i,s)\mathrm{d}s\Big)\Big| \\
\ll\ & \max_{1\le i\le n}|\widetilde{h}_i|\cdot\max_{1\le k\le n}\Big|\sum_{i=1}^{k}\int_{A_j} E_m(t_{\pi_i},s)\mathrm{d}s\Big|\cdot\max_{1\le l\le n}\Big|\sum_{i=1}^{l} V_{j_i}\Big| \\
=\ & O(2^{-m})\cdot O(1)\cdot O(n^{1/2}\log n)=O((\log n)^2), \qquad \text{a.s.,} && (6.3)\\
&\Big|\sum_{i=1}^{n}\Big(\sum_{p=1}^{n} u_p\int_{A_p} E_m(t_i,s)\mathrm{d}s\Big)\Big(\sum_{j=1}^{n} V_j\int_{A_j} E_m(t_i,s)\mathrm{d}s\Big)\Big| \\
=\ & O(1)\max_{1\le i,j\le n}\Big|\int_{A_j} E_m(t_i,s)\mathrm{d}s\Big|\cdot\max_{1\le m\le n}\Big|\sum_{i=1}^{m} u_{\pi_i}\Big| \\
&\cdot\max_{1\le k,j\le n}\Big|\sum_{i=1}^{k}\int_{A_j} E_m(t_{\pi_i},s)\mathrm{d}s\Big|\cdot\max_{1\le l\le n}\Big|\sum_{i=1}^{l} V_{j_i}\Big| \\
=\ & O\Big(\frac{2^m}{n}\Big)\cdot O(n^{1/2}\log n)\cdot O(n^{1/2}\log n)\cdot O(1)=O(\sqrt{n\log n}), \qquad \text{a.s..} && (6.4)
\end{aligned}
$$

Thus, from (6.1)-(6.4) it follows that $I_{2n}=O(\sqrt{n\log n})$. Similarly,

$$
\begin{aligned}
|I_{3n}| \le\ & \Big|\sum_{i=1}^{n} u_i\widetilde{g}_i\Big|+\Big|\sum_{i=1}^{n}\widetilde{h}_i\widetilde{g}_i\Big|+\Big|\sum_{i=1}^{n}\Big(\sum_{j=1}^{n} u_j\int_{A_j} E_m(t_i,s)\mathrm{d}s\Big)\widetilde{g}_i\Big| \\
\ll\ & \max_{1\le i\le n}|\widetilde{g}_i|\cdot\max_{1\le k\le n}\Big|\sum_{i=1}^{k} u_{j_i}\Big|+n\max_{1\le i\le n}|\widetilde{h}_i|\cdot\max_{1\le i\le n}|\widetilde{g}_i| \\
&+\max_{1\le i\le n}|\widetilde{g}_i|\cdot\max_{1\le k,j\le n}\Big|\sum_{i=1}^{k}\int_{A_j} E_m(t_{\pi_i},s)\mathrm{d}s\Big|\cdot\max_{1\le l\le n}\Big|\sum_{i=1}^{l} u_{j_i}\Big|=O(\sqrt{n\log n}),\\
|I_{12n}| \ll\ & \max_{1\le i\le n}|\widetilde{h}_i|\cdot\max_{1\le m\le n}\Big|\sum_{i=1}^{m} V_{j_i}\Big|=O((\log n)^{5/2}),\\
|I_{13n}| \ll\ & \max_{1\le i,j\le n}\Big|\int_{A_j} E_m(t_{\pi_i},s)\mathrm{d}s\Big|\cdot\max_{1\le m\le n}\Big|\sum_{i=1}^{m} V_{\pi_i}\Big|\cdot\max_{1\le k\le n}\Big|\sum_{i=1}^{k} u_{j_i}\Big|=O(\sqrt{n\log n}).
\end{aligned}
$$

Therefore $I_{12n}-I_{13n}-I_{2n}+I_{3n}=O(\sqrt{n\log n})$, a.s.. Next, we prove

$$
I_{11n}=O(\sqrt{n\log n}), \qquad \text{a.s..} \tag{6.5}
$$

Here, we will use a similar arguments as in Liang et al. (2006). Note that

$$
\frac{I_{11n}}{\sqrt{n\log n}}=\frac{1}{\sqrt{n\log n}}\sum_{i=1}^{n} u_i\sum_{j=-n}^{n} c_je_{i-j}+\frac{1}{\sqrt{n\log n}}\sum_{i=1}^{n} u_i\sum_{|j|>n} c_je_{i-j}:=A_{1n}+A_{2n}. \tag{6.6}
$$

By $\sum\limits_{i=1}^{n}|u_i| = O(n)$ and $\mathsf{E}e_0^2 < \infty$, for any $\alpha > 0$, we have

$$\begin{aligned}\mathsf{P}(|A_{2n}| > \alpha) &\le \frac{1}{\alpha^2 n\log n}\mathsf{E}\Big\{\sum_{i=1}^{n} u_i \sum_{|j|>n} c_j e_{i-j}\Big\}^2\\ &\le \frac{1}{\alpha^2 n\log n}\mathsf{E}\Big\{\sum_{i_1=1}^{n}|u_{i_1}|\sum_{i_2=1}^{n}|u_{i_2}|\cdot\Big|\sum_{|j_1|>n} c_{j_1}e_{i_1-j_1}\cdot\sum_{|j_2|>n} c_{j_2}e_{i_2-j_2}\Big|\Big\}\\ &\ll \frac{1}{n(\log n)^2}\Big(n\sqrt{\log n}\sum_{|j|>n}|c_j|\Big)^2.\end{aligned}$$

An application of the Borel-Cantelli lemma together with $n\sqrt{\log n}\sum\limits_{|j|>n}|c_j| < \infty$ implies

$$A_{2n} = o(1), \qquad \text{a.s.}. \tag{6.7}$$

Note that $A_{1n} = (n\log n)^{-1/2}\sum\limits_{l=1-n}^{2n}\Big(\sum\limits_{i=\max(1,l-n)}^{\min(n,n+l)} u_i c_{i-l}\Big)e_l := \sum\limits_{l=1-n}^{2n} b_{nl}e_l$ and

$$\max_{1-n\le l\le 2n}|b_{nl}| \le \max_{1\le i\le n}|u_i|\Big(\sum_{j=-\infty}^{\infty}|c_j|\Big)\Big/\sqrt{n\log n} = O(n^{-1/(2+\delta)}),$$

$$\begin{aligned}\sum_{l=1-n}^{2n} b_{nl}^2 &= \frac{1}{n\log n}\Big\{\sum_{l=1-n}^{2n}\sum_{i=\max(1,l-n)}^{\min(n,n+l)} u_i^2 c_{i-l}^2 + 2\sum_{l=1-n}^{2n}\sum_{s=\max(1,l-n)}^{\min(n,n+l)-1}\sum_{t=s+1}^{\min(n,n+l)} u_s u_t c_{s-l}c_{t-l}\Big\}\\ &\le \frac{1}{n\log n}\Big\{\sum_{i=1}^{n}u_i^2\Big(\sum_{j=-\infty}^{\infty}c_j^2\Big) + 2\sum_{i=1}^{n}u_i^2\Big(\sum_{j=-\infty}^{\infty}|c_j|\Big)^2\Big\} = O((\log n)^{-1}).\end{aligned}$$

Hence, according to Lemma 5.1, we have $A_{1n} = O(1)$ a.s., which, together with (6.6)-(6.7), yields (6.5).

(b) From the proof in (a), it suffices to show that

$$I_{11n} + I_{12n} - I_{13n} - I_{2n} + I_{3n} = o(n^{1-\gamma}\log n), \qquad \text{a.s.}.$$

Following the line as in (a), from the assumptions one can obtain that $I_{12n} - I_{13n} - I_{2n} + I_{3n} = o(n^{1-\gamma}\log n)$, a.s.. Now, we verify that $I_{11n} = o(n^{1-\gamma}\log n)$, a.s.. Write

$$\frac{I_{11n}}{n^{1-\gamma}\log n} = \frac{1}{n^{1-\gamma}\log n}\Big\{\sum_{i=1}^{n}u_i\sum_{j=-n}^{n}c_j e_{i-j} + \sum_{i=1}^{n}u_i\sum_{|j|>n}c_j e_{i-j}\Big\} := A'_{1n} + A'_{2n}.$$

It is easy to see that

$$\mathsf{P}(|A'_{2n}| > \alpha) \le \frac{C}{n(\log n)^2}\Big(n^{1/2+\gamma}\sum_{|j|>n}|c_j|\Big)^2,$$

which yields that $A'_{2n} = o(1)$ a.s. from $\sup\limits_{n\ge 1} n^{1/2+\gamma}\sum\limits_{|j|>n}|c_j| < \infty$.

Since $A'_{1n} = \dfrac{1}{n^{1-\gamma}\log n}\sum\limits_{l=1-n}^{2n}\Big(\sum\limits_{i=\max(1,l-n)}^{\min(n,n+l)} u_i c_{i-l}\Big)e_l := \sum\limits_{l=1-n}^{2n} b'_{nl}e_l$ and

$$\max_{1-n\le l\le 2n}|b'_{nl}| = O(n^{-1/(2+\delta)}),\qquad \sum_{l=1-n}^{2n} b_{nl}^2 = o((\log n)^{-1}).$$

Hence, according to Lemma 5.1, we have $A'_{1n} = o(1)$ a.s..

This completes the proof of Theorem 4.1. □

Proof of Theorem 4.2 Observe that

$$\begin{aligned}
&\sup_{0\le t\le 1}|\widehat{g}_n(t)-g(t)|\\
\le\ & \sup_{0\le t\le 1}\Big|\sum_{i=1}^n V_i\int_{A_i}E_m(t,s)\mathrm{d}s\Big| + \sup_{0\le t\le 1}\Big|\sum_{i=1}^n x_i(\beta-\widehat{\beta}_n)\int_{A_i}E_m(t,s)\mathrm{d}s\Big|\\
&+\sup_{0\le t\le 1}\Big|\sum_{i=1}^n g(t_i)\int_{A_i}E_m(t,s)\mathrm{d}s - g(t)\Big|\\
:=\ & B_{1n}+B_{2n}+B_{3n}.
\end{aligned}$$

From (A4) and Lemma 5.4 we have

$$\begin{aligned}
&\sum_{i=1}^n\Big|\int_{A_i}E_m(t,s)\mathrm{d}s\Big| \le \sum_{i=1}^n\int_{A_i}|E_m(t,s)|\mathrm{d}s = O(1);\\
&\max_{1\le i\le n}\Big|\int_{A_i}E_m(t,s)\mathrm{d}s\Big| = O\Big(\frac{2^m}{n}\Big) = O(n^{-2/3}).
\end{aligned}$$

Note that $E_0(t,s)$ uniformly satisfies Lipschitz condition of order 1 on t, so it is easy to verify that $|E_m(t_1,s)-E_m(t_2,s)|\le C2^{2m}|t_1-t_2|$ for $s,\ t_1,t_2\in[0,1]$. Hence, in view of Lemma 5.4 we can obtain that

$$\max_{1\le i\le n}\Big|\int_{A_i}E_m(t_1,s)\mathrm{d}s - \int_{A_i}E_m(t_2,s)\mathrm{d}s\Big| \le C\frac{2^{2m}}{n}|t_1-t_2| = O(|t_1-t_2|).$$

From (A4) and Lemma 5.2, choosing $p=3$, $\alpha=1/3$, $s=2/3$, we have

$$B_{1n} = O(n^{-1/3}\log n)\qquad \text{a.s..}$$

By (A1) (ii), (A2), (A4), (3.2), and Theorem 4.1 (b), where $\delta=1$, $\gamma=1/3$, it follows

$$\begin{aligned}
B_{2n}\ &\le\ C|\beta-\widehat{\beta}_n|\sup_{0\le t\le 1}\Big\{\sum_{i=1}^n\int_{A_i}|E_m(t,s)|\mathrm{d}s\cdot\max_{1\le i\le n}|h(t_i)|\\
&\qquad +\max_{1\le i\le n}\int_{A_i}|E_m(t,s)|\mathrm{d}s\cdot\max_{1\le k\le n}\Big|\sum_{i=1}^k u_{j_i}\Big|\Big\}\\
&=\ o(n^{-1/3}\log n)\qquad\text{a.s..}
\end{aligned}$$

Finally, from Remark 2 (b) we have $B_{3n} = O(2^{-m}) = o(n^{-1/3}\log n)$. Therefore, the proof is completed. □

§7. Simulations

In this Section, we carry out a simulation to study the finite sample performance of the estimators $\widehat{\beta}$ and $\widehat{g}$ of β and g, respectively. The observations are generated from

$$y_i = 3.5x_i + \sin(2\pi t_i) + V_i,$$

where V_i is an AR(1) process $V_i = 0.5V_{i-1} + e_i$ and e_i are i.i.d. $U(-0.5, 0.5)$ random variables, $t_i = (i-0.5)/n$, x_i are generated from $x_i = t_i^2 + u_i$ and $u_i = (i-1)/(n-1) - 0.5$.

We choose the scale function $\phi(x) = I(0 \le x \le 1)$. For a given sample size n, taking $2^m = n^{1/3}$. We generate 1000, 2000 and 5000 samples, respectively, from the above model. Biases and mean square errors (MSE) of the estimator $\widehat{\beta}$ are given in following Table based on 50 replications.

n	Bias($\widehat{\beta}$)	MSE($\widehat{\beta}$)
1000	-0.0488	0.2855
2000	-0.1097	0.1217
5000	-0.0127	0.0626

From the Table, it can be seen that MSE of the estimator $\widehat{\beta}$ reduces obviously as increasing of the sample size n.

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相依MA(∞)误差下半参数模型小波估计的收敛速度

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考虑半参数回归模型$y_i = x_i\beta + g(t_i) + V_i$ $(1 \le i \le n)$, 其中(x_i, t_i)是已知的设计点, 斜率参数β是未知的, $g(\cdot)$是未知函数, 误差$V_i = \sum\limits_{j=-\infty}^{\infty} c_j e_{i-j}$, $\sum\limits_{j=-\infty}^{\infty} |c_j| < \infty$并且$e_i$是负相关的随机变量. 在适当的条件下, 我们研究了β与$g(\cdot)$小波估计量的强收敛速度. 结果显示$g(\cdot)$的小波估计量达到最优收敛速度. 同时, 对β小波估计量也作了模拟研究.

关 键 词: 半参数回归模型, 负相关, 小波估计量, 收敛速度.

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