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# A Easy and Feasible Way to Construct the Joint-Life Status Life Table: Method and Theory *

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**Abstract**

The future lifetimes of involved insured persons in the multiple-life model are always assumed to be independent in almost all actuarial textbooks. In this paper we consider the two-life model and assume that the future lifetimes are positively dependent. We use PQD (positively quadrant dependent) to describe such dependence, and give a easy method to construct the joint-life status life table.

**Keywords:** Future lifetime, comonotonicity, PQD.

**AMS Subject Classification:** 60K10.

## §1. Introduction

Consider two lives: $(x)$ with initial age $x$ and $(y)$ with initial age $y$, we denote their future lifetimes by $T(x)$ and $T(y)$ respectively. In the two-life model, any status $(u)$ can be defined by defining its future lifetime $T(u)$ based on $T(x)$ and $T(y)$.

We denote by ${}_tp_{u+s}$ the conditional probability that the status $(u)$ is still intact at time $t+s$, give that the status existed at time 0 and was intact at time $s$, i.e. ${}_tp_{u+s} = \mathsf{P}\{T(u)>t+s|T(u)>s\}$. Most other symbols in individual life model can be similarly defined in two-life model. For example, ${}_tq_{u+s}=1-{}_t\,p_{u+s}$ is the mortality probability of status $(u)$; $f_{T(u)}(t)=(\mathrm{d}/\mathrm{d}t)_tq_u$ is the pdf of status $u$; $\mu_u(t)=f_{T(u)}(t)/{}_tp_u=-(\mathrm{d}/\mathrm{d}t)\ln{}_tp_u$ is mortality force function of status $(u)$.

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The most commonly used statuses are the joint-life status, denoted by $(u)$ or $(x:y)$, and the last-survivor status, denoted by $(v)$ or $(\overline{x:y})$. They are defined as follows:

$$T(u) = \min\{T(x), T(y)\}, \tag{1.1}$$

$$T(v) = \max\{T(x), T(y)\}. \tag{1.2}$$

It is easy to see that the joint-life status $(u)$ is still intact if and only if $(x)$ and $(y)$ are both alive. While the last-survivor status is still alive as long as one of $(x)$ and $(y)$ is still alive. In spite of the dependence structure between $T(x)$ and $T(y)$, there are several equalities, such as

$$T(u) + T(v) = T(x) + T(y), \tag{1.3}$$

$${}_tp_u + {}_tp_v = {}_tp_x + {}_tp_y, \tag{1.4}$$

$$f_{T(u)}(t) + f_{T(v)}(t) = f_{T(x)}(t) + f_{T(v)}(y). \tag{1.5}$$

But be careful that such equality on the force of mortality functions does not hold.

In most actuarial textbooks, such as Gerber (1997), $T(x)$ and $T(y)$ are assumed to be independent. Under this assumption, ${}_tp_u$ and $\mu_u(t)$ can be expressed explicitly:

$${}_tp_u = {}_tp_x \cdot {}_tp_y, \tag{1.6}$$

$$\mu_u(t) = \mu_x(t) + \mu_y(t). \tag{1.7}$$

So if the life tables of $(x)$ and $(y)$ are given, the life table of status $(u)$ can be easily derived by (1.6), and so does status $(v)$ by (1.4).

But actually, the future lifetimes of the two insured persons who buy a two-life insurance or a two-life annuity are deemed to be dependent more or less. For example, most insured persons are a couple, and it is obvious that husband and wife's future lifetimes are usually positive dependent. So it is meaningful to construct life table of statuses without the independence assumption.

This paper use correlation order to compare random variable pairs dependence. In Section 2 we introduce some concepts of correlation order, comonotonicity and PQD. In Section 3 we derive the distribution of the joint-life status future lifetime when $T(x)$ and $T(y)$ are comonotonic. In Section 4 we derive the stochastic bound of the joint-life status future lifetime when $T(x)$ and $T(y)$ are PQD. In Section 5 we introduce a easy method to construct the life table of the joint-life status $(u)$.

## §2. Comonotonicity, PQD and Stochastic Order

Since the marginal distributions of $T(x)$ and $T(y)$ are definite, the dependence structure between them determines the distribution of the status $(u)$. We first introduce a special dependence structure called comonotonicity, and we will prove that it is the strongest positive dependence structure later in a certain sense.

**Definition 2.1** Two random variables $X$ and $Y$ are said to be comonotonic if $(X(\omega_1)-X(\omega_2))(Y(\omega_1)-Y(\omega_2))\geq 0$ holds almost sure.

It is known that any random variable is a function of event $\omega$, so random variable $X$ is also written as $X(\omega)$. The comonotonicity between $X$ and $Y$ means that if $\omega$ change from $\omega_1$ to $\omega_2$, the value of $X(\omega)$ and $Y(\omega)$ change in the same direction. This is the intuitive meaning of comonotonicity.

There are some equivalent statements on comonotonicity, here are some.

**Theorem 2.1** (Dhaene et al. (2002)) $X$ and $Y$ are comonotonic if and only if one of the following holds:

(1) $(X(\omega_1)-X(\omega_2))(Y(\omega_1)-Y(\omega_2))\geq 0$ holds almost sure;

(2) For all $x, y$, we have

$$F_{X,Y}(x,y)=\min\{F_X(x),F_Y(y)\}, \tag{2.1}$$

where $F_{X,Y}(x,y)$ is the joint cdf of $(X,Y)$, $F_X(x)$ and $F_Y(y)$ are the marginal cdf of $X$ and $Y$ respectively;

(3) For $U\sim \text{Uniform}(0,1)$, we have

$$(X,Y)\stackrel{d}{=}(F_X^{-1}(U),F_Y^{-1}(U)), \tag{2.2}$$

where '$\stackrel{d}{=}$' means having the same distribution, $F_X^{-1}(p)\triangleq\inf\{x|F_X(x)\geq p\}$ is called the inverse distribution function of $X$;

(4) There exists a random variable $Z$ and non-decreasing function $f, g$, such that

$$(X,Y)\stackrel{d}{=}(f(Z),g(Z)). \tag{2.3}$$

We denote by $R(F_X,F_Y)$ the set of random variable pairs of which the marginal cdf's are same as the cdf's of $X$ and $Y$, i.e.

$$R(F_X,F_Y)=\{(U,V)|F_U=F_X,F_V=F_Y\}.$$

Now we introduce a partial order called correlation order to compare the dependence between the components of each element in $R(F_X,F_Y)$.

**Definition 2.2** (Dhaene et al. (1996)) For any elements $(X_1, Y_1)$ and $(X_2, Y_2)$ in $R(F_X, F_Y)$, $(X_1, Y_1)$ is said to be less correlated than $(X_2, Y_2)$, written as $(X_1, Y_1) \le_{\text{cor}} (X_2, Y_2)$, if

$$\mathsf{Cov}\,(f(X_1), g(Y_1)) \le \mathsf{Cov}\,(f(X_2), g(Y_2)) \tag{2.4}$$

holds for any non-decreasing functions $f$ and $g$ for which the covariances exist.

There is a feasible criterion to judge the correlation order between two pairs of random variables which have the same marginal cdf's.

**Theorem 2.2** (Dhaene et al. (1996)) For any elements $(X_1, Y_1)$ and $(X_2, Y_2)$ in $R(F_X, F_Y)$, $(X_1, Y_1) \le_{\text{cor}} (X_2, Y_2)$ if and only if

$$F_{X_1,Y_1}(x, y) \le F_{X_2,Y_2}(x, y) \tag{2.5}$$

holds for any $x, y$.

**Theorem 2.3** $X$ and $Y$ are comonotonic, then

$$(X', Y') \le_{\text{cor}} (X, Y) \tag{2.6}$$

holds for all $(X', Y') \in R(F_X, F_Y)$.

**Proof** From Theorem 2.1, we find that

$$\begin{aligned} F_{X',Y'}(x, y) &= \mathsf{P}\{X' \le x, Y' \le y\} \\ &\le \min\{\mathsf{P}\{X' \le x\}, \mathsf{P}\{Y' \le y\}\} \\ &= \min\{F_X(x), F_Y(y)\} \\ &= F_{X,Y}(x, y). \end{aligned}$$

From Theorem 2.2 we can say that $(X', Y') \in R(F_X, F_Y)$. □

From Theorem 2.3, we find that if we use correlation order to gauge the dependence between the components of random variable pairs which have the same marginal cdf's, then comonotonicity is the most 'positively dependent' or most 'positively correlated'.

Denote by $(X^\perp, Y^\perp)$ the independent version of $(X, Y)$, i.e. $X^\perp$ and $Y^\perp$ are independent, $X^\perp \stackrel{d}{=} X$, $Y^\perp \stackrel{d}{=} Y$.

**Definition 2.3** (Dhaene et al. (1996)) $X$ and $Y$ are said to be positively quadrant dependent (PQD) if $(X^\perp, Y^\perp) \le_{\text{cor}} (X, Y)$.

We can see that if $X$ and $Y$ are PQD, then the dependence between $X$ and $Y$ is more positively dependent than independence. From Theorem 2.2, it is obvious that $X$ and $Y$

are PQD if and only if

$$F_{X,Y}(x,y) \geq F_X(x)F_Y(y), \qquad \text{for all } \ x, y. \tag{2.7}$$

Now we introduce a tool to compare random variables.

**Definition 2.4** (Shaked et al. (1994)) $X$ is said to be stochastically smaller than $Y$, written as $X \leq_{\rm st} Y$, if

$$\mathsf{E}f(X) \leq \mathsf{E}f(Y) \tag{2.8}$$

holds for any non-decreasing function $f$.

**Theorem 2.4** (Shaked et al. (1994)) $X \leq_{\rm st} Y$ if and only if

$$\overline{F}_X(x) \leq \overline{F}_Y(x), \qquad \text{for all } \ x, \tag{2.9}$$

where $\overline{F}_X(x) = 1 - F_X(x)$.

## §3. The Distribution of $T(u)$ under Comonotonicity

Now suppose $T(x)$ and $T(y)$ are comonotonic and we are to derive the expression of ${}_tp_u$ and $\mu_u(t)$.

**Theorem 3.1** If $T(x)$ and $T(y)$ are comonotonic then

$${}_tp_u = \min\{{}_tp_x, {}_tp_y\}. \tag{3.1}$$

**Proof** Let $U \sim \text{Uniform}(0,1)$, then from Theorem 2.1 we find that

$$\begin{aligned}
{}_tp_u &= \mathsf{P}\{T(u) > t\} \\
&= \mathsf{P}\{\min\{T(x), T(y)\} > t\} \\
&= \mathsf{P}\{F^{-1}_{T(x)}(U) > t, F^{-1}_{T(y)}(U) > t\} \\
&= \mathsf{P}\{U > F_{T(x)}(t), U > F_{T(y)}(t)\} \\
&= \mathsf{P}\{U > \max\{F_{T(x)}(t), F_{T(y)}(t)\}\} \\
&= 1 - \mathsf{P}\{U \leq \max\{{}_tq_x, {}_tq_y\}\} \\
&= 1 - \max\{{}_tq_x, {}_tq_y\} \\
&= \min\{{}_tp_x, {}_tp_y\}. \qquad \square
\end{aligned}$$

From (3.1) we can easily find that

$${}_tq_u = \max\{{}_tq_x, {}_tq_y\}, \tag{3.2}$$

It can be explained intuitively. We know that the joint-life status will be no longer intact as long as one of the two insured persons dies. And both of the insured persons are exposed to the risk of death at the same time, which force the mortality probability of $(u)$ to be no less than that of any individual insured. More positively dependent between $T(x)$ and $T(y)$, the difference between ${}_tq_u$ and $\max\{{}_tq_x, {}_tq_y\}$ will be smaller. But from Theorem 2.3 we know that comonotonicity is the strongest positive dependence in the sense of correlation order, so the difference is smallest. From Theorem 3.1, the smallest difference happen to be zero.

**Theorem 3.2** If $T(x)$ and $T(y)$ are comonotonic, then

$$\mu_u(t) = \mu_x(t) I_{\{{}_tq_x \geq {}_tq_y\}} + \mu_y(t) I_{\{{}_tq_x < {}_tq_y\}}, \tag{3.3}$$

where $I_A$ is the indicator function of $A$.

**Proof** From Theorem 3.1 we find that

$$\begin{aligned}
\mu_u(t) &= -\frac{\mathrm{d}}{\mathrm{d}t} \ln {}_tp_u \\
&= -\frac{\mathrm{d}}{\mathrm{d}t} \ln(\min\{{}_tp_x, {}_tp_y\}) \\
&= -\frac{\mathrm{d}}{\mathrm{d}t} \ln({}_tp_x I_{\{{}_tq_x \geq {}_tq_y\}} + {}_tp_y I_{\{{}_tq_x < {}_tq_y\}}) \\
&= I_{\{{}_tq_x \geq {}_tq_y\}} \cdot \Big( -\frac{\mathrm{d}}{\mathrm{d}t} \ln {}_tp_x \Big) + \Big( I_{\{{}_tq_x < {}_tq_y\}} \cdot \Big( -\frac{\mathrm{d}}{\mathrm{d}t} \ln {}_tp_y \Big) \Big) \\
&= \mu_x(t) I_{\{{}_tq_x \geq {}_tq_y\}} + \mu_y(t) I_{\{{}_tq_x < {}_tq_y\}}. \qquad \square
\end{aligned}$$

We know that the probability that $(u)$ 'dies' in the instance $(t, t + \mathrm{d}t]$ is

$$\mathsf{P}\{t < T(u) \leq t + \mathrm{d}t\} = {}_tp_u \mu_u(t) \mathrm{d}t. \tag{3.4}$$

From Theorem 3.1 and Theorem 3.2 we find that

$$\mathsf{P}\{t < T(u) \leq t + \mathrm{d}t\} = {}_tp_u (\mu_x(t) I_{\{{}_tq_x \geq {}_tq_y\}} + \mu_y(t) I_{\{{}_tq_x < {}_tq_y\}}) \mathrm{d}t. \tag{3.5}$$

So we can say that, if $T(x)$ and $T(y)$ are comonotonic, only one individual insured risk exposure to death adds such exposure of $(u)$ in any instance $(t, t + \mathrm{d}t]$, and the insured which adds the exposure of $(u)$ is the one that has the bigger mortality probability in time period $(0, t]$.

## §4. The Stochastic Bounds for $T(u)$

Suppose $T(x)$ and $T(y)$ are PQD, symbols $(T^{\perp}(x), T^{\perp}(y))$ and $(T^c(x), T^c(y))$ stand for the independent and comonotonic versions of $(T(x), T(y))$ respectively. All other symbols with '$\perp$' or '$c$' are defined similarly.

**Theorem 4.1** Let $(T_1(x), T_1(y))$ and $(T_2(x), T_2(y))$ are two elements of $R(F_{T(x)}, F_{T(y)})$, their corresponding joint-statuses future life are $T_1(u)$ and $T_2(u)$. If $(T_1(x), T_1(y)) \le_{\rm cor} (T_2(x), T_2(y))$ then $T_1(u) \le_{\rm st} T_2(u)$.

**Proof** From Theorem 2.2 we have

$$F_{T_1(x),T_1(y)}(a,b) \le F_{T_2(x),T_2(y)}(a,b), \qquad \text{for all } a,b.$$

It is equivalent to

$$\mathsf{P}\{T_1(x) > a, T_1(y) > b\} \le \mathsf{P}\{T_2(x) > a, T_2(y) > b\}, \qquad \text{for all } a,b.$$

Let $a = b = t$, we get

$$\mathsf{P}\{T_1(u) > t\} \le \mathsf{P}\{T_2(u) > t\}, \qquad \text{for all } t.$$

From Theorem 2.4 we can say that $T_1(u) \le_{st} T_2(u)$. $\square$

**Corollary 4.1** If $T(x)$ and $T(y)$ are PQD, then

$$T^{\perp}(u) \le_{\rm st} T(u) \le_{\rm st} T^c(u). \tag{4.1}$$

**Proof** From Theorem 2.1 and Theorem 2.2 and Definition 2.3, we find that

$$(T^{\perp}(x), T^{\perp}(y)) \le_{\rm cor} (T(x), T(y)) \le_{\rm cor} (T^c(x), T^c(y)).$$

Then it is obvious that $T^{\perp}(u) \le_{\rm st} T(u) \le_{\rm st} T^c(u)$ from Theorem 4.1. $\square$

## §5. Construct the Life Table of $(u)$

Recall that $\mu_u^{\perp}(t) = \mu_x(t) + \mu_y(t)$ when $T(x)$ and $T(y)$ are independent, and $\mu_u^c(t) = \mu_x(t) I_{\{{}_tq_x \ge {}_tq_y\}} + \mu_y(t) I_{\{{}_tq_x < {}_tq_y\}}$ when $T(x)$ and $T(y)$ are comonotonic. It is obvious that $\mu_u^c(t) \le \mu_u^{\perp}(t)$ for all $t$. Now consider the weight average function of $\mu_u^c(t)$ and $\mu_u^{\perp}(t)$

$$\mu_u^w(t) = w\mu_u^c(t) + (1-w)\mu_u^{\perp}(t), \qquad 0 \le w \le 1. \tag{5.1}$$

For any $w \in [0,1]$, $\mu_u^w(t)$ is still a force of mortality of status $(u)$. Since

$${}_tp_u = e^{-\int_0^t \mu_u(s)\mathrm{d}s}, \qquad t \ge 0, \tag{5.2}$$

${}_tp_u$ and $\mu_u(t)$ are determined by each other, and we can find that

$${}_tp_u^w = ({}_tp_u^c)^w ({}_tp_u^{\perp})^{1-w}. \tag{5.3}$$

It is obvious that ${}_tp_u^{\perp} \le {}_tp_u^w \le {}_tp_u^c$.

If $T(x)$ and $T(y)$ are PQD, then from Corollary 4.1 we see that ${}_tp_u^{\perp} \le {}_tp_u \le {}_tp_u^c$. So our idea is to replace ${}_tp_u$ with ${}_tp_u^w$ by adjusting the weight $w$ to a likely value.

In practice, what we have are the life tables of $(x)$ and $(y)$, in other words the distributions of $T(x)$ and $T(y)$ are discrete and we have troubles in dealing with force of mortality. Most textbooks, such as Gerber (1997) and Bowers et al. (1999), give three interpolation methods to solve this fractional age problem. Anyway, in this paper, what we are interested in is to construct the life table of $(u)$, and we just concentrate in the integer ages. Fortunately, all interpolation methods don't change survival probability at integer point, So we can use (5.3) to construct the life table of $(u)$ directly.

So in order to construct the life table of $(u)$, we should first estimate the dependence of $T(x)$ and $T(y)$ and determine the value of $w$, then calculate ${}_tp_u^w$ following the formulas (1.6), (3.1) and (5.3).

To end this paper, we give a simple example to illustrate the method. Suppose the mortality rates of $(x)$ and $(y)$ are

Mortality rates of $(x)$ and $(y)$

| $n$ | | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $q_n$ | $(x)$ | 0.10 | 0.05 | 0.08 | 0.10 | 0.15 | 0.20 | 0.30 | 0.40 | 0.70 | 1.00 |
| | $(y)$ | 0.12 | 0.04 | 0.09 | 0.10 | 0.12 | 0.21 | 0.25 | 0.50 | 0.75 | 1.00 |

We consider the joint-life status of $(u) = (3:2)$, we can construct the life table of $(u)$ with $w = 0.4$ as following:

Example: Life table of (3:2) with $w = 0.4$

| $n$ | $q_{x+n}$ | $q_{y+n}$ | ${}_np_x$ | ${}_np_y$ | ${}_np_u^{\perp}$ | ${}_np_u^c$ | ${}_np_u^w$ | $q_{u+n}^w$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.10 | 0.09 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.1495 |
| 1 | 0.15 | 0.10 | 0.9000 | 0.9100 | 0.8190 | 0.9000 | 0.8505 | 0.2021 |
| 2 | 0.20 | 0.12 | 0.7650 | 0.8190 | 0.6265 | 0.7650 | 0.6786 | 0.2591 |
| 3 | 0.30 | 0.21 | 0.6120 | 0.7207 | 0.4411 | 0.6120 | 0.5028 | 0.3923 |
| 4 | 0.40 | 0.25 | 0.4284 | 0.5694 | 0.2439 | 0.4284 | 0.3056 | 0.4951 |
| 5 | 0.70 | 0.50 | 0.2570 | 0.4270 | 0.1098 | 0.2570 | 0.1543 | 0.8021 |
| 6 | 1.00 | 0.75 | 0.0771 | 0.2135 | 0.0165 | 0.0771 | 0.0305 | 1.0000 |

Where the values of each fields are calculated as following:

1. $q_{x+n}$ and $q_{y+n}$ are from the life tables of $(x)$ and $(y)$, the maximum $n$ is the one that makes either $q_{x+n}$ or $q_{y+n}$ equals to 1;
2. ${}_0p_x = {}_0p_y = 1$, ${}_np_x = {}_{n-1}p_x \cdot (1 - q_{x+n-1})$, ${}_np_y = {}_{n-1}p_y \cdot (1 - q_{y+n-1})$;
3. ${}_np_u^{\perp} = {}_np_x \cdot {}_np_y$, ${}_np_u^c = \min\{{}_np_x, {}_np_y\}$;
4. ${}_np_u^w = ({}_np_u^c)^w \cdot ({}_np_u^{\perp})^{1-w}$, $q_{u+n}^w = ({}_np_u^w - {}_{n+1}p_u^w)/{}_np_u^w$.

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# 一种构造连生状态生命表的简单易行方法: 方法和理论


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在多生命模型中, 几乎所有精算学教科书都假设被保险人的剩余寿命之间相互独立. 本文中我们研究两生命模型. 我们认为剩余寿命是正相依的, 并用正象限相依描述相依性, 给出了一种简单方法构造联合生命表.



**关键词**: 剩余寿命, 同单调, 正象限相依.

**学科分类号**: O212.5.