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Empirical Likelihood in Partial Linear Model under m-Dependent Errors *

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Abstract

In this paper, the empirical likelihood method is extended to partial linear models with fixed designs under m-dependent errors. We show that not the usual empirical likelihood but the blockwise empirical likelihood works in this situation, and the blockwise empirical log-likelihood ratio is asymptotically chi-squared distributed. A simulation study is reported to show its efficiency.

Keywords: Partial linear model, m-dependent errors, blockwise empirical likelihood.

AMS Subject Classification: 62G10, 62G20, 62M10.

§1. Introduction

Consider the following partial linear model:

$$y_i = x_i'\beta + g(t_i) + e_i, \qquad i = 1, 2, \ldots, n, \tag{1.1}$$

where y_i is the scalar response, the p-vector x_i is the ith fixed design point and t_i is the ith fixed scalar design point, β is a vector of unknown parameters to be estimated, $g(\cdot)$ is an unknown function defined on the closed interval I of R, and the prime $(')$ denotes the transpose operator.

Model (1.1) with independent identically distributed (i.i.d.) errors was first introduced by Engle et al. (1986), and further studied by Heckman (1986), Speckman (1988),

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Chen (1988), Hamilton and Truong (1997) and Mammen and Van de Geer (1997), amongst others.

On the other hand, the empirical likelihood method is one of the important methods for constructing confidence regions in nonparametric setting, and its general property was subsequently studied by Owen (1990). The empirical likelihood ratio statistic has much in common with its conventional parametric counterpart. In particular, its logarithm has a chi-squared limiting distribution. The empirical likelihood method has been studied extensively because of its generality and efficiency (see Owen (1991), Chen (1993, 1994) and Qin and Lawless (1994), amongst others). It should be noted that the above work seems to focus on independent data and the usual empirical likelihood can not be used properly for dependent data. Recently, Kitamura (1997) and Zhang, et al. (1999) use the blockwise empirical likelihood to deal with strong dependent data.

In this paper, we shall attempt to apply the blockwise empirical likelihood method to the above partial linear model (1.1) under the condition that the random errors $\{e_i, 1 \leq i \leq n\}$ are stationary m-dependent variables of mean zero (refer to Yu and Tu (1993) for the definition of m-dependent random variables).

The paper is organized as follows. In Section 2, we introduce the empirical likelihood and blockwise empirical likelihood methods, and study their large sample properties. Some simulation results will be presented in Section 3 to evaluate the performance of the proposed method. Proofs are given Section 4.

§2. Main Results

We now discuss how to apply the empirical likelihood method to model (1.1) and construct an empirical log-likelihood confidence region for the regression coefficient β. Here, we assume that the random errors $\{e_i, 1 \leq i \leq n\}$ are stationary m-dependent variables, $\mathsf{E}e_i = 0$, $\mathsf{E}e_i^2 = \sigma^2$, $\mathsf{E}|e_i|^j = \mu_j < \infty$, for $j = 3, 4$.

For model (1.1), if β is known to be the true parameter, then by $\mathsf{E}e_i = 0$, we have

$$g(t_i) = \mathsf{E}(y_i - x_i'\beta), \qquad i = 1, 2, \ldots, n.$$

Hence a natural nonparametric estimator of $g(\cdot)$ given β is

$$\widehat{g}_n(t) = \sum_{i=1}^n W_{ni}(t)(y_i - x_i'\beta),$$

where $W_{ni}(\cdot)\,(1 \leq i \leq n)$ are some weight functions defined on I.

At first, we consider the ordinary empirical likelihood ratio statistic for β. Let $\widetilde{x}_i = x_i - \sum\limits_{j=1}^n W_{nj}(t_i)x_j$, $\widetilde{y}_i = y_i - \sum\limits_{j=1}^n W_{nj}(t_i)y_j$, $\omega_i = \widetilde{x}_i(y_i - x_i'\beta - \widehat{g}_n(t_i)) = \widetilde{x}_i(\widetilde{y}_i - \widetilde{x}_i'\beta)$, the (log) empirical likelihood ratio statistic is defined as

$$l(\beta) = 2\sum_{i=1}^{n}\log\{1+\lambda'(\beta)\omega_i\}, \tag{2.1}$$

where $\lambda(\beta) \in R^p$ is determined by

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\omega_i}{1+\lambda'(\beta)\omega_i} = 0. \tag{2.2}$$

We need the following assumption.

Assumption A

(1) $0 \le W_{nj}(t) \le 1$ for $\forall\, t \in I$ and $\sum\limits_{j=1}^{n} W_{nj}(t) = 1$ for $\forall\, t \in I$.

(2) $\max\limits_{1\le i,j\le n} |W_{nj}(t_i)| = \mathrm{O}(n^{-1/2}(\log n)^{-1})$.

(3) $\max\limits_{1\le i\le n}\sum\limits_{j=1}^{n}|W_{nj}(t_i)| = \mathrm{O}(1)$, $\max\limits_{1\le i\le n}\Big|\sum\limits_{j=1}^{n} W_{nj}(t_i) - 1\Big| = \mathrm{o}(n^{-1/2})$.

(4) $\max\limits_{1\le i\le n}\sum\limits_{j=1}^{n} W_{nj}(t_i)|t_i - t_j| I(|t_i - t_j| > (n\log n)^{-1/2}) = \mathrm{o}(n^{-1/2})$.

(5) $\max\limits_{1\le i\le n}\|\widetilde{x}_i\| = \mathrm{O}(n^{1/2}(\log n)^{-1})$, $\limsup\limits_{n\longrightarrow\infty} 1/(\sqrt{n}\log n)\max\limits_{1\le m\le n}\Big\|\sum\limits_{i=1}^{m}\widetilde{x}_{j_i}\Big\| < \infty$, where $(j_1, j_2, \ldots, j_n)$ is any permutation of $(1, 2, \ldots, n)$ and $\|\cdot\|$ denotes the Euclidean norm.

(6) $\max\limits_{1\le j\le n}\Big\|\sum\limits_{i=1}^{n} W_{nj}(t_i)\widetilde{x}_i\Big\| = \mathrm{o}(1)$.

(7) $(1/n^2)\sum\limits_{i=1}^{n}\|\widetilde{x}_i\|^4 \to 0$ as $n \to +\infty$.

(8) As $n \to +\infty$, there exist $A_0 > 0$ and $A > 0$ such that

$$\frac{1}{n}\sum_{i=1}^{n}\widetilde{x}_i\widetilde{x}_i' = A_0 + \mathrm{o}(1).$$

$$\frac{1}{n}\Big\{\sum_{i=1}^{n}\widetilde{x}_i\widetilde{x}_i'\sigma^2 + \sum_{j=1}^{m-1}\sum_{i=1}^{n-j}(\widetilde{x}_i\widetilde{x}_{i+j}' + \widetilde{x}_{i+j}\widetilde{x}_i')\mathsf{E}(e_ie_{i+j})\Big\} = A + \mathrm{o}(1).$$

$$n^{-3/2}\sum_{i=1}^{n}\sqrt{\widetilde{x}_i'\widetilde{x}_i\widetilde{x}_i\widetilde{x}_i'} = \mathrm{o}(1).$$

(9) Let σ_1 and σ_p denote the largest and smallest eigenvalues of A_0, respectively. There exist positive constants C_1 and C_2 such that $C_1 \le \sigma_p \le \sigma_1 \le C_2$.

(10) $g(\cdot)$ satisfies the first-order Lipschitz condition on I.

Remark 1 A commonly used weight function $W_{nj}(t)$ is given by

$$W_{ni}^{(1)}(t) = \frac{1}{h_n}\int_{s_{i-1}}^{s_i} K\Big(\frac{t-s}{h_n}\Big)\mathrm{d}s, \qquad W_{ni}^{(2)}(t) = K\Big(\frac{t-t_i}{h_n}\Big)\Big[\sum_{j=1}^{n} K\Big(\frac{t-t_j}{h_n}\Big)\Big]^{-1},$$

where $s_i = (t_i + t_{i-1})/2$, $i = 1, \ldots, n-1$, $s_0 = 0$, $s_n = 1$, $K(\cdot)$ is the Parzen-Rosenblatt kernel function (cf. Parzen (1962)), and h_n is a bandwidth parameter verifying suitable conditions.

Remark 2 A(5) is assumed in Gao et al. (1994), Chen et al. (1998) and Härdle et al. (2000). A(6)-A(7) are assumed in Wang and Jing (1999). The conditions are common conditions that the design points satisfy to discuss the properties of the empirical likelihood ratio statistic in partially linear models.

Remark 3 The first condition in A(8) is very common. Since $\{e_i\}$ are m-dependent random variables, it can be shown that

$$\mathsf{E}\Big(\sum_{i=1}^{n}\widetilde{x}_i e_i\Big(\sum_{i=1}^{n}\widetilde{x}_i e_i\Big)'\Big)=\sum_{i=1}^{n}\widetilde{x}_i\widetilde{x}_i'\sigma^2+\sum_{j=1}^{m-1}\sum_{i=1}^{n-j}(\widetilde{x}_i\widetilde{x}_{i+j}'+\widetilde{x}_{i+j}\widetilde{x}_i')\mathsf{E}(e_ie_{i+j}).$$

It is reasonable to assume that there exists $A>0$ such that the second condition in A(8) holds. If $\max\limits_{1\le i\le n}\|\widetilde{x}_i\|=\mathrm{o}(n^{-1/2})$, then the third condition in A(8) holds.

The first result in this paper is as follow.

Theorem 2.1 Let β_0 be the true value of β. Suppose that Assumption A is true, then

$$l(\beta_0)\rightarrow_{\mathcal{D}} Z^TA_0^{-1}Z, \qquad \text{as } \ n\to\infty,$$

where $Z\sim N_p(0,A)$ and "$\rightarrow_{\mathcal{D}}$" denotes convergence in distribution.

As we do not know A_0 and A, the above result could not be used in practice. We will use the blockwise empirical likelihood to overcome this shortcoming of the ordinary empirical likelihood.

Let $h=(\log n)^{2-r}$, $0<r<1$, $g=[n/h]$ and put

$$\omega_i^{(1)}=\frac{1}{h}\sum_{j=1}^{h}\omega_{(i-1)h+j}, \qquad i=1,2,\ldots,g.$$

We consider the following blockwise empirical likelihood ratio:

$$R^{(1)}(\beta)=\sup_{p_1,p_2,\ldots,p_g}\Big\{\prod_{i=1}^{g}(gp_i)\Big|\sum_{i=1}^{g}p_i=1,p_i\ge 0,\sum_{i=1}^{g}p_i\omega_i^{(1)}=0\Big\}.$$

It is easy to obtain the (log) blockwise empirical likelihood ratio statistic

$$l^{(1)}(\beta)=2\sum_{i=1}^{g}\log\{1+t'(\beta)\omega_i^{(1)}\}, \tag{2.3}$$

where $t(\beta)\in R^p$ is determined by

$$\frac{h}{g}\sum_{i=1}^{g}\frac{\omega_i^{(1)}}{1+t'(\beta)\omega_i^{(1)}}=0. \tag{2.4}$$

This method to obtain the likelihood ratio statistic is called the blockwise empirical likelihood. The intuitive background of the blockwise empirical likelihood is understandable. As the sample size increased, the proportion of each sample became smaller and smaller, when the difference between them is not too large, you can increase the sample group length, and determine the right configuration of the each group according to the average contribution that the each group made to the overall. To obtain the large sample distribution of $l^{(1)}(\beta_0)$, we also need another assumption.

Assumption B

(11) $[1/(gh)]\sum_{i=1}^{gh}\widetilde{x}_{i,l}'^2\to\Gamma_0$, $[1/(g^2h)]\sum_{i=1}^{gh}\widetilde{x}_{i,l}'^4\to 0$, for $l=1,2,\ldots,p$ as $n\to+\infty$, where $\widetilde{x}_{i,l}$ denotes the l-th component of $\widetilde{x}_i$.

(12) As $n\to+\infty$, there exists $A>0$ such that

$$\frac{1}{gh}\Big\{\sum_{i=1}^{gh}\widetilde{x}_i\widetilde{x}_i'\sigma^2+\sum_{j=1}^{m-1}\sum_{i=1}^{gh-j}(\widetilde{x}_i\widetilde{x}_{i+j}'+\widetilde{x}_{i+j}\widetilde{x}_i')\mathsf{E}(e_ie_{i+j})\Big\}=A+\mathrm{o}(1),$$

$$\frac{1}{gh}\sum_{j=1}^{m-1}\sum_{k=1}^{g-1}\sum_{i=1}^{j}(\widetilde{x}_{kh-j+i}\widetilde{x}_{kh+i}'+\widetilde{x}_{kh+i}\widetilde{x}_{kh-j+i}')\mathsf{E}(e_{kh+i}e_{kh-j+i})=\mathrm{o}(1).$$

The second result in this paper is as follows.

Theorem 2.2　Under the conditions of Theorem 2.1 and suppose that Assumption B is true, then

$$l^{(1)}(\beta_0)\to_{\mathcal{D}}\chi_p^2,\qquad \text{as}\ \ n\to\infty.$$

§3. Simulation Study

We report results from a simulation study designed to evaluate the performance of the proposed empirical likelihood method.

We now describe how to carry out the empirical likelihood method. The design points x_i's in our simulation studies are taken to be $x_i=\Phi^{-1}(i/(n+1))$. On the other hand, the design points t_i's are generated from a uniform distribution $U[0,1]$ with a fixed seed 10 and so they will remain fixed in the simulation. The function $g(\cdot)$ is chosen to be $g(t)=t^2$. The kernel function $k(t)$ is biweight kernel function

$$k(x)=\frac{15}{16}(1-x^2)^2,\qquad |x|\le 1.$$

Also, we have selected the bandwidth h_n to be $n^{-1/2}(\log n)^{-1}$. It is easy to check that all conditions A(1)-A(4) in the paper are satisfied with the above choices. In our simulation studies, we generate ε_i's i.i.d. from the standard normal distribution $N(0,1)$.

Model 1　$e_i=\varepsilon_i+0.5\varepsilon_{i-1}-0.3\varepsilon_{i-2}$, so e_i's are 2-dependent random series.

Model 2　$e_i=\varepsilon_i+0.6\varepsilon_{i-1}-0.4\varepsilon_{i-2}-0.2\varepsilon_{i-3}$, so e_i's are 3-dependent random series.

The samples sizes are choosen to be 500 and 1000, respectively. We use the blockwise empirical likelihood method and choose $h=(\log n)^{2-1/2}$. The nominal levels are taken to be $\alpha = 0.10$ and 0.05, respectively. The coverage probabilities are calculated for the blockwise empirical likelihood method based on 500 simulated data. The results presented in Table 1 and Table 2.

From Table 1 and Table 2, we see that the blockwise empirical likelihood method performs well. The coverage probabilities do not vary greatly for different values of β. It is interesting to note that the coverage accuracies tend to increase as the sample size n gets larger. However, this may not be the case all the time. The reason is that the design points (x_i, t_i)'s are different for each different sample size n.

Table 1 Coverage probabilities for β in Model 1

		$\beta=-3.5$	$\beta=-1.5$	$\beta=-0.5$	$\beta=0$	$\beta=0.5$	$\beta=1.5$	$\beta=3.5$
$n=500$	$\alpha=0.10$	0.870	0.854	0.856	0.886	0.886	0.874	0.882
	$\alpha=0.05$	0.936	0.936	0.934	0.954	0.954	0.940	0.942
$n=1000$	$\alpha=0.10$	0.900	0.890	0.860	0.910	0.900	0.890	0.910
	$\alpha=0.05$	0.954	0.960	0.950	0.950	0.956	0.960	0.950

Table 2 Coverage probabilities for β in Model 2

		$\beta=-3.5$	$\beta=-1.5$	$\beta=-0.5$	$\beta=0$	$\beta=0.5$	$\beta=1.5$	$\beta=3.5$
$n=500$	$\alpha=0.10$	0.880	0.890	0.890	0.910	0.860	0.830	0.900
	$\alpha=0.05$	0.940	0.950	0.944	0.950	0.950	0.940	0.950
$n=1000$	$\alpha=0.10$	0.900	0.910	0.910	0.930	0.890	0.910	0.910
	$\alpha=0.05$	0.960	0.960	0.956	0.970	0.956	0.950	0.960

§4. Proof of the Main Results

To prove the theorems, we first give two Lemmas.

Lemma 4.1 Let $X_1, X_2, \ldots, X_n$ be univariate m-dependent r.v. series with $\mathsf{E}X_i = 0$, $\mathsf{E}|X_i|^r < \infty$ for $i = 1, 2, \ldots, n$ and some $r \geq 2$, then

$$\mathsf{E}\Big|\sum_{i=1}^n X_i\Big|^r \leq C_r n^{r/2-1} \sum_{i=1}^n \mathsf{E}|X_i|^r,$$

where the constant C_r only depends upon r and m.

The proof can be found in Qin et al. (2005).

Lemma 4.2 Let $X_1, X_2, \ldots, X_n$ be univariate m-dependent r.v.s., $\mathsf{E}X_i = 0$, $\mathsf{E}|X_i|^3 < \infty$ $(i = 1, 2, \ldots, n)$. Then

$$\sup_x \left|\mathsf{P}\left(\frac{\sum\limits_{i=1}^n X_i}{B_n} < x\right) - \Phi(x)\right| \leq \frac{C(m+1)^3 \sum\limits_{i=1}^n \mathsf{E}|X_i|^3}{B_n^3},$$

where $B_n = \sqrt{\mathsf{E}\Big(\sum\limits_{i=1}^{n} X_i\Big)^2}$ and $\Phi(x)$ is the distribution function of standard normal random variable.

The proof can be found in Shergin (1979).

Proof of Theorem 2.1 First of all, we will show the following results:

$$\omega_n = \max_{1\le i\le n}\|\omega_i\| = \mathrm{o}_p(n^{1/2}), \tag{4.1}$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\omega_i \to_{\mathcal{D}} N(0, A), \tag{4.2}$$

$$a'\Big(\frac{1}{n}\sum_{i=1}^{n}\omega_i\Big) = \mathrm{O}_p(n^{-1/2}), \qquad \forall\, a\in R^p, \tag{4.3}$$

$$S = \frac{1}{n}\sum_{i=1}^{n}\omega_i\omega_i' = A_0 + \mathrm{o}_p(1), \tag{4.4}$$

where $\to_{\mathcal{D}}$ denotes the convergence in distribution.

It is clear, for any $\varepsilon>0$, that

$$\begin{aligned}\mathsf{P}\Big(\frac{1}{\sqrt{n}}\omega_n>\varepsilon\Big) &\le \sum_{i=1}^{n}\mathsf{P}\Big(\|\widetilde{x}_i\|\Big(\Big|g(t_i)-\sum_{j=1}^{n}W_{nj}(t_i)g(t_j)\Big|+|e_i|+\Big|\sum_{j=1}^{n}W_{nj}(t_i)e_j\Big|\Big)>\sqrt{n}\varepsilon\Big)\\ &\le \frac{C}{n^2\varepsilon^4}\sum_{i=1}^{n}\|\widetilde{x}_i\|^4\Big(\Big|g(t_i)-\sum_{j=1}^{n}W_{nj}(t_i)g(t_j)\Big|^4+\mathsf{E}|e_i|^4+\mathsf{E}\Big|\sum_{j=1}^{n}W_{nj}(t_i)e_j\Big|^4\Big),\end{aligned}$$

where C is a const not depending on n.

Let $\widetilde{g}(t_i) = g(t_i) - \sum\limits_{j=1}^{n} W_{nj}(t_i)g(t_j)$. Using A(1), A(4) and A(10), it is easy to prove $\max\limits_{1\le i\le n}|\widetilde{g}_i(t_i)| = \mathrm{o}(n^{-1/2})$.

By applying Lemma 4.1 and condition A(2), we obtain

$$\mathsf{E}\Big|\sum_{j=1}^{n}W_{nj}(t_i)e_j\Big|^4 = Cn\sum_{j=1}^{n}W_{nj}^4(t_i)\mu_4 = \mathrm{o}(1).$$

Hence, from the assumptions of the model and A(7), we have (4.1).

Note that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\omega_i = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widetilde{x}_ie_i + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widetilde{x}_i\widetilde{g}(t_i) - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widetilde{x}_i\sum_{j=1}^{n}W_{nj}(t_i)e_j.$$

By A(5) and using Abel's inequality, we have

$$\Big\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widetilde{x}_i\widetilde{g}(t_i)\Big\| \le \frac{p}{\sqrt{n}}\max_{1\le i\le n}|\widetilde{g}(t_i)|\max_{1\le m\le n}\Big\|\sum_{i=1}^{m}\widetilde{x}_{j_i}\Big\| = \mathrm{o}(n^{-1/2}(\log n)) = \mathrm{o}(1),$$

where $(j_1, j_2, \ldots, j_n)$ is any permutation of $(1, 2, \ldots, n)$. For any $a\in R^p$, $a\ne 0$, obviously,

$$\frac{1}{n}\mathsf{E}\Big(a'\sum_{i=1}^{n}\widetilde{x}_i\sum_{j=1}^{n}W_{nj}(t_i)e_j\Big) = 0.$$

By applying Lemma 4.1 and condition A(6), we have

$$\begin{aligned}\frac{1}{n}\mathsf{E}\Big(a'\sum_{i=1}^n \widetilde{x}_i\sum_{j=1}^n W_{nj}(t_i)e_j\Big)^2 &\le C\frac{1}{n}\sum_{j=1}^n \mathsf{E}\Big(a'\sum_{i=1}^n \widetilde{x}_i W_{nj}(t_i)\Big)^2 e_j^2\\ &\le C\frac{1}{n}\|a\|^2\sigma^2\sum_{j=1}^n\Big\|\sum_{i=1}^n W_{nj}(t_i)\widetilde{x}_i\Big\|^2 = \mathrm{o}(1).\end{aligned}$$

That is

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n \widetilde{x}_i\sum_{j=1}^n W_{nj}(t_i)e_j = \mathrm{o}_p(1).$$

To prove (4.2), it suffices to show that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n \widetilde{x}_i e_i \to_{\mathcal{D}} N(0,A).$$

For any $a\in R^p$, $a\ne 0$, by applying Lemma 4.2, we have

$$\sup_x\left|\mathsf{P}\left(\frac{\sum_{i=1}^n a'\widetilde{x}_i e_i}{B_{1n}}<x\right)-\Phi(x)\right| \le \frac{C(m+1)^3\sum_{i=1}^n \mathsf{E}|a'\widetilde{x}_i e_i|^3}{B_{1n}^3},$$

where $B_{1n}=\sqrt{\mathsf{E}\Big(\sum_{i=1}^n a'\widetilde{x}_i e_i\Big)^2}$.

By A(8), we have

$$\begin{aligned}\frac{1}{n}B_{1n}^2 &= \frac{1}{n}\mathsf{E}\Big(\sum_{i=1}^n a'\widetilde{x}_i e_i\Big)^2\\ &= a'\Big(\frac{1}{n}\Big\{\sum_{i=1}^n \widetilde{x}_i\widetilde{x}_i^T\sigma^2+\sum_{j=1}^{m-1}\sum_{i=1}^{n-j}(\widetilde{x}_i\widetilde{x}_{i+j}^T+\widetilde{x}_{i+j}\widetilde{x}_i^T)\mathsf{E}(e_ie_{i+j})\Big\}\Big)a\\ &= a'Aa+\mathrm{o}(1).\end{aligned}$$

Noticing that

$$\begin{aligned}\mathsf{E}|a'\widetilde{x}_i e_i|^3 &= a'(\widetilde{x}_i\widetilde{x}_i^T)a|a'\widetilde{x}_i|\mathsf{E}|e_i|^3 \le a'(\widetilde{x}_i\widetilde{x}_i')a\|\widetilde{x}_i\|\|a\|\mathsf{E}|e_i|^3\\ &= \|a\|\mu_3 a'(\sqrt{\widetilde{x}_i'\widetilde{x}_i}(\widetilde{x}_i\widetilde{x}_i'))a.\end{aligned}$$

Hence, by A(8), we obtain

$$\frac{C(m+1)\sum_{i=1}^n \mathsf{E}|a'\widetilde{x}_i e_i|^3}{B_{1n}^3} \le \frac{C(m+1)\|a\|\mu_3(1/n^{-3/2})\sum_{i=1}^n a'(\sqrt{\widetilde{x}_i'\widetilde{x}_i}(\widetilde{x}_i\widetilde{x}_i'))a}{((1/n)B_{1n}^2)^{3/2}} = \mathrm{o}(1).$$

Thus, we get

$$\frac{\sum_{i=1}^n a'\widetilde{x}_i e_i}{B_{1n}} \to_{\mathcal{D}} N(0,1). \tag{4.5}$$

That is

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widetilde{x}_i e_i \rightarrow_{\mathcal{D}} N(0,A).$$

Therefore, (4.2) is proved. From (4.5), we also get (4.3).

Note that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\omega_i\omega_i' &= \frac{1}{n}\sum_{i=1}^{n}\widetilde{x}_i\widetilde{x}_i' e_i^2+\frac{1}{n}\sum_{i=1}^{n}\widetilde{x}_i\widetilde{x}_i'\Big(\sum_{j=1}^{n}W_{nj}(t_i)e_j\Big)^2+\frac{1}{n}\sum_{j=1}^{n}\widetilde{x}_i\widetilde{x}_i'\widetilde{g}^2(t_i)\\ &\quad -\frac{2}{n}\sum_{i=1}^{n}\widetilde{x}_i\widetilde{x}_i' e_i\Big(\sum_{j=1}^{n}W_{nj}(t_i)e_j\Big)+\frac{2}{n}\sum_{i=1}^{n}\widetilde{x}_i\widetilde{x}_i' e_i\widetilde{g}(t_i)\\ &\quad -\frac{2}{n}\sum_{i=1}^{n}\widetilde{x}_i\widetilde{x}_i'\Big(\sum_{j=1}^{n}W_{nj}(t_i)e_j\Big)\widetilde{g}(t_i)\\ &= R_{n1}+R_{n2}+R_{n3}-R_{n4}+R_{n5}-R_{n6}.\end{aligned}$$

For any $a\in R^p$, $a\neq 0$, by A(8), we have

$$\mathsf{E}a'R_{n1}a=a'A_0a+\mathrm{o}(1),$$

by applying Lemma 4.1 and condition A(7), we obtain

$$\begin{aligned}\mathsf{E}(a'R_{n1}a-\mathsf{E}(a'R_{n1}a))^2 &\le \mathsf{E}\Big(\frac{1}{n}\sum_{i=1}^{n}a'\widetilde{x}_i\widetilde{x}_i^T ae_i^2\Big)^2\\ &\le \frac{C}{n^2}\sum_{i=1}^{n}\mathsf{E}(a'\widetilde{x}_i\widetilde{x}_i^T ae_i^2)^2\\ &\le \frac{C}{n^2}\|a\|^4\mu_4\sum_{i=1}^{n}\|\widetilde{x}_i\|^4=\mathrm{o}(1).\end{aligned}$$

That is

$$R_{n1}=A_0+\mathrm{o}_p(1). \tag{4.6}$$

By applying Lemma 4.1, condition A(2) and A(8), for any $a\in R^p$, $a\neq 0$, we have

$$\mathsf{E}|a'R_{n2}a|\le a'\frac{1}{n}\sum_{i=1}^{n}\widetilde{x}_i\widetilde{x}_i' aC\sum_{j=1}^{n}W_{nj}^2(t_i)\sigma^2=\mathrm{o}(1).$$

That is

$$R_{n2}=\mathrm{o}_p(1). \tag{4.7}$$

By applying Lemma 4.1, conditions A(4), A(8) and A(10), for any $a\in R^p$, $a\neq 0$, we have

$$|a'R_{n3}a|\le\Big(\max_{1\le i\le n}|\widetilde{g}(t_i)|\Big)^2 a'\frac{1}{n}\sum_{i=1}^{n}\widetilde{x}_i\widetilde{x}_i' a=\mathrm{o}(1).$$

That is

$$R_{n3}=\mathrm{o}(1). \tag{4.8}$$

By using Swartz inequality, Lemma 4.1 and A(2), for any $a \in R^p$, $a \neq 0$, we have

$$\begin{aligned}\mathsf{E}|a'R_{n4}a| &\leq a'\frac{2}{n}\sum_{i=1}^{n}\widetilde{x}_i\widetilde{x}_i'a(\mathsf{E}e_i^2)^{1/2}\Big(\mathsf{E}\Big(\sum_{j=1}^{n}W_{nj}(t_i)e_j\Big)^2\Big)^{1/2}\\ &\leq a'\frac{2}{n}\sum_{i=1}^{n}\widetilde{x}_i\widetilde{x}_i'a\sigma C\Big(\sum_{i=1}^{n}W_{nj}^2(t_i)\mathsf{E}e_j^2\Big)^{1/2}=\mathrm{o}(1).\end{aligned}$$

That is

$$R_{n4}=\mathrm{o}_p(1). \tag{4.9}$$

Similarly, from the conditions A(4) and A(10), for any $a \in R^p$, $a \neq 0$, we can show that

$$R_{n5}=\mathrm{o}_p(1). \tag{4.10}$$

Now observe that, for any p-dimensional constant vector a,

$$|a'R_{n6}a| \leq |a'R_{n2}a| + |a'R_{n3}a|.$$

This together with (4.7) and (4.8) yields

$$R_{n6}=\mathrm{o}_p(1). \tag{4.11}$$

(4.4) then follows from Equations (4.6)-(4.11).

Combining the results (4.1)-(4.4), similar to the proof of Theorem 1 in Owen (1990), we have

$$l(\beta)=\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\omega_i\Big)'\Big(\frac{1}{n}\sum_{i=1}^{n}\omega_i\omega_i'\Big)^{-1}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\omega_i\Big)+\mathrm{o}_p(1).$$

In addition to (4.2) and (4.4), the proof of Theorem 2.1 is completed. □

Proof of Theorem 2.2 At first, we will show the following results:

$$\omega_n^*=\max_{1\leq i\leq g}\|\omega_i^{(1)}\|=\mathrm{o}_p(\sqrt{g/h}), \tag{4.12}$$

$$\sqrt{\frac{h}{g}}\sum_{i=1}^{g}\omega_i^{(1)} \to_{\mathcal{D}} N(0,A), \tag{4.13}$$

$$a^T\Big(\frac{1}{g}\sum_{i=1}^{g}\omega_i^{(1)}\Big)=\mathrm{O}_p((gh)^{-1/2}), \qquad \forall\, a \in R^p, \tag{4.14}$$

$$S^{(1)}=\frac{h}{g}\sum_{i=1}^{g}\omega_i^{(1)}\omega_i^{(1)'}=A+\mathrm{o}_p(1). \tag{4.15}$$

Noticing that

$$\sqrt{\frac{h}{g}}\sum_{i=1}^{g}\omega_i^{(1)}=\frac{1}{\sqrt{gh}}\sum_{i=1}^{gh}\omega_i.$$

So with the same reason as in proof of Theorem 2.1, (4.13) and (4.14) hold. So we only need to prove (4.12) and (4.15) in the following.

It is clear, for any $\varepsilon > 0$, that

$$\begin{aligned}
\mathsf{P}(\sqrt{h/g}\omega_n^* > \varepsilon) &\le \sum_{i=1}^{g} \mathsf{P}\Big(\sqrt{h/g}\Big\|\frac{1}{h}\sum_{j=1}^{h}\omega_{(i-1)h+j}\Big\| > \varepsilon\Big) \\
&\le \frac{1}{(gh)^2\varepsilon^4}\sum_{i=1}^{g}\mathsf{E}\Big(\Big\|\sum_{j=1}^{h}\omega_{(i-1)h+j}\Big\|^4\Big) \\
&\le \frac{C}{(gh)^2\varepsilon^4}\sum_{i=1}^{g}\sum_{k=1}^{p}\mathsf{E}\Big(\sum_{j=1}^{h}\omega_{(i-1)h+j,k}\Big)^4 \\
&\le \frac{C}{(gh)^2\varepsilon^4}\sum_{k=1}^{p}\sum_{i=1}^{g}\Big\{\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,k}\widetilde{g}(t_{(i-1)h+j})\Big)^4 \\
&\quad + \mathsf{E}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,k}e_{(i-1)h+j}\Big)^4 \\
&\quad + \mathsf{E}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,k}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)\Big)^4\Big\}
\end{aligned}$$

By applying B(11) and $\max\limits_{1\le i\le n}|\widetilde{g}(t_i)| = \mathrm{o}(n^{-1/2})$, we have

$$\frac{1}{(gh)^2}\sum_{i=1}^{g}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,k}\widetilde{g}(t_{(i-1)h+j})\Big)^4 \le \Big\{\max_{1\le i\le n}|\widetilde{g}(t_i)|\Big\}^4 h^2\frac{1}{g^2h}\sum_{i=1}^{gh}\widetilde{x}_{i,k}^4 = \mathrm{o}(1/g^2).$$

By applying Lemma 4.1 and condition B(11), we obtain

$$\frac{1}{(gh)^2}\sum_{i=1}^{g}\mathsf{E}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,k}e_{(i-1)h+j}\Big)^4 \le \frac{1}{g^2h}\sum_{i=1}^{gh}\widetilde{x}_{i,k}^4\mu_4 = \mathrm{o}(1),$$

Similarly, from the conditions A(2), B(11) and Lemma 4.1 we can show that

$$\frac{1}{(gh)^2}\sum_{i=1}^{g}\mathsf{E}\Big\{\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,k}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)\Big\}^4 = \mathrm{o}(1).$$

Thus we have (4.12).

It can be seen that

$$\begin{aligned}
S^{(1)} &= \frac{1}{gh}\sum_{i=1}^{g}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j}\widetilde{g}(t_{(i-1)h+j})\Big)\Big(\sum_{j=1}^{h}\widetilde{x}'_{(i-1)h+j}\widetilde{g}(t_{(i-1)h+j})\Big) \\
&\quad + \frac{1}{gh}\sum_{i=1}^{g}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j}e_{(i-1)h+j}\Big)\Big(\sum_{j=1}^{h}\widetilde{x}'_{(i-1)h+j}e_{(i-1)h+j}\Big) \\
&\quad + \frac{1}{gh}\sum_{i=1}^{g}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)\Big) \\
&\quad \cdot\Big(\sum_{j=1}^{h}\widetilde{x}'_{(i-1)h+j}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)\Big) \\
&\quad + \frac{1}{gh}\sum_{i=1}^{g}\Big[\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j}\widetilde{g}(t_{(i-1)h+j})\Big)\Big(\sum_{j=1}^{h}\widetilde{x}'_{(i-1)h+j}e_{(i-1)h+j}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j}e_{(i-1)h+j}\Big)\Big(\sum_{j=1}^{h}\widetilde{x}'_{(i-1)h+j}\widetilde{g}(t_{(i-1)h+j})\Big)\Big] \\
&-\frac{1}{gh}\sum_{i=1}^{g}\Big[\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j}\widetilde{g}(t_{(i-1)h+j})\Big)\Big(\sum_{j=1}^{h}\widetilde{x}'_{(i-1)h+j}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)\Big) \\
&+\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)\Big)\Big(\sum_{j=1}^{h}\widetilde{x}'_{(i-1)h+j}\widetilde{g}(t_{(i-1)h+j})\Big)\Big] \\
&-\frac{1}{gh}\sum_{i=1}^{g}\Big[\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j}e_{(i-1)h+j}\Big)\Big(\sum_{j=1}^{h}\widetilde{x}'_{(i-1)h+j}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)\Big) \\
&+\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)\Big)\Big(\sum_{j=1}^{h}\widetilde{x}'_{(i-1)h+j}e_{(i-1)h+j}\Big)\Big] \\
={}& L_1+L_2+L_3+L_4-L_5-L_6.
\end{aligned}
$$

By B(11), we have

$$
\begin{aligned}
&\Big\{\frac{1}{gh}\sum_{i=1}^{g}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j}\widetilde{g}(t_{(i-1)h+j})\Big)\Big(\sum_{j=1}^{h}\widetilde{x}'_{(i-1)h+j}\widetilde{g}(t_{(i-1)h+j})\Big)\Big\}_{kl}^2 \\
\le{}& \frac{C}{gh^2}\Big\{\max_{1\le i\le n}|\widetilde{g}(t_i)|\Big\}^4\sum_{i=1}^{g}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,k}\Big)^2\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,l}\Big)^2 \\
\le{}& \frac{C}{gh^2}\Big\{\max_{1\le i\le n}|\widetilde{g}(t_i)|\Big\}^4\sum_{i=1}^{g}\sqrt{h^6\Big(\sum_{j=1}^{h}\widetilde{x}^4_{(i-1)h+j,k}\Big)\Big(\sum_{j=1}^{h}\widetilde{x}^4_{(i-1)h+j,l}\Big)} \\
={}& \frac{Ch}{g}\Big\{\max_{1\le i\le n}|\widetilde{g}(t_i)|\Big\}^4\sum_{i=1}^{g}\Big(\sum_{j=1}^{h}\widetilde{x}^4_{(i-1)h+j,k}+\sum_{j=1}^{h}\widetilde{x}^4_{(i-1)h+j,l}\Big) \\
={}& Cgh^2\Big\{\max_{1\le i\le n}|\widetilde{g}(t_i)|\Big\}^4\Big(\frac{1}{g^2h}\sum_{i=1}^{gh}\widetilde{x}^4_{i,k}+\frac{1}{g^2h}\sum_{i=1}^{gh}\widetilde{x}^4_{i,l}\Big)=\mathrm{o}(1/g),
\end{aligned}
$$

That is $L_1=\mathrm{o}(1)$.

By B(12), we have $\mathsf{E}L_2=A+\mathrm{o}(1)$. By applying Lemma 4.1 and condition B(11), we obtain

$$
\begin{aligned}
&\mathsf{E}\Big[\frac{1}{gh}\sum_{i=1}^{g}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j}e_{(i-1)h+j}\Big)\Big(\sum_{j=1}^{h}\widetilde{x}'_{(i-1)h+j}e_{(i-1)h+j}\Big)\Big]_{kl}^2 \\
\le{}& \frac{C}{g^2h^2}\sum_{i=1}^{g}\mathsf{E}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,k}e_{(i-1)h+j}\Big)^2\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,l}e_{(i-1)h+j}\Big)^2 \\
\le{}& \frac{C}{g^2h^2}\sum_{i=1}^{g}\sqrt{\mathsf{E}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,k}e_{(i-1)h+j}\Big)^4\mathsf{E}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,l}e_{(i-1)h+j}\Big)^4} \\
\le{}& \frac{C}{g^2h^2}\sum_{i=1}^{g}\sqrt{h^2\Big(\sum_{j=1}^{h}\widetilde{x}^4_{(i-1)h+j,k}\mu_4\Big)\Big(\sum_{j=1}^{h}\widetilde{x}^4_{(i-1)h+j,l}\mu_4\Big)} \\
\le{}& \frac{C}{g^2h}\sum_{i=1}^{g}\Big(\sum_{j=1}^{h}\widetilde{x}^4_{(i-1)h+j,k}\mu_4+\sum_{j=1}^{h}\widetilde{x}^4_{(i-1)h+j,l}\mu_4\Big)=\frac{C}{g^2h}\Big(\sum_{i=1}^{gh}\widetilde{x}^4_{i,k}+\sum_{i=1}^{gh}\widetilde{x}^4_{i,l}\Big)\mu_4 \\
={}& \mathrm{o}(1).
\end{aligned}
$$

That is $L_2 = A + \mathrm{o}_p(1)$.

By applying Lemma 4.1 and condition B(11), we know

$$\begin{aligned}
&\mathsf{E}\Big|\Big[\frac{1}{gh}\sum_{i=1}^{g}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)\Big)\\
&\cdot\Big(\sum_{j=1}^{h}\widetilde{x}^{T}_{(i-1)h+j}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)\Big)\Big]_{km}\Big|\\
\le\ &\frac{1}{gh}\sum_{i=1}^{g}\mathsf{E}\Big|\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,k}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)\Big|\\
&\cdot\Big|\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,m}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)\Big|\\
\le\ &\frac{1}{gh}\sum_{i=1}^{g}\Big\{h\sum_{j=1}^{h}\widetilde{x}^2_{(i-1)h+j,k}\mathsf{E}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)^2\\
&+h\sum_{j=1}^{h}\widetilde{x}^2_{(i-1)h+j,m}\mathsf{E}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)^2\Big\}\\
\le\ &\frac{1}{g}\sum_{i=1}^{g}\Big\{\sum_{j=1}^{h}\widetilde{x}^2_{(i-1)h+j,k}\sum_{l=1}^{n}W^2_{nl}(t_{(i-1)h+j})\mathsf{E}e_l^2\\
&+\sum_{j=1}^{h}\widetilde{x}^2_{(i-1)h+j,m}\sum_{l=1}^{n}W^2_{nl}(t_{(i-1)h+j})\mathsf{E}e_l^2\Big\}\\
\le\ &\frac{n\sigma^2}{g}\Big\{\max_{1\le i,j\le n}|W_{nj}(t_i)|\Big\}^2\Big\{\sum_{i=1}^{g}\Big(\sum_{j=1}^{h}\widetilde{x}^2_{(i-1)h+j,k}+\sum_{j=1}^{h}\widetilde{x}^2_{(i-1)h+j,m}\Big)\Big\}\\
=\ &\mathrm{o}(1).
\end{aligned}$$

That is $L_3 = \mathrm{o}_p(1)$.

Let $L_4 = L_{41} + L_{42}$. Note that $\mathsf{E}L_{41} = 0$, by applying Lemma 4.1 and condition B(11), we have

$$\begin{aligned}
&\mathsf{E}\Big\{\frac{1}{gh}\sum_{i=1}^{g}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j}\widetilde{g}(t_{(i-1)h+j})\Big)\Big(\sum_{j=1}^{h}\widetilde{x}'_{(i-1)h+j}e_{(i-1)h+j}\Big)\Big\}^2_{k,l}\\
\le\ &\frac{C}{g^2h^2}\sum_{i=1}^{g}\sqrt{\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,k}\widetilde{g}(t_{(i-1)h+j})\Big)^4\mathsf{E}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,l}e_{(i-1)h+j}\Big)^4}\\
\le\ &\frac{C}{g^2h^2}\sum_{i=1}^{g}\sqrt{h^4\Big\{\max_{1\le i\le n}|\widetilde{g}(t_i)|\Big\}^4\sum_{j=1}^{h}\widetilde{x}^4_{(i-1)h+j,k}\sum_{j=1}^{h}\widetilde{x}^4_{(i-1)h+j,l}\mu_4}\\
\le\ &\frac{C}{g^2}\Big\{\max_{1\le i\le n}|\widetilde{g}(t_i)|\Big\}^2\Big(\sum_{i=1}^{gh}\widetilde{x}^4_{i,k}+\sum_{i=1}^{gh}\widetilde{x}^4_{i,l}\Big)=\mathrm{o}(1/g).
\end{aligned}$$

Therefore, $L_{41} = \mathrm{o}_p(1)$, and we also can have $L_{42} = \mathrm{o}_p(1)$. Similarly, by applying Lemma 4.1 and condition B(11), we can show that $L_5 = \mathrm{o}_p(1)$.

Let $L_6 = L_{61} + L_{62}$. By applying Lemma 4.1 and condition B(11), we have

$$\begin{aligned}
&\mathsf{E}\Big|\Big\{\frac{1}{gh}\sum_{i=1}^{g}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j}e_{(i-1)h+j}\Big)\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)\Big)\Big\}_{km}\Big| \\
\le\ & \frac{1}{gh}\sum_{i=1}^{g}\mathsf{E}\Big|\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,k}e_{(i-1)h+j}\Big|\Big|\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,m}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)\Big| \\
\le\ & \frac{1}{gh}\sum_{i=1}^{g}\sqrt{\mathsf{E}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,k}e_{(i-1)h+j}\Big)^2\mathsf{E}\Big(\sum_{j=1}^{h}\widetilde{x}_{(i-1)h+j,m}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)\Big)^2} \\
\le\ & \frac{C}{gh}\sum_{i=1}^{g}\sqrt{h\Big(\sum_{j=1}^{h}\widetilde{x}^2_{(i-1)h+j,k}\mathsf{E}e^2_{(i-1)h+j}\Big)\Big(\sum_{j=1}^{h}\widetilde{x}^2_{(i-1)h+j,m}\mathsf{E}\Big(\sum_{l=1}^{n}W_{nl}(t_{(i-1)h+j})e_l\Big)^2\Big)} \\
=\ & \frac{C}{g\sqrt{h}}\sum_{i=1}^{g}\sqrt{\sigma^2\sum_{j=1}^{h}\widetilde{x}^2_{(i-1)h+j,k}\Big(\sum_{j=1}^{h}\widetilde{x}^2_{(i-1)h+j,m}\sum_{l=1}^{n}W^2_{nl}(t_{(i-1)h+j})\mathsf{E}e^2_l\Big)} \\
\le\ & \frac{C\sigma^2}{g\sqrt{h}}\sum_{i=1}^{g}\sqrt{\sum_{j=1}^{h}\widetilde{x}^2_{(i-1)h+j,k}\sum_{j=1}^{h}\widetilde{x}^2_{(i-1)h+j,m}n\Big\{\max_{1\le i,j\le n}|W_{nj}(t_i)|\Big\}^2} \\
\le\ & \frac{C\sigma^2\sqrt{n}}{g\sqrt{h}}\Big\{\max_{1\le i,j\le n}|W_{nj}(t_i)|\Big\}\sum_{i=1}^{g}\Big(\sum_{j=1}^{h}\widetilde{x}^2_{(i-1)h+j,k}+\sum_{j=1}^{h}\widetilde{x}^2_{(i-1)h+j,m}\Big) \\
=\ & C\sigma^2\sqrt{nh}\Big\{\max_{1\le i,j\le n}|W_{nj}(t_i)|\Big\}\frac{1}{gh}\Big(\sum_{i=1}^{gh}\widetilde{x}^2_{i,k}+\sum_{i=1}^{gh}\widetilde{x}^2_{i,m}\Big) = \mathrm{o}(1).
\end{aligned}$$

Therefore, $L_{61} = \mathrm{o}_p(1)$. Similarly, we know $L_{62} = \mathrm{o}_p(1)$.

(4.15) then follows from above.

We now start the proof of Theorem 2.2. Write $t(\beta) = \rho\theta$, where $\rho \ge 0$ and $\|\theta\| = 1$. From (2.4), similar to the proof of Theorem 1 in Qwen (1990), we have

$$0 \ge \frac{\rho\theta' S^{(1)}\theta}{1+\rho\omega_n^*} - \frac{h}{g}\Big|\sum_{j=1}^{p}e_j'\sum_{i=1}^{g}\omega_i^{(1)}\Big|, \tag{4.16}$$

where e_j is the unit vector in the jth coordinate direction. From (4.14), the second term in (4.16) is $\mathrm{O}_p(\sqrt{h/g})$. Easily, we can see that $\theta' S^{(1)}\theta \ge \mathrm{mineig}(A) + \mathrm{o}_p(1)$. It follows that

$$\frac{\rho}{1+\rho\omega_n^*} = \mathrm{O}_p(\sqrt{h/g}),$$

by (4.12), we have $\rho = \mathrm{O}_p(\sqrt{h/g})$, that is

$$\|t(\beta)\| = \mathrm{O}_p(\sqrt{h/g}). \tag{4.17}$$

Let $\gamma_i = t'(\beta)\omega_i^{(1)}$, then

$$\max_{1\le i\le g}\|\gamma_i\| = \|t(\beta)\|\omega_n^* = \mathrm{O}_p(\sqrt{h/g})\mathrm{o}_p(\sqrt{g/h}) = \mathrm{o}_p(1). \tag{4.18}$$

From (2.4),

$$0=\frac{h}{g}\sum_{i=1}^{g}\omega_i^{(1)}-S^{(1)}t(\beta)+\frac{h}{g}\sum_{i=1}^{g}\frac{\omega_i^{(1)}\gamma_i^2}{1+\gamma_i}. \tag{4.19}$$

Note that the final term in (4.19)

$$\begin{aligned}\Big\|\frac{h}{g}\sum_{i=1}^{g}\frac{\omega_i^{(1)}\gamma_i^2}{1+\gamma_i}\Big\| &= \Big\|\frac{h}{g}\sum_{i=1}^{g}\omega_i^{(1)}t'(\beta)\omega_i^{(1)}\omega_i^{(1)\prime}t(\beta)\Big\||1+\gamma_i|^{-1}\\ &\le \|t(\beta)\|^2\omega_n^*\|S^{(1)}\||1+\gamma_i|^{-1}=\mathrm{o}_p(\sqrt{h/g}).\end{aligned}$$

Therefore, we may write

$$t(\beta)=S^{(1)}\Big(\frac{h}{g}\sum_{i=1}^{g}\omega_i^{(1)}\Big)+\beta^{(1)},\qquad \text{where }\ \|\beta^{(1)}\|=\mathrm{o}_p(\sqrt{h/g}). \tag{4.20}$$

By (4.18) we may expand $\log(1+\gamma_i)=\gamma_i-\gamma_i^2/2+\eta_i$, where, for some finite $B>0$,

$$\mathsf{P}(|\eta_i|\le B|\gamma_i|^3,1\le i\le g)\to 1\qquad \text{as }\ n\to\infty.$$

Therefore,

$$l^{(1)}(\beta)=\frac{g}{h}\Big(\frac{h}{g}\sum_{i=1}^{g}\omega_i^{(1)}\Big)'S^{(1)-1}\Big(\frac{h}{g}\sum_{i=1}^{g}\omega_i^{(1)}\Big)-\frac{g}{h}\beta^{(1)\prime}S^{(1)}\beta^{(1)}+2\sum_{i=1}^{g}\eta_i. \tag{4.21}$$

Since $(g/h)\beta^{(1)\prime}S^{(1)}\beta^{(1)}=\mathrm{o}_p(1)$, and

$$\begin{aligned}\Big|\sum_{i=1}^{g}\eta_i\Big| &\le B\sum_{i=1}^{g}|t'(\beta)\omega_i^{(1)}|^3\le B\|t'(\beta)\|\omega_n^*\sum_{i=1}^{g}(t'(\beta)\omega_i^{(1)})^2\\ &= \frac{g}{h}B\|t(\beta)\|\omega_n^*t'(\beta)S^{(1)}t(\beta)\\ &\le \frac{g}{h}B\|t(\beta)\|^3\omega_n^*\mathrm{maxeig}(A)+\mathrm{o}_p(1)=\mathrm{o}_p(1).\end{aligned}$$

Therefore, from (4.21), (4.13) and (4.15) we obtain Theorem 2.2. □

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m-相依误差下部分线性模型的经验似然统计推断

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本文中, 我们针对误差为m-相依序列的固定设计的部分线性模型, 运用经验似然方法和分组经验似然方法, 构造了回归参数的对数经验似然比检验统计量, 并且证明了分组经验似然比检验统计量在参数取真值时是渐近地服从卡方分布的. 模拟计算表明分组经验似然方法的有效性.

关 键 词: 部分线性模型, m-相依误差, 分组经验似然.
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