应用概率统计　第二十九卷
第一期　2013年2月

Chinese Journal of Applied Probability
and Statistics Vol.29 No.1 Feb. 2013

A Note on the Central Limit Theorem for Strong Mixing Sequences *

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Abstract

In this paper, we derive a central limit theorem for $\sum\limits_{i=1}^{n} a_{ni}\xi_i$, where $\{\xi_i\}$ is a strong mixing sequences, and $\{a_{ni}\}$ is a triangular array of real numbers. To show the application of the central limit theorem, we establish a central limit theorem for a partial sum of a linear process.

Keywords: Central limit theorem, strong mixing, linear process.

AMS Subject Classification: 60F05, 60G10.

§1. Introduction

Suppose $\{\xi_i : i \in \mathbf{Z}\}$ is a real-valued random variable sequence on a probability space $(\Omega, \mathcal{B}, \mathsf{P})$. Let $\mathcal{F}_n^m$ denote the σ-field generated by $(\xi_i : m \le i \le n)$, and

$$\alpha(n) = \sup\{|P(AB) - P(A)P(B)| : A \in \mathcal{F}_{-\infty}^m, B \in \mathcal{F}_{m+n}^{\infty}\}.$$

The sequence $\{\xi_i\}$ is said to be α-mixing or strong mixing if $\alpha(n) \to 0$ as $n \to \infty$.

The α-mixing sequence was introduced by Rosenblatt (1956) and has been commonly employed in establishing limiting results for time series and random fields. For example, Doukhan et al. (1994), Billingsley (1995), Merlevàde and Peligrad (2000) studied some sufficient conditions for the central limit theorem (CLT) of strong mixing sequence.

It is well known that the CLT and functional central limit theorem (FCLT) for linear process have been extensively studied in the literature. Such as, Wang et al. (2002), Lee (1997), Kim and Baek (2001), Kim and Ko (2003), Ko et al. (2006), Ko and Kim (2008), Moon (2008), Haydn (2009).

*The work is supported by the National Natural Science Foundation of China (11061029).
Received September 20, 2010.

In this paper, let $\{\xi_i\}$ be a sequence of zero-mean strong mixing random variables, and $\{a_{ni}, 1 \le i \le n\}$ be a triangular array of real numbers. Many statistical procedures produce estimators of the type

$$S_n = \sum_{i=1}^{n} a_{ni}\xi_i. \tag{1.1}$$

Set $\{a_k\}$ be a sequence of real numbers. We define a linear process by

$$X_m = \sum_{j=-\infty}^{\infty} a_{m+j}\xi_j. \tag{1.2}$$

In time-series analysis, this process is of great importance. Many important time-series models, such as the causal ARMA process (Brockwell and Davis, 1987, p.89), have type (1.2).

Peligrad and Utev (1997) obtained the CLT for the model (1.1), and the conditions of Theorem 2.2 (c) in Peligrad and Utev (1997) are that:

(A1) For a certain $\delta > 0$, $\{|\xi_i|^{2+\delta}\}$ is an uniformly integrable family, $\inf_i \mathsf{Var}\,(\xi_i) > 0$, $\mathsf{Var}\left(\sum_{i=1}^{n} a_{ni}\xi_i\right) = 1$, and $\sum_{n=1}^{\infty} n^{2/\delta}\alpha(n) < \infty$.

(A2) $\sup_n \sum_{i=1}^{n} a_{ni}^2 < \infty$, and $\max_{1\le i\le n} |a_{ni}| \to 0$ as $n \to \infty$.

However, they are restrictive for some cases. So it's the main purpose of our paper to establish a CLT for the sum of (1.1) with α-mixing innovations under weaker conditions, and also get the CLT for the linear process (1.2).

Throughout this paper, it is supposed that all limits are taken as $n \to \infty$, unless specified otherwise. The paper is organized as follows. Section 2 contains our main results. Proofs of the main results are provided in Section 3.

§2. The Main Results

We will prove the following results.

Theorem 2.1 (A3) Let $\{\xi_i : i \ge 1\}$ be an α-mixing sequence of random variables with $\mathsf{E}\xi_i = 0$, $\inf_i \mathsf{Var}\,(\xi_i) > 0$, $\mathsf{Var}\left(\sum_{i=1}^{n} a_{ni}\xi_i\right) = 1$, and $\mathsf{E}|\xi_i|^{2+\delta} < \infty$. Suppose that $\theta > (2+\delta)/\delta$ and $\alpha(n) \le Cn^{-\theta}$ for some $C > 0$.

(A4) Let $\{a_{ni}, 1 \le i \le n\}$ be a triangular array of real numbers, such that

$$\sum_{i=1}^{n} |a_{ni}| < \infty, \qquad \text{and} \qquad \max_{1\le i\le n} |a_{ni}| = O(n^{-\beta}),$$

where $\beta > 0$. Then

$$\sum_{i=1}^{n} a_{ni}\xi_i \xrightarrow{\mathcal{D}} N(0,1).$$

As a corollary the above theorem we obtain the following:

Corollary 2.1 Let $\{\xi_i\}$ be a centered sequence of α-mixing random variables satisfying condition (A3) in Theorem 2.1, and $\{a_{ni}, 1 \le i \le n\}$ be a triangular array of real numbers such that

$$\sum_{i=1}^{n} \frac{|a_{ni}|}{\sigma_n} < \infty \quad \text{and} \quad \max_{1\le i\le n} \frac{|a_{ni}|}{\sigma_n} = O(n^{-\beta}), \qquad \text{where } \sigma_n^2 = \mathsf{Var}\Big(\sum_{i=1}^{n} a_{ni}\xi_i\Big), \ \ \beta > 0.$$

Then

$$\frac{1}{\sigma_n}\sum_{i=1}^{n} a_{ni}\xi_i \xrightarrow{\mathcal{D}} N(0,1).$$

Theorem 2.2 Let $\{a_j, j \in \mathbf{Z}\}$ be a sequence of real numbers such that $\sum\limits_{j} |a_j| < \infty$, and $\{\xi_j, j \in \mathbf{Z}\}$ be a centered sequence of strong mixing random variables satisfying condition (A3) in Theorem 2.1. Set $S_n^* = \sum\limits_{m=1}^{n} X_m$, where $X_m = \sum\limits_{j=-\infty}^{\infty} a_{m+j}\xi_j$. Moreover, assume

$$\inf_{n\ge 1} n^{-1}\sigma_n^{*2} > 0, \qquad \text{where } \sigma_n^{*2} = \mathsf{Var}\,(S_n^*).$$

Then

$$\frac{S_n^*}{\sigma_n^{*2}} \xrightarrow{\mathcal{D}} N(0,1).$$

Remark 1 (i) In Theorem 2.1, $\alpha(n) = O(n^{-\theta})$, $\theta > (2+\delta)/\delta$. By it, we can get $\sum\limits_{n=1}^{\infty} \alpha^{2/(2+\delta)}(n) < \infty$. In addition, in Theorem 2.2 (c) of Peligrad and Utev (1997), $\sum\limits_{n=1}^{\infty} n^{2/\delta}\alpha(n) < \infty$ implies $\alpha(n) = O(n^{-\theta})$, $\theta > (2+\delta)/\delta$. Then, the mixing rate in Theorem 2.1 is almost the same as the one in Theorem 2.2 (c) of Peligrad and Utev (1997), but, the operation of the former is more easy and convenient than the latter.

(ii) In Theorem 2.2 (c) of Peligrad and Utev (1997), $\sup\limits_{n} \sum\limits_{i=1}^{n} a_{ni}^2 < \infty$. It is obviously stronger than the corresponding form $\sum\limits_{i=1}^{n} |a_{ni}| < \infty$ in our Theorem 2.1. But $\max\limits_{1\le i\le n} |a_{ni}| = O(n^{-\beta})$ in Theorem 2.1 is a special case of $\max\limits_{1\le i\le n} |a_{ni}| \to 0$ in Theorem 2.2 (c) of Peligrad and Utev (1997).

Hence, in certain senses, our result improves on Theorem 2.2 (c) of Peligrad and Utev (1997).

Remark 2 The proof method of our Theorem 2.1 differs from that of Theorem 2.2 (c) in Peligrad and Utev (1997). In this paper, we use the small-block and large-block argument which is used to show limit theorems for mixing random variables. But, Theorem 2.2 (c) in Peligrad and Utev (1997) was proofed by truncating the variables.

§3. Proofs of Main Results

To prove our theorems, we first give the following lemmas.

Lemma 3.1 (Xing et al. (2009)) Let $1 < r \le 2$, $\delta > 0$ and $\{\xi_i : i \ge 1\}$ be an α-mixing sequence of random variables with $\mathsf{E}\xi_i = 0$ and $\mathsf{E}|\xi_i|^{r+\delta} < \infty$. Assume $\theta > (r-1)(r+\delta)/\delta$ and $\alpha(n) \le Cn^{-\theta}$ for some $C > 0$. Then, for any $\varepsilon > 0$, there exists a positive constant $K = K(\varepsilon, r, \delta, \theta, C)$ such that

$$\mathsf{E} \max_{1\le j\le n} |S_j|^r \le K\Big\{n^{\varepsilon}\sum_{i=1}^{n}\mathsf{E}|\xi_i|^r + \Big(\sum_{i=1}^{n}\|\xi_i\|_{r+\delta}^2\Big)^r\Big\}.$$

Lemma 3.2 (Volkonskii and Rozanov (1959)) Let $\{Z_1, \ldots, Z_k\}$ be α-mixing random variables measurable with respect to the σ-algebras $\mathcal{F}_{i_1}^{j_1}, \ldots, \mathcal{F}_{i_k}^{j_k}$ respectively, with $1 \le i_1 < j_1 < \cdots < j_L < n$, $i_{l+1} - j_l \ge w \le 1$ and $|Z_j| \le 1$ for $j = 1, \ldots, k$. Then

$$\Big|\mathsf{E}\Big(\prod_{j=1}^{k}\Big) - \prod_{j=1}^{k}\mathsf{E}(Z_j)\Big| \le 16(k-1)\alpha(w).$$

Proof of Theorem 2.1 We use the small-block and large-block argument. We can choose $p = p_n$, $q = q_n$, $k = k_n$ as follows:

Let $k \sim n^a$, $p \sim n^{1-a}$, $q \sim n^c$, where $\max\{1-\beta, (2+\delta)/(4+\delta)\} < a < \min\{1-\varepsilon c, \beta - \varepsilon c\}$ and $a, c, \varepsilon > 0$. Then, we can show that

$$kq^{\varepsilon}\max_{1\le i\le n}|a_{ni}| \to 0, \quad p^{\varepsilon}\max_{1\le i\le n}|a_{ni}| \to 0, \quad kp^{\varepsilon}\max_{1\le i\le n}|a_{ni}| < \infty, \quad (k-1)\alpha(q+1) \to 0. \tag{3.1}$$

Denote by

$$Y_i = \sum_{j=(i-1)(p+q)+1}^{(i-1)(p+q)+p} a_{nj}\xi_j, \qquad Y_i^* = \sum_{j=(i-1)(p+q)+p+1}^{i(p+q)} a_{nj}\xi_j.$$

For $k(q+q)+1 \le n \le (k+1)(p+q)$, S_n may be split as

$$S_n = \sum_{i=1}^{k} Y_i + \sum_{i=1}^{k} Y_i^* + \sum_{i=k(p+q)+1}^{n} a_{ni}\xi_i. \tag{3.2}$$

Hence, we need only to prove that

$$\mathsf{E}\Big(\sum_{i=1}^{k} Y_i^*\Big)^2 \to 0, \qquad \mathsf{E}\Big(\sum_{i=k(p+q)+1}^{n} a_{ni}\xi_i\Big)^2 \to 0, \tag{3.3}$$

$$\mathsf{E}\Big(\sum_{i=1}^{k} Y_i\Big)^2 \to 1, \tag{3.4}$$

$$\Big|\mathsf{E}\exp\Big(it\sum_{i=1}^{k} Y_i\Big) - \prod_{i=1}^{k}\mathsf{E}\exp(itY_i)\Big| \to 0, \tag{3.5}$$

$$\Lambda_n(\varepsilon) = \sum_{i=1}^{k}\mathsf{E}Y_i^2 I(|Y_i|>\gamma) \to 0, \qquad \forall\,\gamma>0. \tag{3.6}$$

We first establish (3.3). Using the C_r-inequality, Lemma 3.1, the condition of (A4), and the first two terms of (3.1), we obtain

$$\begin{aligned}
\mathsf{E}\Big(\sum_{i=1}^{k} Y_i^*\Big)^2 &\le k\sum_{i=1}^{k}\mathsf{E}\Big(\sum_{j=(i-1)(p+q)+p+1}^{i(p+q)} a_{ni}\xi_j\Big)^2\\
&\le Kk\sum_{i=1}^{k}\Big(q^\varepsilon\sum_{j=(i-1)(p+q)+p+1}^{i(p+q)}\mathsf{E}(a_{ni}\xi_j)^2 + \sum_{j=(i-1)(p+q)+p+1}^{i(p+q)}\|a_{ni}\xi_j\|_{2+\delta}^2\Big)\\
&\le Ck\sum_{i=1}^{k}\Big(q^\varepsilon\sum_{j=(i-1)(p+q)+p+1}^{i(p+q)} a_{nj}^2\Big)\\
&\le Ckq^\varepsilon\Big(\sum_{i=1}^{k}\sum_{j=(i-1)(p+q)+p+1}^{i(p+q)}|a_{nj}|\Big)\max_{1\le j\le n}|a_{nj}|\\
&\le Ckq^\varepsilon\max_{1\le j\le n}|a_{nj}| \to 0,
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&\mathsf{E}\Big(\sum_{i=k(p+q)+1}^{n} a_{ni}\xi_i\Big)^2\\
\le\;& K\Big((n-k(q+p)-1)^\varepsilon\sum_{i=k(p+q)+1}^{n}\mathsf{E}(a_{ni}\xi_i)^2 + \sum_{i=k(p+q)+1}^{n}\|a_{ni}\xi_i\|_{2+\delta}^2\Big)\\
\le\;& Cp^\varepsilon\max_{1\le i\le n}|a_{ni}| \to 0.
\end{aligned} \tag{3.8}$$

Therefore, (3.7) and (3.8) imply that the last two terms on the right-hand side of (3.2) are asymptotically negligible, and (3.3) holds.

Now, we consider (3.4). Following the method of (3.7) and by the third term of (3.1), we can conclude that

$$\mathsf{E}\Big(\sum_{i=1}^{k} Y_i\Big)^2 \le Ckp^\varepsilon\max_{1\le j\le n}|a_{nj}| < \infty. \tag{3.9}$$

Note that

$$\begin{aligned} 1 &= \mathsf{E}S_n = \mathsf{E}\Big(\sum_{i=1}^{k} Y_i + \sum_{i=1}^{k} Y_i^* + \sum_{i=k(p+q)+1}^{n} a_{ni}\xi_i\Big)^2 \\ &= \mathsf{E}\Big(\sum_{i=1}^{k} Y_i\Big)^2 + \mathsf{E}\Big(\sum_{i=1}^{k} Y_i^*\Big)^2 + \mathsf{E}\Big(\sum_{i=k(p+q)+1}^{n} a_{ni}\xi_i\Big)^2 + 2\mathsf{Cov}\Big(\sum_{i=1}^{k} Y_i, \sum_{i=1}^{k} Y_i^*\Big) \\ &\quad + 2\mathsf{Cov}\Big(\sum_{i=1}^{k} Y_i^*, \sum_{i=k(p+q)+1}^{n} a_{ni}\xi_i\Big) + 2\mathsf{Cov}\Big(\sum_{i=1}^{k} Y_i, \sum_{i=k(p+q)+1}^{n} a_{ni}\xi_i\Big). \end{aligned} \tag{3.10}$$

Therefore, by the Cauchy-Schwarz inequality, (3.4) follows from (3.7), (3.8) and (3.10).

As to (3.5), let $Z_i = \exp(itY_i)$, which is $\mathcal{F}_a^b$-measurable, where $a = (i-1)(p+q)+1$ and $b = (i-1)(p+q)+p$, then $|Z_i| = 1$. Therefore, applying Lemma 3.2 to $\{Z_i\}$, and the last term of relation (3.1), we obtain

$$\Big|\mathsf{E}\exp\Big(it\sum_{i=1}^{k} Y_i\Big) - \prod_{i=1}^{k}\mathsf{E}\exp(itY_i)\Big| \le 16(k-1)\alpha(q+1) \to 0.$$

Next, we prove (3.6).

$$\begin{aligned} \Lambda_n(\varepsilon) &\le \sum_{i=1}^{k}\mathsf{E}Y_i^2 = \sum_{i=1}^{k}\mathsf{E}\Big(\sum_{j=(i-1)(p+q)+1}^{(i-1)(p+q)+p} a_{ni}\xi_j\Big)^2 \\ &\le \sum_{i=1}^{k}\Big(p^{\varepsilon}\sum_{j=(i-1)(p+q)+1}^{(i-1)(p+q)+p}\mathsf{E}(a_{ni}\xi_j)^2 + \sum_{j=(i-1)(p+q)+1}^{(i-1)(p+q)+p}\|a_{ni}\xi_j\|_{2+\delta}^2\Big) \\ &\le C\sum_{i=1}^{k}\Big(p^{\varepsilon}\sum_{j=(i-1)(p+q)+1}^{(i-1)(p+q)+p} a_{nj}^2\Big) \\ &\le Cp^{\varepsilon}\Big(\sum_{i=1}^{k}\sum_{j=(i-1)(p+q)+1}^{(i-1)(p+q)+p}|a_{nj}|\Big)\max_{1\le j\le n}|a_{nj}| \\ &\le Cp^{\varepsilon}\max_{1\le j\le n}|a_{nj}| \to 0. \qquad \square \end{aligned} \tag{3.11}$$

Proof of Corollary 2.1 Let $a_{ni}^* = a_{ni}/\sigma_n$, then the conditions in Theorem 2.1 hold true for $\{a_{ni}^*\}$. Hence, the proof is complete by Theorem 2.1. $\square$

Proof of Theorem 2.2 We first note that

$$S_n^* = \sum_{i=1}^{n} X_i = \sum_{j=-\infty}^{\infty}\Big(\sum_{i=1}^{n} a_{i+j}\Big)\xi_j.$$

In order to apply Theorem 2.1, we choose W_n such that $\sum\limits_{|j|>W_n} a_j^2 < n^{-3}$, and take $l_n = W_n + n$. Then

$$\frac{S_n^*}{\sigma_n^{*2}} = \sum_{|j|\le W_n}\Big(\sum_{i=1}^{n} a_{i+j}\Big)\xi_j/\sigma_n^* + \sum_{|j|>W_n}\Big(\sum_{i=1}^{n} a_{i+j}\Big)\xi_j/\sigma_n^* =: T_n + U_n.$$

Similarly to the arguments as in (3.8) in Peligrad and Utev (1997) or (3.9) in Ko et al. (2006), we also have the following estimate

$$\begin{aligned}\mathsf{Var}(U_n) &\le \sum_{|j|>k_n}\Big(\sum_{k=1}^{n} a_{k+j}/\sigma_n^*\Big)^2 \le n\sigma_n^{*-2}\sum_{|j|>k_n}\sum_{k=1}^{n} a_{k+j}^2\\ &\le n^2\sigma_n^{*-2}\sum_{|j|>k_n-n} a_j^2 \le n^{-1}\sigma_n^{*-2}\to 0. \qquad (3.12)\end{aligned}$$

Thus, we obtain $U_n \to 0$ in probability.

Next, we have to prove that $T_n \xrightarrow{\mathcal{D}} N(0,1)$. Put $a_{ni}=\sum\limits_{j=1}^{n} a_{i+j}/\sigma_n^*$. By Theorem 2.1, it is suffices to show that

$$\sup_{-\infty<i<\infty}\sum_{j=1}^{n} a_{i+j}/\sigma_n^* \to 0. \qquad (3.13)$$

As the proof of Corollary 2.1 in Peligrad and Utev (1997, p.448-449), (3.13) holds. This establishes the result of Theorem 2.2. □

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一个强混合样本中心极限定理的注记

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本文在$\{\xi_i\}$为强混合样本, $\{a_{ni}\}$是实三角阵列下, 得到了一个新的关于线性和$\sum\limits_{i=1}^{n} a_{ni}\xi_i$的中心极限定理. 并利用该中心极限定理, 进一步建立了线性过程部分和的中心极限定理.

关 键 词: 中心极限定理, 强混合, 线性过程.

学 科 分 类 号: O211.5.