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The Gerber-Shiu Function in a Risk Model Perturbed by Diffusion *

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Abstract

In this paper we consider a risk process perturbed by diffusion and obtain the integro-differential equation for the Gerber-Shiu expected discounted penalty function. Furthermore, we prove that the Gerber-Shiu function satisfies a certain renewal equation in the case of the generalized Erlang(2).

Keywords: Gerber-Shiu function, integro-differential equations, Laplace transform, generalized Lundberg's fundamental equation.

AMS Subject Classification: 60G35.

§1. Introduction

Consider a risk process

$$U(t)=u+ct-\sum_{i=1}^{K_1(t)}X_i-\sum_{j=1}^{K_2(t)}Y_j+\sigma B(t), \tag{1.1}$$

where $u>0$ is initial surplus, c is the positive constant premium income rate, $\{X_i, i\geq 1\}$ are the claim sizes form the first class, assumed to be i.i.d. positive random variables with a common p.d.f. $f_X(x)$, while $\{Y_j, j\geq 1\}$ are the claim sizes form the second class, assumed to be i.i.d. random variables with a common p.d.f. $f_Y(x)$, the claim number process $K_1(t)$ is a Poisson process with parameter μ, i.e. the corresponding claim inter-arrival times, denoted by $\{L_i\}_{i\geq 1}$, are i.i.d. exponentially distributed r.v. with parameter μ; By contrast, the claim number process $K_2(t)$ is a renewal process with i.i.d. claim inter-arrival times $\{W_i\}_{i\geq 1}$, which are independent of $\{L_i\}_{i\geq 1}$ and generalized Erlang(n) distributed, i.e. the W_i's are distributed as the sum of n independent and exponentially

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distributed r.v.'s $S_n := V_1 + V_2 + \cdots + V_n$, where V_i $(i = 1, \cdots, n)$ may have different exponential parameters $\lambda_i > 0$. $\{B(t) : t \geq 0\}$ is a standard Brownian motion, σ is a positive constant.

We assume that $\{X_i, i \geq 1\}$, $\{Y_j, j \geq 1\}$, $K_1(t)$, $K_2(t)$ and $B(t)$ are independent each other. Now define

$$T = \inf\{t \geq 0, U(t) \leq 0\}, \qquad T = \infty \text{ if the set is empty,}$$

to be the time of the ruin, and

$$\psi(u) = \mathsf{P}(T < \infty | U(0) = u),$$
$$\phi_s(u) = \mathsf{E}(e^{-\delta T} w(U(T-), |U(T)|) I_{(T<\infty, U(T)<0)} | U(0) = u)$$

to be the ruin probability and the expected discounted penalty function if the ruin is caused by a claim, respectively. Let

$$\phi_d(u) = \mathsf{E}(e^{-\delta T} I_{(T<\infty, U(T)=0)} | U(0) = u)$$

be the Laplace transform of the ruin time T due to the oscillations. Then

$$\phi(u) = \phi_s(u) + \phi_d(u)$$

is the Gerber-Shiu expected discounted penalty function.

Recently, many studies tackle ruin problems for Sparre Anderson models, in which claims occur according to a more general renewal process. Li and Garrido (2005) consider a risk process with claim i.i.d. inter-arrival times distributed as generalized Erlang(n) ($n \in N^+$) with perturbed diffusion, extending the classical risk model and the Erlang(2) model of Dickson (1998), Dickson and Hipp (2001) and Cheng and Tang (2003). Gerber and Shiu (2005) extend the model to generalized Erlang(n) waiting times. In this paper, we consider the expected discounted penalty function for a risk process perturbed by a diffusion. As in Li and Garrido (2005) we assume that claim process is generalized Erlang(n) process. An outline of the paper is as follows. For the risk model (1.1), we first obtain the integro-differential equation for the Gerber-Shiu expected discounted penalty function in Section 2, and in Section 3 we get the renewal equation for integro-differential equation when $K_2(t)$ is the generalized Erlang(2) process case.

§2. Integro-Differential Equations

For the model (1.1), we obtain the integro-differential equation for the Gerber-Shiu expected discounted penalty function as follows:

Theorem 2.1 Let I, D and P denote the identity operator, differential operator and Integration operator, respectively. Let $P(H(u)) = \int_0^u H(u-x)f_X(x)\mathrm{d}x$. It is assumes that $\phi(u)$, $\phi_s(u)$, and $\phi_d(u)$ are $2n$th differentiable. Then $\phi_s(u), \phi_d(u)$ satisfies the following equations for $u \geq 0$,

$$\Big(\prod_{i=1}^{n}\Big(\gamma_i(D)-\frac{\mu}{\lambda_i}P\Big)\Big)\phi_s(u)+\Omega(D)w_1(u)=\int_0^u \phi_s(u-x)f_Y(x)\mathrm{d}x+w_2(u), \tag{2.1}$$

with $\phi_s(0)=0$,

$$\prod_{i=1}^{n}\Big(\gamma_i(D)-\frac{\mu}{\lambda_i}P\Big)\phi_d(u)=\int_0^u \phi_d(u-x)f_Y(x)\mathrm{d}x, \tag{2.2}$$

with $\phi_d(0)=1$, where

$$\gamma_i(D)=\Big(1+\frac{\delta+\mu}{\lambda_i}\Big)I-\frac{c}{\lambda_i}D-\frac{\sigma^2}{2\lambda_i}D^2, \qquad i=1,2,\cdots,n,$$
$$\Omega(D)=\Big(\sum_{i=0}^{n-1}\frac{\mu}{\lambda_{n-i}}\prod_{k=n-i}^{n}\Big(\gamma_k(D)-\frac{\mu}{\lambda_{n-i}}P\Big)\Big),$$
$$w_1(u)=\int_u^\infty w(u,x-u)f_X(x)\mathrm{d}x, \qquad w_2(u)=\int_u^\infty w(u,x-u)f_Y(x)\mathrm{d}x.$$

Proof Fixed the number $j=0,1,\cdots,n-1$ of exponential r.v.'s of the sum $S_j := V_1+V_2+\cdots+V_j$, $j=1,2,\cdots,n-1$, with $S_0=0$. We define

$$\phi_{s,j}(u)=\mathsf{E}(e^{-\delta(T-t)}w(U(T-),|U(T)|)I_{(T<\infty,U(T)<0)}|S_j=t,U(0)=u),$$

with $\phi_{s,0}(u)=\phi_s(u)$ and $\phi_{s,j}(0)=0$, $j=1,2,\cdots,n-1$. Then we consider infinitesimal interval $[S_j, S_j+dt]$ for $j=1,2,\cdots,n-2$, we get

$$\begin{aligned}
&\phi_{s,j}(u)\\
=\ &e^{-\delta t}\{\mathsf{P}(L_1>dt,V_{j+1}>dt)\mathsf{E}(\phi_{s,j}(u+cdt+\sigma B(dt)))\\
&+\mathsf{P}(L_1>dt,V_{j+1}<dt)\mathsf{E}(\phi_{s,j+1}(u+cdt+\sigma B(dt)))\}\\
&+e^{-\delta t}\Big\{\mathsf{P}(L_1<dt,V_{j+1}>dt)\Big[\int_0^{u+cdt+\sigma B(dt)}\mathsf{E}(\phi_{s,j}(u+cdt+\sigma B(dt)-x))f_X(x)\mathrm{d}x\\
&+\int_{u+cdt+\sigma B(dt)}^\infty w(u+cdt+\sigma B(dt),x-u-cdt-\sigma B(dt))f_X(x)\mathrm{d}x\Big]\\
&+\mathsf{P}(L_1<dt,V_{j+1}<dt)\Big[\int_0^{u+cdt+\sigma B(dt)}\mathsf{E}(\phi_{s,j+1}(u+cdt+\sigma B(dt)-x))f_X(x)\mathrm{d}x\\
&+\int_{u+cdt+\sigma B(dt)}^\infty w(u+cdt+\sigma B(dt),x-u-cdt-\sigma B(dt))f_X(x)\mathrm{d}x\Big]\Big\}.
\end{aligned} \tag{2.3}$$

Substituting

$$\begin{aligned}
&\mathsf{P}(L_1 > dt, V_{j+1} > dt) = 1 - (\mu + \lambda_{j+1})dt + o(dt),\\
&\mathsf{P}(L_1 > dt, V_{j+1} < dt) = (1 - \mu dt)\lambda_{j+1}dt + o(dt),\\
&\mathsf{P}(L_1 < dt, V_{j+1} > dt) = \mu dt(1 - \lambda_{j+1}dt) + o(dt),\\
&\mathsf{P}(L_1 < dt, V_{j+1} < dt) = \mu dt\lambda_{j+1}dt = o(dt),\\
&e^{-\delta t} = 1 - \delta t + o(dt),
\end{aligned}$$

and

$$\mathsf{E}(\phi_{s,j}(u + cdt + \sigma B(dt))) = \phi_{s,j}(u) + \Big[c\phi'_{s,j}(u) + \frac{\sigma^2}{2}\phi''_{s,j}(u)\Big]dt + o(dt)$$

into (2.3), letting $dt \to 0$ and rearranging it, we get

$$\begin{aligned}
&(\delta + \lambda_{j+1} + \mu)\phi_{s,j}(u) - c\phi'_{s,j}(u) - \frac{\sigma^2}{2}\phi''_{s,j}(u)\\
= \ &\lambda_{j+1}\phi_{s,j+1}(u) + \mu\Big(\int_0^u \phi_{s,j}(u-x)f_X(x)\mathrm{d}x + \int_u^\infty w(u, x-u)f_X(x)\mathrm{d}x\Big),
\end{aligned}$$

so

$$\phi_{s,j+1}(u) = \Big[\Big(1 + \frac{\delta+\mu}{\lambda_{j+1}}\Big)I - \frac{c}{\lambda_{j+1}}D - \frac{\sigma^2}{2\lambda_{j+1}}D^2 - \frac{\mu}{\lambda_{j+1}}P\Big]\phi_{s,j}(u) + \frac{\mu}{\lambda_{j+1}}w_1(u), \tag{2.4}$$

where $w_1(u) = \int_u^\infty w(u, x-u)f_X(x)\mathrm{d}x$. Similarly, for $j = n-1$, we have

$$\begin{aligned}
&\int_0^u \phi_{s,n}(u-x)f_Y(x)\mathrm{d}x + w_2(u)\\
= \ &\Big[\Big(1 + \frac{\delta+\mu}{\lambda_n}\Big)I - \frac{c}{\lambda_n}D - \frac{\sigma^2}{2\lambda_n}D^2 - \frac{\mu}{\lambda_n}P\Big]\phi_{s,n-1}(u) + \frac{\mu}{\lambda_n}w_1(u),
\end{aligned}$$

where $w_2(u) = \int_u^\infty w(u, x-u)f_Y(x)\mathrm{d}x$, I, D and P denote the identity operator, differential operator and Integration operator in turn.

For $\phi_{s,0}(u) = \phi_s(u)$, we get

$$\Big(\prod_{i=1}^n \gamma_i(D) - \frac{\mu}{\lambda_i}P\Big)\phi_s(u) + \Omega(D)w_1(u) = \int_0^u \phi_s(u-x)f_Y(x)\mathrm{d}x + w_2(u),$$

where

$$\begin{aligned}
\gamma_i(D) &= \Big(1 + \frac{\delta+\mu}{\lambda_i}\Big)I - \frac{c}{\lambda_i}D - \frac{\sigma^2}{2\lambda_i}D^2, \qquad i = 1, 2, \cdots, n,\\
\Omega(D) &= \Big(\sum_{i=0}^{n-1}\frac{\mu}{\lambda_{n-i}}\prod_{k=n-i}^{n}\Big(\gamma_k(D) - \frac{\mu}{\lambda_{n-i}}P\Big)\Big).
\end{aligned}$$

Finally, we have $\phi_s(0) = 0$, since $\mathsf{P}(T < \infty, U(T) < 0 | U(0) = 0) = 0$.

To verify the equation for $\phi_d(u)$, we define

$$\phi_{d,j}(u) = \mathsf{E}(e^{-\delta(T-t)} I_{(T<\infty, U(T)=0)} | S_j = t, U(0) = u), \qquad j = 1, 2, \cdots, n-1,$$

with $\phi_{d,0}(u) = \phi_d(u)$. Similar to the arguments above, we get

$$\phi_{d,j+1}(u) = \Big[\Big(1 + \frac{\delta+\mu}{\lambda_{j+1}}\Big) I - \frac{c}{\lambda_{j+1}} D - \frac{\sigma^2}{2\lambda_{j+1}} D^2 - \frac{\mu}{\lambda_{j+1}} P\Big] \phi_{d,j}(u),$$

for $j = 1, 2, \cdots, n-2$, and for $j = n-1$,

$$\int_0^u \phi_{d,n}(u-x) f_Y(x) \mathrm{d}x = \Big[\Big(1 + \frac{\delta+\mu}{\lambda_n}\Big) I - \frac{c}{\lambda_n} D - \frac{\sigma^2}{2\lambda_n} D^2 - \frac{\mu}{\lambda_n} P\Big] \phi_{d,n-1}(u).$$

Then

$$\Big(\prod_{i=1}^n \Big(\gamma_i(D) - \frac{\mu}{\lambda_i} P\Big)\Big) \phi_d(u) = \int_0^u \phi_d(u-x) f_Y(x) \mathrm{d}x.$$

We also have $\phi_d(0) = 1$, since $\mathsf{P}(T < \infty, U(T) = 0 | U(0) = 0) = 1$. $\square$

Remark 1 (1) If $X_i = 0$, $i = 1, 2, \cdots, n$, the equation (2.1) and (2.2) yield (2) and (3) of Li and Garrido (2005).

(2) If $\sigma = 0$, the equation (2.4) yields (6) of Li and Lu (2004).

§3. Renewal Equation

We first study the roots of the generalized Lundberg's fundamental equation.

Theorem 3.1 For $\delta > 0$ and $n \in N$, the equation

$$\Big(\prod_{i=1}^n \Big(\gamma_i(s) - \frac{\mu}{\lambda_i} f_X(s)\Big)\Big) - f_Y(s) = 0 \tag{3.1}$$

has exactly n roots, say $\rho_1(\delta), \rho_2(\delta), \cdots, \rho_n(\delta)$ with positive real parts $\Re(\rho_j) > 0$.

Proof Since

$$\Big(1 + \frac{\delta+\mu}{\lambda_i}\Big) - \frac{c}{\lambda_i} s - \frac{\sigma^2}{2\lambda_i} s^2 = 0$$

has exactly two solutions

$$s_1 = -\frac{c}{\sigma^2} - \sqrt{\frac{c^2}{\sigma^4} + \frac{(\lambda_i + \mu + \sigma)}{\sigma^2}} < 0 \quad \text{and} \quad s_2 = -\frac{c}{\sigma^2} + \sqrt{\frac{c^2}{\sigma^4} + \frac{(\lambda_i + \mu + \sigma)}{\sigma^2}} > 0.$$

On the half circle in the complex plane given by $|z| = r$ (for $r > 0$ fixed) and $\Re(z) \geq 0$, we have $|\gamma_i(s)| > \mu/\lambda_j$ if r is sufficiently large. While for s on the imaginary axis we have that $|\gamma_i(s)| \geq 1 + (\delta+\mu)/\lambda_j > \mu/\lambda_j$ which is on the contour boundary of the half circle and the imaginary axis, $|\gamma_i(s)| > |(\mu/\lambda_i) f_X(s)|$. Then we conclude that on the right half plane the number of the roots to $\gamma_i(s) - (\mu/\lambda_i) f_X(s)) = 0$ equals to that of $\gamma_i(s) = 0$. So on the right half plane the number of the roots to $\prod_{i=1}^{n}(\gamma_i(s) - (\mu/\lambda_i) f_X(s)) = 0$ equals to that of $\prod_{i=1}^{n} \gamma_i(s) = 0$, similar to the proof above, the number of the roots to $\Big(\prod_{i=1}^{n}(\gamma_i(s) - (\mu/\lambda_i) f_X(s))\Big) - f_Y(s) = 0$ equals to that of $\prod_{i=1}^{n}(\gamma_i(s) - (\mu/\lambda_i) f_X(s)) = 0$. Because the latter has exactly the same number of positive roots as that of $\prod_{i=1}^{n} \gamma_i(s) = 0$ and the result follows. $\square$

We are now ready to solve the integro-differential equations (2.1) and (2.2). We only consider $K_2(t)$ is the generalized Erlang(2) process. In the following of the paper, we assume that $\psi(u), \psi_s(u), \psi_d(u)$ are 4th differentiable. As in Gerber and Shiu (2005), we first use the concept of divided differences. For distinct numbers $r_1, r_2, \cdots, r_n$, the k-th divided difference $h[r_1, r_2, \cdots, r_n, s]$ of a function h is defined recursively as follows:

$$
\begin{aligned}
&h(s) = h(r_1) + (s - r_1) h[r_1, s],\\
&h[r_1, s] = h[r_1, r_2] + (s - r_2) h[r_1, r_2, s],\\
&h[r_1, r_2, \cdots, r_{k-1}, s] = h[r_1, r_2, \cdots, r_k] + (s - r_k) h[r_1, r_2, \cdots, r_k, s].
\end{aligned}
$$

Note that $h(s)$ is a polynomial of degree n. The following result also holds:

$$
h[r_1, r_2, \cdots, r_k] = \sum_{j=1}^{k} \frac{h(r_j)}{\tau_k'(r_j; r_1, r_2, \cdots, r_k)},
$$

where $\tau_k'(r_j; r_1, r_2, \cdots, r_k) = \prod_{i=1}^{k}(s - r_i)$ is a polynomial.

Next, as in Dickson (2001), we define the operator T_r of a real-valued function f, with respect to a complex number r by

$$
T_r f(x) = \int_x^{\infty} e^{-r(y-x)} f(y) \mathrm{d}y, \qquad x \geq 0.
$$

Properties of this operator can be found in Gerber and Shiu (2005),

(a) $(T_r\varphi)(0) = \widehat{\varphi}(r)$, and

$$
\Big[\Big(\prod_{j=1}^{m} T_{r_j}\Big)\varphi\Big](0) = (-1)^{m-1}\widehat{\varphi}[r_1, r_2, \cdots, r_m]. \tag{3.2}
$$

(b) By interchanging the order of integration, we have

$$
T_r(\varphi * \eta) = \varphi * (T_r\eta) + (T_r\eta)(0)(T_r\varphi). \tag{3.3}
$$

Now we consider the generalized Erlang(2) case, then equation (2.1) changes to

$$\prod_{i=1}^{2}\Big(\gamma_i(D)-\frac{\mu}{\lambda_i}P\Big)\phi_s(u)=\int_0^u\phi_s(u-x)f_Y(x)\mathrm{d}x+w(u), \tag{3.4}$$

where

$$w(u)=w_2(u)-\frac{\mu}{\lambda_2}w_1(u)-\frac{\mu}{\lambda_2}\Big(\gamma_1(D)-\frac{\mu}{\lambda_1}P\Big)w_1(u).$$

Since

$$\begin{aligned}(DP)\phi_s(u) &= D\Big(\int_0^u\phi_s(u-x)f_X(x)\mathrm{d}x\Big)\\ &= \int_0^u\phi_s'(u-x)f_X(x)\mathrm{d}x+\phi_s(0)f_X(u)=(PD)\phi_s(u).\end{aligned}$$

From the definition of P, (3.4) yields the equation

$$h(D)\phi_s(u)-\mu\Big(\frac{\gamma_1(D)}{\lambda_2}+\frac{\gamma_2(D)}{\lambda_1}\Big)(\phi_s*f_X)(u)+\frac{\mu^2}{\lambda_1\lambda_2}\phi_s*f_X^{2*}(u)=\phi_s*f_Y(u)+w(u), \tag{3.5}$$

where $h(D)=\gamma_1(D)\gamma_2(D)$.

For the first step, we write $h(s)=h(\rho_1)+(s-\rho_1)h[\rho_1,s]$ then

$$T_{\rho_1}h(D)\phi_s(u)=h(\rho_1)T_{\rho_1}\phi_s(u)-h[\rho_1,D]\phi_s(u), \tag{3.6}$$

since $(\beta I+D)(\varphi(u)*e^{-\beta u})=\varphi(u)$. By (3.3) and (3.6),

$$\begin{aligned}& T_{\rho_1}\Big(\frac{\gamma_1(D)}{\lambda_2}+\frac{\gamma_2(D)}{\lambda_1}\Big)(\phi_s*f_X)(u)\\ = & \Big(\frac{\gamma_1(\rho_1)}{\lambda_2}+\frac{\gamma_2(\rho_1)}{\lambda_1}\Big)(\phi_s*(T_{\rho_1}f_X))(u)+\Big(\frac{\gamma_1(\rho_1)}{\lambda_2}+\frac{\gamma_2(\rho_1)}{\lambda_1}\Big)T_{\rho_1}f_X(0)T_{\rho_1}\phi_s(u)\\ & -\Big(\frac{\gamma_1[\rho_1,D]}{\lambda_2}+\frac{\gamma_2[\rho_1,D]}{\lambda_1}\Big)(\phi_s*f_X)(u),\end{aligned} \tag{3.7}$$

$$T_{\rho_1}(\phi_s*f_X^{2*})(u)=\phi_s*(T_{\rho_1}f_X^{2*})(u)+T_{\rho_1}f_X^{2*}(0)T_{\rho_1}\phi_s(u), \tag{3.8}$$

$$T_{\rho_1}(\phi_s*f_Y)(u)=\phi_s*(T_{\rho_1}f_Y)(u)+T_{\rho_1}f_Y(0)T_{\rho_1}\phi_s(u). \tag{3.9}$$

By (3.5), (3.6), (3.7), (3.8) and (3.9), applying the operator T_{ρ_1} on both sides of (3.5), we obtain

$$\begin{aligned}& \phi_s*(T_{\rho_1}f_Y)(u)+T_{\rho_1}w(u)\\ = & -h[\rho_1,D]\phi_s(u)-\mu\Big(\frac{\gamma_1(\rho_1)}{\lambda_2}+\frac{\gamma_2(\rho_1)}{\lambda_1}\Big)(\phi_s*(T_{\rho_1}f_X))(u)\\ & +\mu\Big(\frac{\gamma_1[\rho_1,D]}{\lambda_2}+\frac{\gamma_2[\rho_1,D]}{\lambda_1}\Big)(\phi_s*f_X)(u)+\frac{\mu^2}{\lambda_1\lambda_2}\phi_s*(T_{\rho_1}f_X^{2*})(u).\end{aligned} \tag{3.10}$$

Because

$$h(\rho_1)-\mu\Big(\frac{\gamma_1(\rho_1)}{\lambda_2}+\frac{\gamma_2(\rho_1)}{\lambda_1}\Big)T_{\rho_1}f_X(0)+\frac{\mu^2}{\lambda_1\lambda_2}T_{\rho_1}f_X^{2*}(0)=T_{\rho_1}f_Y(0),$$

i.e.

$$h(\rho_1)-\mu\Big(\frac{\gamma_1(\rho_1)}{\lambda_2}+\frac{\gamma_2(\rho_1)}{\lambda_1}\Big)\widehat{f}_X(\rho_1)+\frac{\mu^2}{\lambda_1\lambda_2}\widehat{f}_X^2(\rho_1)=\widehat{f}_Y(\rho_1).$$

For the second step, we write

$$\begin{aligned}&h[\rho_1,s]=h[\rho_1,r_2]+(s-\rho_2)h[\rho_1,\rho_2,s],\\&T_{\rho_2}h(\rho_1,D)\phi_s(u)=h[\rho_1,\rho_2]T_{\rho_2}\phi_s(u)-h[\rho_1,\rho_2,D]\phi_s(u),\end{aligned}\tag{3.11}$$

$$\begin{aligned}&T_{\rho_2}\Big(\Big(\frac{\gamma_1[\rho_1,D]}{\lambda_2}+\frac{\gamma_2[\rho_1,D]}{\lambda_1}\Big)\Big)(\phi_s*f_X)(u)\\
=&\Big(\frac{\gamma_1[\rho_1,\rho_2]}{\lambda_2}+\frac{\gamma_2[\rho_1,\rho_2]}{\lambda_1}\Big)(\phi_s*(T_{\rho_2}f_X))(u)\\
&+\Big(\frac{\gamma_1[\rho_1,\rho_2]}{\lambda_2}+\frac{\gamma_2[\rho_1,\rho_2]}{\lambda_1}\Big)T_{\rho_2}f_X(0)T_{\rho_2}\phi_s(u)\\
&-\Big(\frac{\gamma_1[\rho_1,\rho_2,D]}{\lambda_2}+\frac{\gamma_2[\rho_1,\rho_2,D]}{\lambda_1}\Big)(\phi_s*f_X)(u).\\
&T_{\rho_2}\Big(\frac{\gamma_1(\rho_1)}{\lambda_2}+\frac{\gamma_2(\rho_1)}{\lambda_1}\Big)(\phi_s*(T_{\rho_1}f_X))(u)\\
=&\Big(\frac{\gamma_1(\rho_1)}{\lambda_2}+\frac{\gamma_2(\rho_1)}{\lambda_1}\Big)[\phi_s*(T_{\rho_2}T_{\rho_1}f_X)(u)-(T_{\rho_2}T_{\rho_1}f_X)(0)T_{\rho_2}\phi_s(u)],\end{aligned}$$

$$\begin{aligned}&T_{\rho_2}(\phi_s*(T_{\rho_1}f_X^{2*}))(u)=\phi_s*(T_{\rho_2}T_{\rho_1}f_X^{2*})(u)-(T_{\rho_2}T_{\rho_1}f_X^{2*})(0)T_{\rho_2}\phi_s(u),\\&T_{\rho_2}(\phi_s*(T_{\rho_1}f_Y))(u)=\phi_s*(T_{\rho_2}T_{\rho_1}f_Y)(u)-(T_{\rho_2}T_{\rho_1}f_Y)(0)T_{\rho_2}\phi_s(u).\end{aligned}$$

We now obtain

$$\begin{aligned}&h[\rho_1,\rho_2,D]\phi_s(u)+\mu\Big(\frac{\gamma_1[\rho_1,\rho_2]}{\lambda_2}+\frac{\gamma_2[\rho_1,\rho_2]}{\lambda_1}\Big)(\phi_s*(T_{\rho_2}f_X))(u)\\
&-\mu\Big(\frac{\gamma_1[\rho_1,\rho_2,D]}{\lambda_2}+\frac{\gamma_2[\rho_1,\rho_2,D]}{\lambda_1}\Big)(\phi_s*f_X)(u)\\
&-\mu\Big(\frac{\gamma_1(\rho_1)}{\lambda_2}+\frac{\gamma_2(\rho_1)}{\lambda_1}\Big)\phi_s*(T_{\rho_2}T_{\rho_1}f_X)(u)+\frac{\mu^2}{\lambda_1\lambda_2}\phi_s*(T_{\rho_2}T_{\rho_1}f_X^{2*})(u)+\Delta T_{\rho_2}\phi_s(u)\\
=&\ \phi_s*(T_{\rho_2}T_{\rho_1}f_Y)(u)+T_{\rho_2}T_{\rho_1}w(u),\end{aligned}$$

where

$$\begin{aligned}\Delta=&-h[\rho_1,\rho_2]+\mu\Big(\frac{\gamma_1(\rho_1)}{\lambda_2}+\frac{\gamma_2(\rho_1)}{\lambda_1}\Big)(T_{\rho_2}T_{\rho_1}f_X)(0)\\
&+\mu\Big(\frac{\gamma_1[\rho_1,\rho_2]}{\lambda_2}+\frac{\gamma_2[\rho_1,\rho_2]}{\lambda_1}\Big)T_{\rho_2}f_X(0)-\frac{\mu^2}{\lambda_1\lambda_2}(T_{\rho_2}T_{\rho_1}f_X^{2*})(0)+(T_{\rho_2}T_{\rho_1}f_Y)(0).\end{aligned}$$

Next we prove $\Delta=0$. Since $h(s)\widehat{f}_X(s)=h(\rho_1)\widehat{f}_X(s)+(s-\rho_1)h[\rho_1,s]\widehat{f}_X(s)$, we have

$$(T_\rho h(s)\widehat{f}_X(s)=h(\rho_1)(T_\rho\widehat{f}_X(s)-h[\rho_1,s]\widehat{f}_X(s). \tag{3.12}$$

By (3.11) and (3.12)

$$\begin{aligned}
h[\rho_1,\rho_2] &= \frac{h(\rho_2)-h(\rho_1)}{\rho_2-\rho_1}\\
&= \frac{\mu\Big(\dfrac{\gamma_1(\rho_1)}{\lambda_2}+\dfrac{\gamma_2(\rho_1)}{\lambda_1}\Big)T_{\rho_1}f_X(0)-\mu\Big(\dfrac{\gamma_1(\rho_1)}{\lambda_2}+\dfrac{\gamma_2(\rho_1)}{\lambda_1}\Big)T_{\rho_2}f_X(0)}{\rho_2-\rho_1}\\
&\quad -\frac{\mu^2}{\lambda_1\lambda_2}(T_{\rho_2}T_{\rho_1}f_X^{2*})(0)+(T_{\rho_2}T_{\rho_1}f_Y)(0)\\
&= \mu\Big(\frac{\gamma_1(\rho_1)}{\lambda_2}+\frac{\gamma_2(\rho_1)}{\lambda_1}\Big)(T_{\rho_2}T_{\rho_1}f_X)(0)+\mu\Big(\frac{\gamma_1[\rho_1,\rho_2]}{\lambda_2}+\frac{\gamma_2[\rho_1,\rho_2]}{\lambda_1}\Big)T_{\rho_2}f_X(0)\\
&\quad -\frac{\mu^2}{\lambda_1\lambda_2}(T_{\rho_2}T_{\rho_1}f_X^{2*})(0)+(T_{\rho_2}T_{\rho_1}f_Y)(0).
\end{aligned}$$

From discussion above we can get the result:

Theorem 3.2 For $u\geq 0$, $\phi_s(u)$ satisfies the following equation

$$\begin{aligned}
&h[\rho_1,\rho_2,D]\phi_s(u)+\mu\Big(\frac{\gamma_1[\rho_1,\rho_2]}{\lambda_2}+\frac{\gamma_2[\rho_1,\rho_2]}{\lambda_1}\Big)(\phi_s*(T_{\rho_2}f_X))(u)\\
&-\mu\Big(\frac{\gamma_1[\rho_1,\rho_2,D]}{\lambda_2}+\frac{\gamma_2[\rho_1,\rho_2,D]}{\lambda_1}\Big)(\phi_s*f_X)(u)\\
=\ &\phi_s*\eta(u)+W(u),
\end{aligned}\tag{3.13}$$

where ρ_1, ρ_2 are the roots of the generalized Lundberg equation with positive real parts, $h(D)=\gamma_1(D)\gamma_2(D)$,

$$\eta(y)=T_{\rho_2}T_{\rho_1}f_Y(y)+\mu\Big(\frac{\gamma_1(\rho_1)}{\lambda_2}+\frac{\gamma_2(\rho_1)}{\lambda_1}\Big)(T_{\rho_2}T_{\rho_1}f_X)(y)-\frac{\mu^2}{\lambda_1\lambda_2}(T_{\rho_2}T_{\rho_1}f_X^{2*})(y)$$

and $W(u)=T_{\rho_2}T_{\rho_1}w(u)$.

Next, as in Geber and Shiu (2005), taking Laplace transform of (3.13) gives

$$\begin{aligned}
&[h[\rho_1,\rho_2,s]+\mu\Big(\frac{\gamma_1[\rho_1,\rho_2]}{\lambda_2}+\frac{\gamma_2[\rho_1,\rho_2]}{\lambda_1}\Big)T_sT_{\rho_2}f_X(0)\\
&-\mu\Big(\frac{\gamma_1[\rho_1,\rho_2,s]}{\lambda_2}+\frac{\gamma_2[\rho_1,\rho_2,s]}{\lambda_1}\Big)\widehat{f}_X(s)-\widehat{\eta}(s)]\widehat{\phi}_s(s)\\
=\ &W(s)+q_2(s)+\mu q_1(s)\widehat{f}_X(s),
\end{aligned}$$

$$(h[\rho_1,\rho_2,s]+\widehat{\eta}_2(s))\widehat{\phi}_s(s)=W(s)+q_2(s)+\mu q_1(s)\widehat{f}_X(s), \tag{3.14}$$

where

$$\widehat{\eta}_2(s)=\mu\Big(\frac{\gamma_1[\rho_1,\rho_2]}{\lambda_2}+\frac{\gamma_2[\rho_1,\rho_2]}{\lambda_1}\Big)T_sT_{\rho_2}f_X(0)-\mu\Big(\frac{\gamma_1[\rho_1,\rho_2,s]}{\lambda_2}+\frac{\gamma_2[\rho_1,\rho_2,s]}{\lambda_1}\Big)\widehat{f}_X(s)-\widehat{\eta}(s).$$

$q_1(s)$, $q_2(s)$ are polynomial of degree 1 or less respectively, with coefficients in term of $\delta, c, \lambda, \rho_i$ for $i=1,2,\cdots,n$ and the derivatives of ϕ_s at 0, $\phi_s^{(k)}(0)$, for $k=0,1,2$.

Since $h[\rho_1,\rho_2,\cdots,\rho_n,s]$ is a polynomial of degree n and the coefficient of s^n is equal to that of s^{2n} in $h(s)$, which is $(-1)^n\sigma^{2n}/\big(\prod\limits_{i=1}^{n}(2\lambda_i)\big)$, then $h[\rho_1,\rho_2,\cdots,\rho_n,s]$ can be factored as

$$h[\rho_1,\rho_2,\cdots,\rho_n,s]=(-1)^n\frac{\sigma^{2n}(s+a_1)(s+a_2)\cdots(s+a_n)}{\prod\limits_{i=1}^{n}(2\lambda_i)},$$

where the $a_1,a_2,\cdots,a_n$ come in pairs of conjugate complex numbers.

Thus equation (3.14) can be rewritten as

$$\widehat{\phi}_s(s)\Big(1+\frac{4\lambda_1\lambda_2\widehat{\eta}_2(s)}{\sigma^4(s+a_1)(s+a_2)}\Big)=\frac{4\lambda_1\lambda_2(W(s)+\mu q_1(s)\widehat{f}_X(s))}{\sigma^4(s+a_1)(s+a_2)}+\sum_{i=1}^{2}\frac{b_i}{s+a_i},\qquad(3.15)$$

where the coefficients b_i are given by $b_i=4\lambda_1\lambda_2q_2(-a_i)/\big[\sigma^4\prod\limits_{j=1,j\neq i}^{2}(a_j-a_i)\big]$.

Thus $\phi_s(u)$ satisfies the following renewal equations

$$\phi_s(u)=\int_0^u\phi_s(u-x)g(x)\mathrm{d}x+H(u)+\sum_{i=1}^{2}b_ie^{-a_iu},\qquad u\geq 0.$$

Where $g(x)=-h_1*h_2*\eta_2(x)$, $H(u)=h_1*h_2*(W(u)+\mu q_1*f_X(u))$ with $h_i(x)=(2\lambda_i/\sigma^2)\cdot e^{-a_ix}$ and $*$ denotes the convolution product.

$\phi_s(u)$ is uniquely determined by the 4-th order integro-differential equation (3.4), if initial conditions $\phi_s^{(k)}(0)$ are given for $k=0,1,2,3$. Taking Laplace transforms on both sides of the integro-differential equation (3.4) yields

$$\Big(\prod_{i=1}^{2}(\gamma_i(s)-\frac{\mu}{\lambda_i}f_X(s))-f_Y(s)\Big)\phi_s(s)=W(s)+q(s),$$

since $\phi_s(u)$ is finite for all complex number s such that $\Re(s)>0$, we have that

$$W(\rho_i)=-q(\rho_i),\qquad i=1,2.$$

Another two conditions are needed. Setting $u=0$ in (2.4) yields

$$\Big\{\Big(\gamma_2(D)-\frac{\mu}{\lambda_2}P\Big)\phi_s(u)+\frac{\mu}{\lambda_2}w_1(u)\Big\}\Big|_{u=0}=0,$$

together with $\phi_s(0)=0$ yield a system of 2×2 linear equations that can be solved for the unknowns $\phi_s^{(k)}(0)$, $k=0,1,2,3$.

$$DP\phi_d(u)=PD\phi_d(u)+f_X(u),$$
$$\prod_{i=1}^{2}\Big(\gamma_i(D)-\frac{\mu}{\lambda_i}P\Big)\phi_d(u)=\int_0^u\phi_d(u-x)f_Y(x)\mathrm{d}x.$$

Then

$$h(D)\phi_d(u)-\mu\Big(\frac{\gamma_1(D)}{\lambda_2}+\frac{\gamma_2(D)}{\lambda_1}\Big)(\phi_d*f_X(u))+\frac{\mu^2}{\lambda_1\lambda_2}\phi_d*f_X^{2*}=\phi_d*f_Y(u)+w_3(u), \quad (3.16)$$

where $h(D)=\gamma_1(D)\gamma_2(D)$,

$$w_3(u)=\frac{\mu\Big(\dfrac{\sigma^2}{2\lambda_2}(\phi_d'(0)f_X(u)+f_X'(u))+\dfrac{c}{\lambda_2}f_X(u)-1-\dfrac{\delta+\mu}{\lambda_2}\Big)}{\lambda_1}.$$

Similarly, taking Laplace transforms on both sides of equation (3.16) yields

$$[h[\rho_1,\rho_2,s]+\widehat{\eta}_2(s)]\widehat{\phi}_d(s)=W_3(s)+Q_2(s)+\mu Q_1(s)\widehat{f}_X(s),$$

where

$$\widehat{\eta}_2(s)=\mu\Big(\frac{\gamma_1[\rho_1,\rho_2]}{\lambda_2}+\frac{\gamma_2[\rho_1,\rho_2]}{\lambda_1}\Big)T_sT_{\rho_2}f_X(0)-\mu\Big(\frac{\gamma_1[\rho_1,\rho_2,s]}{\lambda_2}+\frac{\gamma_2[\rho_1,\rho_2,s]}{\lambda_1}\Big)\widehat{f}_X(s)-\widehat{\eta}(s)$$

and

$$\eta(y)=T_{\rho_2}T_{\rho_1}f_Y+\mu\Big(\frac{\gamma_1(\rho_1)}{\lambda_2}+\frac{\gamma_2(\rho_1)}{\lambda_1}\Big)(T_{\rho_2}T_{\rho_1}f_X)-\frac{\mu^2}{\lambda_1\lambda_2}(T_{\rho_2}T_{\rho_1}f_X^{2*}),$$

$Q_1(s),Q_2(s)$ are polynomial of degree 1 or less respectively.

$$\begin{aligned}&\phi_d(s)\Big(1+\frac{(2\lambda)^2\eta_2(s)}{\sigma^4(s+a_1)(s+a_2)}\Big)\\
=\;&\frac{(2\lambda)^2(W_3(s)+\mu Q_1(s)f_X(s))}{\sigma^4(s+a_1)(s+a_2)}+\frac{(2\lambda)^2Q_2(s)}{\sigma^4(s+a_1)(s+a_2)}\\
=\;&\frac{(2\lambda)^2(W_3(s)+\mu Q_1(s)f_X(s))}{\sigma^4(s+a_1)(s+a_2)}+\sum_{i=1}^{2}\frac{c_i}{s+a_i}, \qquad (3.17)\end{aligned}$$

where the coefficients c_i are given by $c_i=4\lambda_1\lambda_2Q_2(-a_i)/\big[\sigma^4\prod\limits_{j=1,j\neq i}^{2}(a_j-a_i)\big]$.

These can be solved for $\phi_s(u)$ and $\phi_d(u)$ by inverting Laplace transforms. Then we have the following theorem.

Theorem 3.3 $\phi_s(u),\phi_d(u)$ and $\phi(u)$ satisfy the following renewal equations

$$\phi_s(u)=\int_0^u\phi_s(u-x)g(x)\mathrm{d}x+H_1(u)+\sum_{i=1}^{n}b_ie^{-a_iu}, \qquad u\geq 0; \qquad (3.18)$$

$$\phi_d(u)=\int_0^u\phi_d(u-x)g(x)\mathrm{d}x+H_2(u)+\sum_{i=1}^{2}c_ie^{-a_iu}, \qquad u\geq 0; \qquad (3.19)$$

$$\phi(u)=\int_0^u\phi(u-x)g(x)\mathrm{d}x+H(u)+\sum_{i=1}^{2}(b_i+c_i)e^{-a_iu}, \qquad u\geq 0, \qquad (3.20)$$

where $g(x)=h_1*h_2*\eta_2(x)$, $H_1(u)=h_1*h_2*(W(u)+\mu q_1*f_X(u))$, $H_2(u)=h_1*h_2*(W(u)+\mu Q_1*f_X(u))$, $H(u)=H_1(u)+H_2(u)$ with $h_i(x)=(2\lambda_i/\sigma^2)e^{-a_ix}$.

Proof Inverting the Laplace transforms (3.15) and (3.17) gives the renewal equations (3.18) and (3.19). Since $\phi(u) = \phi_s(u) + \phi_d(u)$, adding (3.18) to (3.19), we obtain (3.20). $\square$

Remark 2 (1) If $X_i = 0$, $i = 1, 2, \cdots, n$, equations (3.18), (3.19) and (3.20) yield (23), (24) and (25) of Li and Garrido (2005) when $n = 2$.

(2) If $Y_i = 0$, $i = 1, 2, \cdots, n$, equation (3.20) yields (17) of Gerber and Landry (1998), while equation (3.19) gives (2.10) of Tsai and Willmot (2002).

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一类带扰动风险模型的Gerber-Shiu函数

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本文研究了一类带扰动风险模型, 得到了此过程下Gerber-Shiu函数的微分积分方程, 并得到了推广Erlang(2)情形下Gerber-Shiu函数满足的更新方程.

关键词: Gerber-Shiu函数, 微分积分方程, Laplace变换, 广义Lundberg方程.

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