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Limit Theorems for the Size of the Clusters of Dependent Percolation Process on $\mathbb{Z}^2$ *

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Abstract

Limit theorems such as central limit theorems, large deviations and large number laws play an important role in probability theory. In this paper, we consider dependent percolation on the planar lattice $\mathbb{Z}^2$. For this model, we not only prove the existence and uniqueness of the infinite cluster but also prove the central limit theorems respectively for the size of biggest cluster and the size of the cluster at the origin in the lattice boxes.

Keywords: Limit theorem, dependent percolation, duality.

AMS Subject Classification: 60K35.

§1. Introduction

In this paper, we will consider dependent percolation process on the $\mathbb{Z}^2$ lattice, in which the directed edges from the vertex x to its nearest neighbors decided by a probability measure μ_x, independently for different x. In the usual bond percolation process on a large (usually infinite) graph G, the edges are undirected and the states of different edges are independent random variables. For example, in standard bond percolation on $\mathbb{Z}^d$, one takes and removes each edge independently with probability $1-p$, thus keeping it with probability $p \in [0,1]$. Now we formally describe the dependent percolation model. Let $(E, \mathcal{E}, \mu)$ be a probability space, where $E = \{0,1\}^4$, $\mathcal{E}$ is the collection of all subsets of E and μ is a probability measure on $(E, \mathcal{E})$ which is log-supermodular on the set $\mathcal{E}$. We say that the measure μ is log-supermodular, if

$$\mu(a)\mu(b) \le \mu(a \wedge b)\mu(a \vee b)$$

for all $a, b \in E$, with $(a_1, a_2, a_3, a_4) \wedge (b_1, b_2, b_3, b_4) = (a_1 \wedge b_1, a_2 \wedge b_2, a_3 \wedge b_3, a_4 \wedge b_4)$ and $(a_1, a_2, a_3, a_4) \vee (b_1, b_2, b_3, b_4) = (a_1 \vee b_1, a_2 \vee b_2, a_3 \vee b_3, a_4 \vee b_4)$. Throughout this paper,

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we assume that μ is log-supermodular and symmetrical. Now we assume that $(\Omega, \mathcal{F}, \mathsf{P})$ is the product probability space as

$$(\Omega, \mathcal{F}, \mathsf{P}) = \prod_{x \in \mathbb{Z}^2} (E_x, \mathcal{E}_x, \mu_x),$$

where $E_x = E$, $\mathcal{E}_x = \mathcal{E}$ and $\mu_x = \mu$ for all $x \in \mathbb{Z}^2$. We write E for the expectation of the measure P. Let $\eta = \{\eta_x, x \in \mathbb{Z}^2\}$ be a family of independent identically distributed random elements of E, each η_x having distribution μ, indexed by the integer lattice. That is, $\eta \in \Omega$ with distribution P. We give some explanation for η_x. $\eta_x = (\eta_x^r, \eta_x^t, \eta_x^l, \eta_x^b) \in E$ denote the states of the directed edges emanating from x, where η_x^r (respectively, $\eta_x^t, \eta_x^l, \eta_x^b$) denotes the state of the edge from x directing to its right (respectively, top, left, bottom) side. For example, $\eta_0 = (1, 0, 0, 1)$ denote the directed edges from $(0, 0)$ to $(1, 0)$ and $(0, -1)$ are open, the other from $(0, 0)$ to $(0, 1)$ and $(-1, 0)$ are closed.

Such percolation-type model was first studied by Kuulasmaa[8], named as "locally dependent random graph". Throughout this paper, we prefer to call it "dependent percolation process". Some results for the model have been obtained by Berg, Grimmett and Schinazi[3], such as the exponential decay for connectivity functions and the cluster-size distribution.

Recently, there are many papers to study limit theorems for classic percolation model on $\mathbb{Z}^d$, see [4] [6] [7] [10] [11]. With application of Mcleish's[9] martingale approach, Zhang[11] proved the central limit theorem and large deviations for the number of connected components obtained in lattice boxes by deleting all closed edges and the boundary of the boxes. Penrose[10] extended the martingale method to obtain a general argument of central limit theorem and with application to the site percolation on $\mathbb{Z}^d$ for the size of open cluster in the lattice boxes. However, there are few literature on the limit theorems for dependent percolation model.

In the present paper, we apply the general central limit theorem (Theorem 2.1 of [10]) to the dependent percolation process in two dimensional lattice $\mathbb{Z}^2$. Some notations and main results are mentioned in Section 2. We do some preliminary work in Section 3, where existence and uniqueness of the infinite cluster are proved. In Section 4, we prove the central limit theorems for the size of biggest cluster and the size of the cluster at the origin in the lattice boxes.

§2. Notations and Main Results

We begin with some jargon. Most of them come from [6] or [10]. A *lattice box* B is a subset of $\mathbb{Z}^2$ as form $B = \mathbb{Z}^2 \cap ([a_1, b_1] \times [a_2, b_2])$. Furthermore, we call B is δ-*comparable* for a constant δ, if

$$\frac{(b_1-a_1)\wedge(b_2-a_2)}{(b_1-a_1)\vee(b_2-a_2)}\geq\delta.$$

For $x=(x_1,x_2)$, $y=(y_1,y_2)\in\mathbb{Z}^2$, the *distance* of x,y is defined as $d(x,y)=|x_1-y_1|\vee|x_2-y_2|$. For a finite set $A\subset\mathbb{Z}^2$, we define the *diameter* of A by

$$\operatorname{diam}(A)=\max_{x,y\in A}d(x,y)$$

and denote by $|A|$ the *cardinality* of A. For $L>0$, the L-bound set of A is defined as $\partial_L A=\{x\in A: d(x,\mathbb{Z}^2\setminus A)\leq L\}$. Specially, set $\partial A=\partial_1 A$ for $L=1$.

We define an order on Ω. For $e=(e_1,e_2,e_3,e_4)$, $f=(f_1,f_2,f_3,f_4)\in E$, we denote by $e\leq f$ if $e_i\leq f_i$ for all $i=1,2,3,4$. Now, for $\eta,\xi\in\Omega$, we denote by $\eta\leq\xi$ if $\eta_x\leq\xi_x$ for all $x\in\mathbb{Z}^2$. We say a configurations set $\Lambda\subset\Omega$ is increasing, if $\xi\in\Lambda$ whenever $\eta\in\Lambda$ and $\eta\leq\xi$.

Since μ is log-supermodular, μ satisfies FKG inequality on partially ordered probability space $(E,\mathcal{E})$. For the detail, one can refer [1]. Now, it follows from Theorem 3 of [2] that the probability P satisfies the FKG inequality.

Throughout this paper, we always assume that $0\in B$ whenever B is a lattice box. We denote by $H(\eta,B)$ a real-valued function for $\eta\in\Omega$ and $B\subset\mathbb{Z}^2$. Let $\eta^0\in\Omega$ be the configuration $\eta\in\Omega$ with the value η_0 at he origin replaced by an independent E-valued variable ζ with distribution μ, but the values at other sites the same. Let $\Delta_0(B)=H(\eta^0,B)-H(\eta,B)$ denote the increment of $H(\cdot,B)$ by only changing the value at the origin. The following definition comes from [10].

Definition 2.1 We say that the function H for the lattice boxes sequence $\{B_n,n\geq 1\}$ satisfies

(1) *stabilization* condition, if there is a random variable $\Delta_0(\infty)$ such that the sequence $\{\Delta_0(B_n),n\geq1\}$ converges to $\Delta_0(\infty)$ in probability.

(2) *bounded moment* condition, if there exists a constant $\gamma>2$ such that the sequence $\{\mathsf{E}|\Delta_0(B_n)|^\gamma,n\geq1\}$ is uniformly bounded.

A finite non-void set $\xi=\{v_0,\cdots,v_n\}\subset\mathbb{Z}^2$ of vertices is called a (self-avoiding) *path* of length n from v_0 to v_n if there is a directed edge from v_i to v_{i+1} for $i=0,\cdots,n-1$, where $v_i\neq v_j$ for $i\neq j$. We write $x\to y$ if there exists an open path from x to y, and $x\to\infty$ if there exists an infinite open path starting at x. For sets X,Y of vertices, we write $X\to Y$ if there exist $x\in X$ and $y\in Y$ such that $x\to y$. We write "$X\to Y$ in Z" if there exists an oriented path from some $x\in X$ to some $y\in Y$ using only vertices of Z.

Define *cluster* on $B\subset\mathbb{Z}^d$ by setting two vertices x,y to be in the same cluster (the same cluster in B) if there are open directed paths in B both from x to y and from y to

x. We write $C_x(B)$ for the cluster in B containing the vertex x, that is

$$C_x(B) = \{y \in B : x \to y \text{ in } B, y \to x \text{ in } B\}.$$

Let $C(B)$ denote the biggest cluster in B, choosing via some rule if the biggest cluster is non-unique. And let $\vec{C}_x(B)$ be the set of vertices which can be reached from x using only vertices of B, that is,

$$\vec{C}_x(B) = \{y \in B : x \to y \text{ in } B\}.$$

Usually, we write $\vec{C}_0(\mathbb{Z}^2)$ as $\vec{C}_0$. The *percolation probability* is defined as

$$\theta \equiv \theta(\mu) = \mathsf{P}(|\vec{C}_0| = \infty).$$

Now, we mention the main theorems in this paper as follows.

Theorem 2.1 Suppose $\theta > 0$. If $\{B_n, n \geq 1\}$ is a δ-comparable sequences of lattice boxes tending to $\mathbb{Z}^2$, then there is a constant $\sigma > 0$ such that

$$\lim_{n\to\infty} |B_n|^{-1} \mathsf{Var}\, |C(B_n)| = \sigma^2 \tag{2.1}$$

and

$$|B_n|^{-1/2}(|C(B_n)| - \mathsf{E}|C(B_n)|) \longrightarrow \mathcal{N}(0, \sigma^2). \tag{2.2}$$

Theorem 2.2 Suppose $\widetilde{\theta} \equiv \widetilde{\theta}(\mu) = \mathsf{P}(|C_0| = \infty) > 0$. For any $\delta > 0$, let $\{B_n, n \geq 1\}$ be a δ-comparable sequence of lattice boxes tending to $\mathbb{Z}^2$. Let $a_n = \mathsf{E}[|C(B_n)|]$ and let σ^2 be the same constant as in the Theorem 2.1. Then, for $t \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \mathsf{P}\Big[\frac{|C_0(B_n)| - a_n}{\sigma\sqrt{|B_n|}} \geq t\Big] = \widetilde{\theta}(1 - \Phi(t)), \tag{2.3}$$

where $\Phi(t) = \displaystyle\int_{-\infty}^{t} (2\pi)^{-1/2} \exp(-s^2/2) \mathrm{d}s$ is the distribution function of the standard normal.

Before to prove the above theorems, we need to do some preliminary work.

§3. Preliminaries

For positive integers j and k, we write $B(j,k)$ for the rectangle $[0,j] \times [0,k]$ and $\mathrm{RL}(j,k)$ for the event that there is an open right-to-left crossing of $B(j,k)$.

Lemma 3.1 If $\mathsf{P}(\mathrm{RL}(k,k)) = \tau$, then

$$\mathsf{P}(\mathrm{RL}(3k/2, k)) \geq (1 - \sqrt{1-\tau})^3. \tag{3.1}$$

Proof The idea of proving the lemma comes from Cox and Durrett[4] and Grimmett[6] (Lemma 9.73 of Chapter 9).

Let $\mathscr{T}$ be the set of directed paths of $B(k,k)$ which traverse $B(k,k)$ from right to left side. For $\pi \in \mathscr{T}$, let π_r denote the portion of this path from $\{k\} \times (0,k)$ until the first time it hits the beeline $\{k/2\} \times (0,k)$. Let π_{rr} be the reflections of π_r through $\{k\} \times (0,k)$. And we denote by $(k/2, y_\pi)$ the last vertex of π_r. We write

$$\mathscr{T}^- = \Big\{\pi \in \mathscr{T} : y_\pi \le \frac{k}{2}\Big\}, \qquad \mathscr{T}^+ = \Big\{\pi \in \mathscr{T} : y_\pi > \frac{k}{2}\Big\}.$$

Let L^- (respectively L^+) be the event that there exists an open path π in $\mathscr{T}^-$ (respectively $\mathscr{T}^+$).

For $\pi \in \mathscr{T}$, we denote by L_π the event that π is the lowest open right-to-left crossing of $B(k,k)$. As Grimmett did in [6], it can be proved that there is a lowest open right-to-left crossing of $B(k,k)$. Here, we omit the proof. Let $\mathcal{A}(\pi_r \cup \pi_{rr})$ be the set of vertices in $(k/2, 3k/2) \times (0,k)$ above $\pi_r \cup \pi_{rr}$ and $\mathcal{B}_\pi$ be the set of vertices in $(0,k) \times (0,k)$ on or below π. We follow a coupling construction, that a new configuration η' is created from the old configuration η by replacing η_x for all $x \in \mathcal{B}_\pi$ by an independent copy, while leaving the status of other edges unchanged. Let M_π^- (respectively M_π^+) be the event for the new configuration η' that there is an open directed path in $\mathcal{A}(\pi_r \cup \pi_{rr})$ starting at $(k/2, 3k/2) \times \{k\}$ and connected to π_r (respectively π_{rr}). See Figure 1.

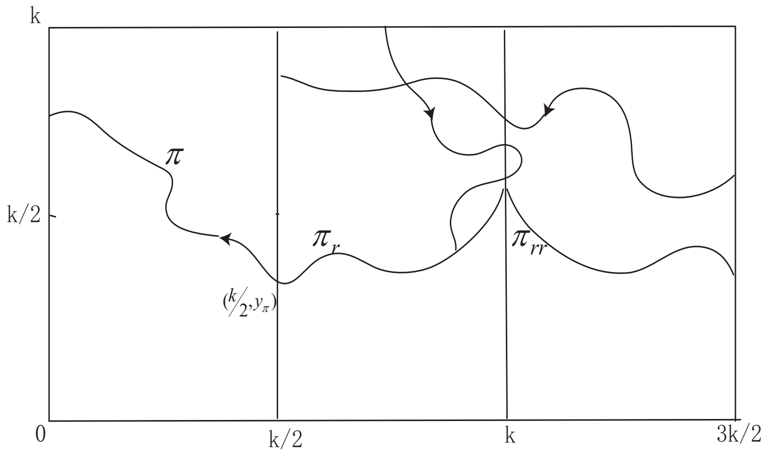

Figure 1

Set

$$G = \bigcup_{\pi \in \mathscr{T}^-} (L_\pi \cap M_\pi^-).$$

We write N^- (respectively N^+) for the event that there exists a open right-to-left crossing of the rectangle $(k/2, 3k/2) \times (0,k)$ which ends at a point with y-coordinate $\ge k/2$

(respectively $\leq k/2$). To prove (3.1), it needs only to show

$$\mathsf{P}(G\cap N^-)\geq(1-\sqrt{1-\tau})^3, \tag{3.2}$$

since the occurrence of G and N^- ensures that there is a open right-to-left crossing in $B(3k/2,k)$.

Let A_π be the set of configurations η of the edges staring at $x\in\mathcal{B}_\pi$ such that π is the lowest open right-to-left crossing of $(0,k)\times(0,k)$. Let L_η be the event that the configuration of the directed edges starting at $x\in\mathcal{B}_\pi$ is η. Now, we have

$$\mathsf{P}(G\cap N^-)=\mathsf{P}\Big(\bigcup_{\pi\in\mathscr{T}^-}(L_\pi\cap M_\pi^-\cap N^-)\Big)=\sum_{\pi\in\mathscr{T}^-}\sum_{\eta\in A_\pi}\mathsf{P}(L_\eta\cap M_\pi^-\cap N^-). \tag{3.3}$$

We apply the square root trick (see P194 of [6]) to the events M_π^+ and M_π^- to find that

$$\mathsf{P}(M_\pi^-)\geq1-\sqrt{1-\mathsf{P}(M_\pi^+\cup M_\pi^-)}\geq1-\sqrt{1-\tau}. \tag{3.4}$$

By FKG inequality, we have

$$\mathsf{P}(M_\pi^-\cap N^-|L_\eta)\geq\mathsf{P}(M_\pi^-|L_\eta)\mathsf{P}(N^-|L_\eta)=\mathsf{P}(M_\pi^-)\mathsf{P}(N^-|L_\eta), \tag{3.5}$$

where the equality comes from the construction of M_π^-, which makes it independent of L_η. Now,

$$\begin{aligned}\mathsf{P}(G\cap N^-)&=\sum_{\pi\in\mathscr{T}^-}\sum_{\eta\in A_\pi}\mathsf{P}(L_\eta\cap M_\pi^-\cap N^-)\\&=\sum_{\pi\in\mathscr{T}^-}\sum_{\eta\in A_\pi}\mathsf{P}(L_\eta)\mathsf{P}(M_\pi^-\cap N^-|L_\eta)\\&\geq\sum_{\pi\in\mathscr{T}^-}\sum_{\eta\in A_\pi}\mathsf{P}(L_\eta)\mathsf{P}(M_\pi^-)\mathsf{P}(N^-|L_\eta)\\&=\sum_{\pi\in\mathscr{T}^-}\sum_{\eta\in A_\pi}\mathsf{P}(M_\pi^-)\mathsf{P}(L_\eta\cap N^-)\\&\geq\sum_{\pi\in\mathscr{T}^-}\mathsf{P}(M_\pi^-)\mathsf{P}(L_\pi\cap N^-)\\&\geq(1-\sqrt{1-\tau})\sum_{\pi\in\mathscr{T}^-}\mathsf{P}(L_\pi\cap N^-)\\&=(1-\sqrt{1-\tau})\mathsf{P}(L^-\cap N^-).\end{aligned} \tag{3.6}$$

By FKG inequality again, we have

$$\mathsf{P}(L^-\cap N^-)\geq\mathsf{P}(L^-)\mathsf{P}(N^-). \tag{3.7}$$

Furthermore by using the square root trick, we can obtain the following two inequalities:

$$\mathsf{P}(L^-)\geq1-\sqrt{1-\mathsf{P}(L^+\cup L^-)}=1-\sqrt{1-\tau} \tag{3.8}$$

and

$$\mathsf{P}(N^-) \geq 1 - \sqrt{1 - \mathsf{P}(N^+ \cup N^-)} = 1 - \sqrt{1-\tau}. \tag{3.9}$$

Therefore, (3.2) follows (3.6)–(3.9). □

Remark 1 For (3.2), we cannot prove it by FKG inequality directly as Grimmett[6] (Lemma 9.73 of Chapter 9) and Cox and Durrett[4]. The reason is that we don't know whether G is an increasing event or not for our case.

The next Lemma is a key to prove Theorems.

Lemma 3.2 Suppose $\theta > 0$. Let $\rho_{j,k} = \mathsf{P}(\mathrm{RL}(j,k))$. Then $1 - \rho_{3k,k}$ decays exponentially in k, that is,

$$\lim_{k\to\infty} \sup k^{-1} \log(1 - \rho_{3k,k}) < 0. \tag{3.10}$$

Proof We first introduce the dual percolation process. Let $\mathbb{Y}^2 = (1/2, 1/2) + \mathbb{Z}^2$ and draw oriented bonds from $y \in \mathbb{Y}^2$ to its four nearest neighboring points. To make the link between the original process and the dual, we associate each dual bond with the bond on the original lattice, which is obtained by rotating the dual bond 90° clockwise around its midpoint, and we declare the dual bond to be closed (respectively, open) if the associated bond on the original lattice was open (respectively, closed).

We claim that, the events that there is an open right-to-left crossing of the rectangle $[0,k] \times [0,k]$, and that there is an open top-to-bottom crossing in the dual lattice $[1/2, k-1/2] \times [1/2, k-1/2]$ are complementary.

The above result is another version of (2.9) in Cox and Durrett[4]. The proof follows from Durrett and Schonmann[5]. First suppose that there is a top-to-bottom crossing on the dual and call it π. By removing loops we can suppose without loss of generality that π is a self-removing one. An application of the Jordan curve theorem shows that π divides the interior of the square into two parts — one we call T_1 which lies to the right of π, and the other we call T_2 which lies to the left of π. If we move along π in the direction of the orientation, T_1 is always on our left and T_2 is on the right. From this, we see that if there is a open path from right to left then any time it crosses from T_1 to T_2 it does so along a bond which is a 90° clockwise rotation of a bond on π but such a bond is closed by the definition of duality so no path exists. Second, we will suppose there is no right-to-left crossing on the square $[0,k] \times [0,k]$ and construct a top-to-bottom one on the dual. Let C be the set of vertices which can be reached from the right edge by a open directed path. Let $D = \{(a,b) \in \mathbb{R}^2 : |a|, |b| \leq 1/2\}$ and orient the boundary of D in a counterclockwise fashion. Let $W = \bigcup\limits_{z\in C} (z+D)$. If we combine the boundaries of the $z+D$ for all $z \in C$, and allow oppositely directed segments to cancel, then the boundaries that remain are closed paths on the dual. One of them $\Gamma =$ the boundary of the component of $\{[1/2, j-1/2]\times[1/2, k-1/2]\}\backslash W$ is the path we want.

Now, we declare that Lemma 3.1 is also true for the dual percolation process, that is,

$$\widetilde{\rho}_{3k/2,k} \geq (1-\sqrt{1-\widetilde{\rho}_{k,k}})^3, \tag{3.11}$$

where $\widetilde{\rho}_{j,k}$ denotes the probability that there is an open right-to-left crossing of the rectangle $[1/2, j-1/2]\times[1/2, k-1/2]$ in the dual percolation model.

To see that (3.11) is legitimate, we note that, although the dual bonds $(x,y)\to(x+1,y)$, $(x+1,y)\to(x+1,y+1)$, $(x+1,y+1)\to(x,y+1)$ and $(x,y+1)\to(x,y)$ are dependent, bonds which go counterclockwise around different squares are independent. Thus, if there is a right-to-left crossing π then all bonds above π are independent of it. Therefore, the dual argument works.

Next, we show that

$$\lim_{k\to\infty}\rho_{k,k}=1. \tag{3.12}$$

Suppose (3.12) is false. By duality, the probability of there being an open top-to-bottom crossing in the dual lattice $[1/2,k-1/2]\times[1/2,k-1/2]$ does not tend to zero. By Lemma 3.1, the probability that there is an open circuit in the annulus $[-3k,3k]^2\backslash[-k,k]^2$ does not tend to zero. Hence, there is an increasing sequence $\{k_n, n\geq 1\}$ such that the probability that there is an open circuit in the annulus $[-3k_n,3k_n]^2\backslash[-k_n,k_n]^2$ is bounded away from zero. Therefore with probability 1 there are infinitely many open dual circuits around the origin, which contradicts the assumption $\theta>0$.

Finally, by Boole inequality (See Figure 2 of [4]), we have

$$1-\rho_{Nk,k}\leq 4(1-\rho_{(N+1)k/2,k}), \qquad \text{for } N\geq 1. \tag{3.13}$$

With an application of Lemma 3.1, along with (3.13), we have $\rho_{2k,k}\to 1$ as $k\to\infty$. Also, if $0<\lambda<1$ and $1-\rho_{2k,k}\leq\lambda/49$, then by (2.6) of [4], $1-\rho_{4k,2k}\leq\lambda^2/49$. Hence by iteration, $1-\rho_{2k,k}$ decays exponentially in k as k increases through powers of 2, and then we can deduce (3.10) using (3.13), as in the proof of Lemma 3.1 of [10]. □

Lemma 3.3 Suppose that $\theta=\theta(\mu)>0$. Then with probability 1 there is exactly one infinite cluster for two dimensional percolation process.

Proof We first prove that there exists an infinite cluster. Indeed, by Lemma 3.2, for large enough k, there are exist right-to-left crossings and left-to-right crossings in the rectangles as forms

$$[-9^N k, 9^N k]\times[-3\cdot 9^{N-1}k, 3\cdot 9^{N-1}k], \qquad N=1,2,\cdots.$$

As the same argument, there are exist top-to-bottom crossings and bottom-to-top crossings in the rectangles as forms

$$[-9^{N-1}k, 9^{N-1}k] \times [-3 \cdot 9^{N-1}k, 3 \cdot 9^{N-1}k], \qquad N = 1, 2, \cdots .$$

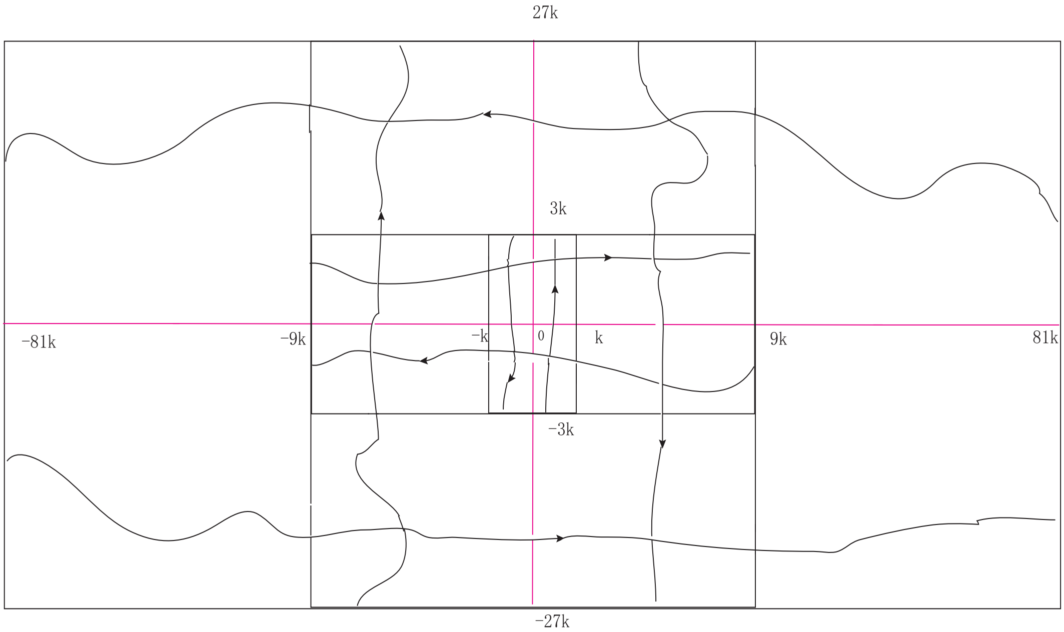

Figure 2

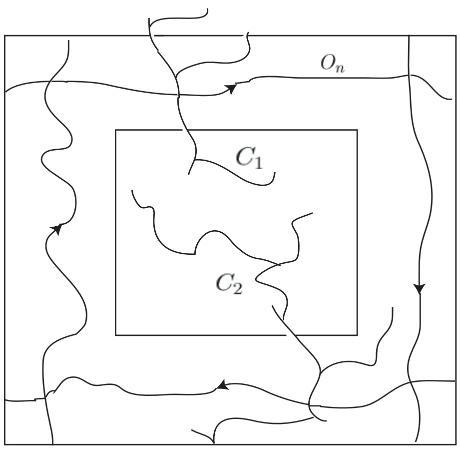

Figure 3

See Figure 2. Now, these crossings in overlapping rectangles must cross each other. Thus the union of all cross sites is contained in a single cluster. Therefore, there exists an infinite cluster with probability 1.

Next, we prove the uniqueness of infinite clusters. By Lemma 3.2, there is an open circuit O_n on the annuli $[-3n, 3n]^2 \setminus [-n, n]^2$ for large enough n. If C_1 and C_2 are two disjoint infinite clusters, one can select a large enough n such that

$$C_1 \cap O_n \neq \emptyset, \qquad C_2 \cap O_n \neq \emptyset.$$

See Figure 3. Now, $C_1 \cup O_n \cup C_2$ is an infinite cluster and C_1 and C_2 are parts of the same cluster, which contradict the assumption that C_1 and C_2 are disjoint. Therefore, the infinite cluster is unique with probability 1. $\square$

Lemma 3.4 Let $\{B_n, n \geq 1\}$ be a δ-comparable sequence of boxes tending to $\mathbb{Z}^d$, and let $b_n = \lceil \mathrm{diam}(B_n)^{1/4} \rceil$. We write A_n for the event that $\mathrm{diam}(C(B_n)) \geq \mathrm{diam}(B_n) - b_n$ and that any open directed path in B_n of diameter at least b_n passes through $C(B_n)$, Then $\mathsf{P}(A_n) \to 1$ as $n \to \infty$.

Proof Set $a_n = \lfloor b_n/2 \rfloor$. We write $\mathcal{R}_n$ for the collection of all nonempty rectangles with $j, l \in \mathbb{Z}$ of the form

$$J_{j,l,n} = ([ja_n, (j+3)a_n] \times [la_n, (l+1)a_n]) \cap B_n$$

and

$$L_{j,l,n} = ([ja_n, (j+1)a_n] \times [la_n, (l+3)a_n]) \cap B_n.$$

Set

$$P_{j,l,n} = \{\text{there are left-to-right and right-to-left crossings of } J_{j,l,n}\}$$

and

$$Q_{j,l,n} = \{\text{there are top-to-bottom and bottom-to-top crossings of } L_{j,l,n}\}.$$

By Lemma 3.2, the probability of the event that all $P_{j,l,n}$ and $Q_{j,l,n}$ with $j, l \in \mathbb{Z}$ happen simultaneously tends to 1 as $n \to \infty$. If this occurs, the open crossing paths in overlapping rectangles must cross each other. This implies A_n. $\square$

§4. Proofs of Theorems

Proof of Theorem 2.1 By Lemma 3.3, there exists exactly one infinite cluster, denoted by C_∞.

By Theorem 2.1 of Penrose[10], it suffices to show that the sequences $\{B_n, n \geq 1\}$ satisfies stabilization and bounded moments condition for the function $H(\eta, B) = |C(B)|$.

We check stabilization only for the configurations $\eta \in \{\xi : \xi_0 = (0,0,0,0)\}$ and $\eta^0 \in \{\xi : \xi_0 \neq (0,0,0,0)\}$. It is not hard to deduce other cases from this one.

Set $\Delta_0(\infty) = C_\infty(\eta^0) - C_\infty(\eta)$. Now $\Delta_0(\infty)$ is the number of vertices added to C_∞ by changing the configuration from η to η^0.

For lattice box B_n, suppose $b_n = \lceil (\mathrm{diam} B_n)^{1/4} \rceil$ and $B_n^o = B_n \setminus \partial_{b_n} B_n$. For large enough n, we have $C_\infty \cap B_n^o \neq \emptyset$. Then A_n (recall that the definition of A_n in Lemma 3.4) occurs and $C(B_n) \cap B_n^o = C_\infty \cap B_n^o$. Therefore, the number of vertices added to $C(B_n)$ as a result of adding the open bonds of η_0^0 is exactly equal to $\Delta_0(\infty)$. That is, $\Delta_0(B_n) = \Delta_0(\infty)$ for large enough n. This proves stabilization.

Next, the bounded moments condition will be checked. Let A_k denote the event that each of four overlapping rectangles consisting of the annulus $[-3k, 3k]^2 \setminus [-k, k]^2$ have long crossings in both directions. See Figure 4.

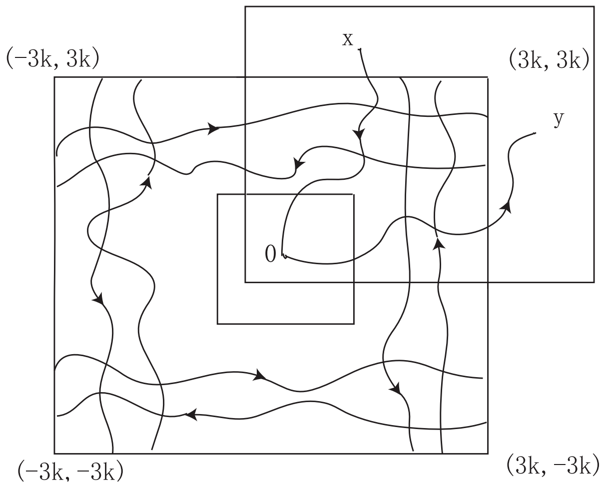

Figure 4

Suppose that $B \cap \{(3k, 3k), (-3k, 3k), (-3k, -3k), (3k, -3k)\} \neq \emptyset$ and A_k occurs. For $x, y \in B \setminus [-3k, 3k]^2$, if there is a directed path through the origin from x to y, then the path must pass through the annulus before arriving to and after away from the origin. Thus there is a directed path from x to y avoiding the origin and even the box $[-k, k]^2$. See Figure 4. The same argument holds for the directed path from y to x. So for $x, y \in B \setminus [-3k, 3k]^2$, x and y are in the same cluster in B for the configuration η if and only if so are them for the configuration η^0. Then, for any cluster C in B for the configuration η (respectively, η^0), $C \setminus [-3k, 3k]^2$ is contained in a same cluster in B for the configuration η^0 (respectively, η). Therefore,

$$|\Delta_0(B)| \leq (6k+1)^2. \tag{4.1}$$

Suppose that $B \cap \{(3k, 3k), (-3k, 3k), (-3k, -3k), (3k, -3k)\} = \emptyset$ and A_k occurs. Noted B is δ-comparable, we have $\mathrm{diam}(B) \leq (6k+1)/\delta$ and

$$|\Delta_0(B)| \leq |B| \leq \frac{(6k+1)^2}{\delta}. \tag{4.2}$$

Therefore, for any δ-comparable lattice box B containing the origin, if A_k occurs, there is a constant λ such that

$$|\Delta_0(B)| \le \lambda k^2. \tag{4.3}$$

By Lemma 3.2, $1-\mathsf{P}(A_k)$ decays exponentially in k. Hence, the sequence $\{\mathsf{E}|\Delta_0(B_n)|^\gamma, n \ge 1\}$ $(\gamma > 2)$ uniformly bounded for the δ-comparable lattice boxes sequence $\{B_n, n \ge 1\}$.

Now, we have completed the proof. $\Box$

Proof of Theorem 2.2 We can prove it as the same argument in the proof of Theorem 3.3 of Penrose[10]. Now we only give the sketch of the proof. We first use Lemma 3.4 to show that

$$\lim_{n\to\infty} \frac{a_n}{|B_n|} = \widetilde{\theta}. \tag{4.4}$$

Indeed, since $a_n = \sum\limits_{x\in B_n} \mathsf{P}\{x \in C(B_n)\}$, it suffices to show that $\mathsf{P}\{x \in C(B_n)\}$ is close to $\widetilde{\theta}$ uniformly for $x \in B_n$ at a distance at least b_n from ∂B_n, recalling that $b_n = \lceil \mathrm{diam}(B_n)^{1/4} \rceil$. Note that

$$\begin{aligned}
\mathsf{P}\{x \in C(B_n)\} &= \mathsf{P}\{\mathrm{diam} C_x(B_n) \ge b_n, x \in C(B_n)\} \\
&\quad + \mathsf{P}\{\mathrm{diam} C_x(B_n) < b_n, x \in C(B_n)\} \\
&= \mathsf{P}\{\mathrm{diam} C_x(B_n) \ge b_n\} - \mathsf{P}\{\mathrm{diam} C_x(B_n) \ge b_n, x \notin C(B_n)\} \\
&\quad + \mathsf{P}\{\mathrm{diam} C_x(B_n) < b_n, x \in C(B_n)\}.
\end{aligned}$$

In the right hand side, the first term tends to $\widetilde{\theta}$, and the second and third term both tend to zero by Lemma 3.4. This completes the proof of (4.4).

Next fixed $t \in \mathbb{R}$, for each n we define the sets $Q_n = [-b_n, b_n]^2$ and the events

$$F_n = \{C_0(Q_n) \cap \partial Q_n \ne \emptyset\},$$

$$G_n = \Big\{ \frac{|C(B_n)| - a_n}{\sigma\sqrt{|B_n|}} \ge t \Big\}; \qquad H_n = \Big\{ \frac{|C_0(B_n)| - a_n}{\sigma\sqrt{|B_n|}} \ge t \Big\}.$$

By Lemma 3.4 and the definitions of F_n and H_n, we know that, if F_n^c occurs for large enough n, then H_n^c occurs. Therefore, for large enough n, we have

$$\mathsf{P}(H_n) = \mathsf{P}(H_n \cap F_n) = \mathsf{P}(G_n \cap F_n) - \mathsf{P}((G_n \backslash H_n) \cap F_n), \tag{4.5}$$

where the last equality because of the fact $H_n \subseteq G_n$. By Lemma 3.4, in the right hand side, the second term tends to zero.

Now, the rest of work is to prove that

$$\lim_{n\to\infty} \mathsf{P}(G_n | F_n) = 1 - \Phi(t) \tag{4.6}$$

noted that $\lim_{n\to\infty} \mathsf{P}(F_n) = \widetilde{\theta}$ and the events F_n and G_n are all increasing.

Since the proof of (4.6) is almost the same as (3.5) in [10], we omit it. $\square$

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$\mathbb{Z}^2$上相依渗流开簇的极限定理

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中心极限定理, 大偏差定理和大数定律等极限定理在概率论中起着很重要的角色. 本文我们研究$\mathbb{Z}^2$上一类相依渗流模型. 对此模型, 我们不仅证明了其无穷开簇的存在唯一性, 而且得到了关于格点盒子类极大开簇的中心极限定理.

关键词: 极限定理, 相依渗流, 对偶.

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