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# Exponential Behaviour of Stochastic 2D Navier-Stokes Equations Driven by Lévy Noise *

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**Abstract**

In this paper, some results on the pathwise exponential stability are established for the weak solutions of stochastic 2D Navier-Stokes equation driven by Lévy noise. Also, some results and comments concerning the stabilizability and stabilization of these equations are stated.

**Keywords:** Stochastic 2D Navier-Stokes equation, Lévy process, exponential stability, stabilization.

**AMS Subject Classification:** Primary: 34D08, 34D25; Secondary: 60H20.



## §1. Introduction

The long-time behaviour of flows is a very interesting and important problem in the theory of fluid dynamics, as the vast literature shows (see [12, 13, 15] and among others), and has been receiving very much attention in probability experts. In the last two decades, many authors have researched exponential behaviour of stochastic partial differential equations (see [1–4, 16], among others, and the references therein).

One of the most studied models is the Navier-Stokes model (and its variants) since it provides a suitable model which covers several important fluids (see Temam (1995, 1988) and the references therein). The Navier-Stokes equations are the fundamental model of the fluids. Despite their great physical importance, existence and uniqueness results for the equations in the three-dimensional case are still not known, and only the two-dimensional (2D in short) situation is amenable to a complete mathematical treatment. In the past years, many researchers studied this equation in the random situations. Most of the works are with Gaussian white noise, e.g. [4, 6, 10, 11] and references therein. As we know, there are a few articles for the non Gaussian white noise, see [7–9, 17, 18] for the Lévy space-time white noise and Poisson random measure.

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In this paper, we shall show some aspects of the effects produced in the long-time behaviour of the solution to a 2D Navier-Stokes equation under the presence of stochastic perturbations driven by Lévy noise as Caraballo, Langa and Taniguchi did in [4], that is, we shall prove the exponential behaviour and stabilizability of the weak solutions of stochastic 2D Navier-Stokes equation driven by Lévy noise with $\sigma$-finite intensity measure.

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t, t \geq 0\}$ satisfying the usual conditions of completeness and right continuity, and let $N(\mathrm{d}t, \mathrm{d}z)$ be a Poisson random measure with intensity measure $\mathrm{d}s\lambda(\mathrm{d}z)$ on $\mathbb{R}^+ \times U$, where $\lambda(\mathrm{d}z)$ is a $\sigma$-finite measure on a measurable space $(U, \mathcal{B}(U))$. We denote $\widetilde{N}(\mathrm{d}t, \mathrm{d}z)$ by $N(\mathrm{d}t, \mathrm{d}z) - \mathrm{d}t\lambda(\mathrm{d}z)$, the compensated Poisson measure. Suppose the operator $Q \in \mathcal{L}(K, K)$, where $K$ is a real and separable Hilbert space. The $K$-valued $W_t$ is a $Q$-Wiener process. Assume $W_t$ and $N(\mathrm{d}t, \mathrm{d}z)$ are independent.

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the torus. We consider a Hilbert space $\mathbb{H}$ which is a closed subspace of $L^2(\mathbb{T}^2, \mathbb{R}^2)$

$$\mathbb{H} = \Big\{u \in L^2(\mathbb{T}^2, \mathbb{R}^2), \mathrm{div} u = 0 \text{ and } \int_{\mathbb{T}^2} u(x)\mathrm{d}x = 0\Big\}.$$

The space $\mathbb{H}$ is endowed with the inner product and the norm of $L^2(\mathbb{T}^2, \mathbb{R}^2)$, which were denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$. A closed sub-Hilbert space $\mathbb{V}$ of $\mathbb{H}$ is defined as

$$\mathbb{V} = \Big\{u \in H^1(\mathbb{T}^2, \mathbb{R}^2), \mathrm{div} u = 0 \text{ and } \int_{\mathbb{T}^2} u(x)\mathrm{d}x = 0\Big\}$$

with the inner product and the norm

$$\langle\!\langle u, v \rangle\!\rangle = \sum_{i,j=1}^{2} \Big\langle \frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j} \Big\rangle, \qquad \|u\| = \langle\!\langle u, u \rangle\!\rangle^{1/2}.$$

Moreover, we set $\mathcal{D}(A) = H^2(\mathbb{T}^2, \mathbb{R}^2) \cap \mathbb{V}$, and

$$Au = -P\triangle u, \qquad u \in \mathcal{D}(A),$$

where $P$ is the orthogonal projector from $L^2(\mathbb{T}^2, \mathbb{R}^2)$ onto $\mathbb{H}$.

It is known that $A$ is a self-adjoint positive operator with compact resolvent in $\mathbb{H}$ and $\mathbb{V}$ coincides with $\mathcal{D}(A^{1/2})$, $\|u\| = |A^{1/2}u|$ for $u \in \mathbb{V}$. We denote by $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ the eigenvalues of $A$ and by $e_1, e_2, \ldots$ the corresponding complete orthonormal system of eigenvectors.

Let $\mathbb{V}^*$ be the dual space of $\mathbb{V}$. Define

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_{\mathbb{T}^2} u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x)\mathrm{d}x$$

and the bilinear operator $B(u,v):\mathbb{V}\times\mathbb{V}\to\mathbb{V}^*$,

$$(B(u,v),w)=b(u,v,w),\qquad u,v,w\in\mathbb{V}.$$

If the fluid flow is incompressible, we have

$$\langle B(u,v),v\rangle=0,\qquad \langle B(u,v),w\rangle=-\langle B(u,w),v\rangle.$$

In this paper, we consider the stochastic 2D Navier-Stokes equation driven by Lévy noise on $\mathbb{T}^2$ for an incompressible fluid flow:

$$\begin{cases}\mathrm{d}X_t+[\nu AX_t+B(X_t)-h(X_t)]\mathrm{d}t=\displaystyle\int_U f(X_{t-},u)\widetilde{N}(\mathrm{d}t,\mathrm{d}u)+g(t,X_t)\mathrm{d}W_t,\\ X_0=x,\end{cases}\tag{1.1}$$

where $X_t$ represents the velocity of the particle at time $t$, the positive parameter $\nu$ is the kinematic viscosity, $h$ is the external force field, and $\displaystyle\int_U f(X_{t-},u)\widetilde{N}(\mathrm{d}t,\mathrm{d}u)+g(t,X_t)\mathrm{d}W_t$ is the random external force field.

For the need in the following, we list here some properties on trilinear form $b$:

$$\begin{aligned}
u\in\mathbb{H},\ v,w\in\mathbb{V}\ &\Rightarrow\ b(u,v,w)=-b(u,w,v),\\
u,v,w\in\mathbb{V}\ &\Rightarrow\ |b(u,v,w)|\le K_1|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2},\\
u\in\mathbb{V},\ v\in\mathcal{D}(A),\ w\in\mathbb{H}\ &\Rightarrow\ |b(u,v,w)|\le K_1|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w|,\\
u\in\mathbb{H},\ v\in\mathbb{V}\ &\Rightarrow\ b(u,v,v)=0,
\end{aligned}$$

where $K_1$ is a positive constant.

**Definition 1.1** Suppose $X_t$ is an $\mathbb{H}$-valued $\mathcal{F}_t$-adapted càdlàg process. If for $t>0$, $\displaystyle\int_0^t[|X_s|+|B(X_s)|+|h(X_s)|]\mathrm{d}s<+\infty$ and for $\xi\in\mathcal{D}(A)$,

$$\begin{aligned}\langle X_t,\xi\rangle &= \langle x,\xi\rangle-\nu\int_0^t\langle X_s,A\xi\rangle\mathrm{d}s-\int_0^t\langle B(X_s),\xi\rangle\mathrm{d}s+\int_0^t\langle h(X_s),\xi\rangle\mathrm{d}s\\ &\quad+\int_0^t\int_U\langle f(X_{s-},z),\xi\rangle\widetilde{N}(\mathrm{d}s,\mathrm{d}z)+\int_0^t\langle\xi,g(s,X_s)\mathrm{d}W_s\rangle,\end{aligned}$$

then $X_t$ is called the weak solution of (1.1).

**Definition 1.2** We say that a weak solution $X_t$ of (1.1) converges to $x_\infty\in\mathbb{H}$ exponentially stable in the mean square if there exist $a>0$ and $M_0=M(X_0)>0$ such that

$$\mathsf{E}|X_t-x_\infty|^2\le M_0e^{-at},\qquad t\ge0.$$

In particular, if $x_\infty$ is a solution to (1.1), then it is said that $x_\infty$ is exponentially stable in the mean square provided that every weak solution to (1.1) converges to $x_\infty$ exponentially in the mean square with the same exponential order $a > 0$.

**Definition 1.3** We say that a weak solution $X_t$ of (1.1) converges to $x_\infty \in \mathbb{H}$ almost surely exponentially if there exists $\gamma > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \ln |X_t - x_\infty| \le -\gamma \qquad \mathsf{P}-\text{a.s.}.$$

In particular, if $x_\infty$ is a solution to (1.1), then it is said that $x_\infty$ is almost surely exponentially stable provided that every weak solution to (1.1) converges to $x_\infty$ almost surely exponentially with the same constant $\gamma$.

## §2. The Exponential Stability of Solutions

We'll introduce the following hypotheses:

(H$_1$) $f(\cdot,\cdot) : \mathbb{H} \times U \to \mathbb{H}$ is a measurable function and there exists $K_2 > 0$ such that

$$\int_U |f(x,z) - f(y,z)|^2 \lambda(\mathrm{d}z) \le K_2 |x-y|^2, \qquad x, y \in \mathbb{H}.$$

(H$_2$) $h(\cdot) : \mathbb{V} \to \mathbb{V}^*$ is a continuous function and there exists $\beta > 0$ such that

$$\|h(u) - h(v)\|_{\mathbb{V}^*} \le \beta \|u - v\|, \qquad u, v \in \mathbb{V}.$$

**Remark 1** Under the condition (H$_2$), Caraballo, Langa and Taniguchi proved the conclusion in [4]: If the function $h$ satisfies that $h(v_m)$ converges to $h(v)$ weakly in $\mathbb{V}^*$ where $\{v_n\}_{n\ge1} \subset \mathbb{V}$ converges to $v \in \mathbb{V}$ weakly in $\mathbb{V}$ and strongly in $\mathbb{H}$, then the equation

$$\nu A u + B(u) = h(u) \qquad (\text{equality in } \mathbb{V}^*) \tag{2.1}$$

has the properties:

(i) If $\nu > \beta$, (2.1) has a stationary solution $u_\infty \in \mathbb{V}$.

(ii) If $\nu > (K_1 \|h(0)\|_{\mathbb{V}^*})/[\sqrt{\lambda_1}(\nu - \beta)] + \beta$, the stationary solution of (2.1) is unique.

Using these we can research the long-time behaviour of weak solutions $X_t$ of (1.1) under some conditions for the kinematic viscosity $\nu$ being sufficiently large. Hence throughout this paper we assume that there exists a unique stationary solution $u_\infty \in \mathbb{V}$ for (2.1).

In the following, we need the conditions:

(H$_3$) $f(u_\infty, z) \equiv 0$, $\forall\, z \in U$.

(H$_4$) $g(\cdot,\cdot):[0,\infty)\times\mathbb{V}\to L(K,\mathbb{H})$ is a continuous function and satisfies that there exist nonnegative integrable functions $\gamma(t)$, $\delta(t)$ and constants $c_1>0$, $\theta>0$, $M_\gamma\geq 1$ and $M_\delta\geq 1$ with

$$\gamma(t)\leq M_\gamma e^{-\theta t},\quad \delta(t)\leq M_\delta e^{-\theta t},\qquad t\geq 0$$

such that

$$\|g(t,u)\|^2_{L_2^0}\leq\gamma(t)+(c_1+\delta(t))|u-u_\infty|^2,$$

where $\|g(t,u)\|^2_{L_2^0}=\mathrm{Tr}(g(t,u)Qg(t,u)^*)$, $u\in\mathbb{V}$ and $L_2^0\mathrel{\widehat=}L_2(Q^{1/2}(K),\mathbb{H})$ is a Hilbert space.

**Theorem 2.1** Suppose $2\nu>2\beta+2\lambda_1^{-1/2}K_1\|u_\infty\|+\lambda_1^{-1}[\delta(s)+c_1+K_2]$ and that conditions (H$_1$)$-$(H$_4$) are satisfied. Then, the weak solution $X_t$ of (1.1) converges to $u_\infty$ exponentially in the mean square. That is, there exists real number $a\in(0,\theta)$, $M_0=M_0(X_0)>0$ such that

$$\mathsf{E}|X_t-u_\infty|^2\leq M_0e^{-at},\qquad t\geq 0.$$

**Proof** Applying the Itô formula to $e^{at}|X_t-u_\infty|^2$ and taking the expectation, we have

$$\begin{aligned}
&e^{at}\mathsf{E}|X_t-u_\infty|^2\\
\leq\ &\mathsf{E}|X_0-u_\infty|^2+\int_0^t\Big(\frac{a}{\lambda_1}-2\nu+2\beta+\frac{2K_1}{\sqrt{\lambda_1}}\|u_\infty\|\Big)e^{as}\mathsf{E}\|X_s-u_\infty\|^2\mathrm{d}s\\
&+\int_0^t e^{as}\Big[\gamma(s)+\frac{c_1+\delta(s)}{\lambda_1}\mathsf{E}\|X_s-u_\infty\|^2\Big]\mathrm{d}s+\int_0^t e^{as}\frac{K_2}{\lambda_1}\mathsf{E}\|X_s-u_\infty\|^2\mathrm{d}s\\
\leq\ &\mathsf{E}|X_0-u_\infty|^2+\int_0^t e^{as}\gamma(s)\mathrm{d}s+\int_0^t k(s)e^{as}\mathsf{E}\|X_s-u_\infty\|^2\mathrm{d}s,
\end{aligned}\tag{2.2}$$

where

$$k(s)=\frac{a}{\lambda_1}-2\nu+2\beta+\frac{2K_1}{\sqrt{\lambda_1}}\|u_\infty\|+\frac{c_1+\delta(s)}{\lambda_1}+\frac{K_2}{\lambda_1}.$$

By the assumptions we have that there exists $a\in(0,\theta)$ such that $k(s)<0$. Hence, (2.2) yields

$$e^{at}\mathsf{E}|X_t-u_\infty|^2\leq\mathsf{E}|X_0-u_\infty|^2+\frac{M_\gamma}{\theta-a}.$$

Put $M_0\mathrel{\widehat=}\mathsf{E}|X_0-u_\infty|^2+M_\gamma/(\theta-a)$. We get

$$\mathsf{E}|X_t-u_\infty|^2\leq M_0e^{-at},\qquad t>0.$$

Which completes the proof. □

**Theorem 2.2** Suppose that all the conditions in Theorem 2.1 are satisfied. Then the weak solution $X_t$ of (1.1) converges to $u_\infty$ almost surely exponentially.

**Proof** Let $N$ be a natural number. By the Burkholds-Davis inequality and the Young inequality, we have that, for any $t \geq N$,

$$\begin{aligned}
&\mathsf{E}\Big[\sup_{N\le t\le N+1}\int_N^t\langle X_s-u_\infty, g(s,X_s)\mathrm{d}W_s\rangle\Big]\\
\le\ & 2\sqrt{6}\mathsf{E}\Big[\int_N^{N+1}|X_s-u_\infty|^2\|g(s,X_s)\|^2_{L_2^0}\mathrm{d}s\Big]^{1/2}\\
\le\ & 48\int_N^{N+1}\mathsf{E}\|g(s,X_s)\|^2_{L_2^0}\mathrm{d}s+\frac{1}{8}\mathsf{E}\Big[\sup_{N\le t\le N+1}|X_t-u_\infty|^2\Big]
\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}
&\mathsf{E}\Big[\sup_{N\le t\le N+1}\int_N^t\int_U\langle X_{s-}-u_\infty, f(X_{s-},z)\rangle\widetilde{N}(\mathrm{d}z,\mathrm{d}s)\Big]\\
\le\ & 3\mathsf{E}\Big[\int_N^{N+1}\int_U|X_s-u_\infty|^2|f(X_{s-},z)|^2\lambda(\mathrm{d}z)\mathrm{d}s\Big]^{1/2}\\
\le\ & 9\int_N^{N+1}K_2\mathsf{E}|X_s-u_\infty|^2\mathrm{d}s+\frac{1}{4}\mathsf{E}\Big[\sup_{N\le t\le N+1}|X_t-u_\infty|^2\Big].
\end{aligned}\tag{2.4}$$

We also have

$$\mathsf{E}\int_N^{N+1}\int_U|f(X_{s-},z)|^2N(\mathrm{d}z,\mathrm{d}s)\le K_2\int_N^{N+1}\mathsf{E}|X_s-u_\infty|^2\mathrm{d}s.\tag{2.5}$$

Therefore, by (2.3)-(2.5), the Itô formula yields

$$\begin{aligned}
&\mathsf{E}\Big[\sup_{N\le t\le N+1}|X_t-u_\infty|^2\Big]\\
\le\ & \mathsf{E}|X_N-u_\infty|^2+97\int_N^{N+1}[\gamma(t)+(c_1+\delta(t))\mathsf{E}|X_t-u_\infty|^2]\mathrm{d}t\\
&+\frac{1}{2}\mathsf{E}\Big[\sup_{N\le t\le N+1}|X_t-u_\infty|^2\Big]+97\int_N^{N+1}K_2\mathsf{E}|X_s-u_\infty|^2\mathrm{d}s.
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{1}{2}\mathsf{E}\Big[\sup_{N\le t\le N+1}|X_t-u_\infty|^2\Big]\ \le\ & \mathsf{E}|X_N-u_\infty|^2+97\int_N^{N+1}\gamma(t)\mathrm{d}t\\
&+97\int_N^{N+1}[c_1+\delta(t)+K_2]\mathsf{E}|X_t-u_\infty|^2\mathrm{d}t.
\end{aligned}$$

Since $\gamma(t)\le M_\gamma e^{-\theta t}$ and $\delta(t)\le M_\delta e^{-\theta t}$, $a\in(0,\theta)$, $M_\gamma\ge 1$, $M_\delta\ge 1$, we obtain, thanks to Theorem 2.1, that there exists $M_1=M_1(X_0)$ such that

$$\mathsf{E}\Big[\sup_{N\le t\le N+1}|X_t-u_\infty|^2\Big]\le M_1e^{-aN}.$$

Finally, using the Borel-Cantelli lemma one can easily finish the proof. □

**Theorem 2.3** Assume that $(\mathrm{H}_1)-(\mathrm{H}_3)$ are satisfied and that $g(\cdot,\cdot):[0,\infty)\times\mathbb{V}\to L(K,\mathbb{H})$ is a continuous function satisfying the following assumptions:

(a) $g(t,u_\infty)\equiv 0$, $t\ge 0$;

(b) $\|g(t,x)-g(t,y)\|_{L_2^0}\le C_g\|x-y\|$, $C_g\ge 0$, $x,y\in\mathbb{V}$.

Let $X$ be a weak solution of (1.1). If

$$2\nu > 2\beta + C_g^2 + \frac{2K_1}{\sqrt{\lambda_1}}\|u_\infty\| + \frac{K_2}{\lambda_1},$$

then we have the followings:

(i) There exists real number $\gamma>0$ such that

$$\mathsf{E}|X_t-u_\infty|^2 \le \mathsf{E}|X_0-u_\infty|^2 e^{-\gamma t}, \qquad t\ge 0.$$

(ii) If $\langle x-y,(g(s,x)-g(s,y))\widetilde{g}\rangle=0$ for $x,y\in\mathbb{V}$, $\widetilde{g}\in K$ and $s\ge 0$, there exists a real number $\gamma>0$ such that

$$\limsup_{t\to+\infty}\frac{1}{t}\ln|X_t-u_\infty| \le -\frac{\gamma}{4}, \qquad \text{a.s..}$$

**Proof** (i) Arguing as before, we have

$$|\langle B(X_s)-B(u_\infty),X_s-u_\infty\rangle| \le K_1\lambda_1^{-1/2}\|u_\infty\|\|X_s-u_\infty\|^2.$$

For $\gamma>0$ being small enough (fixed later), applying the Itô formula and taking expectation, as above calculation we obtain

$$\begin{aligned}
& e^{\gamma t}\mathsf{E}|X_t-u_\infty|^2 \\
\le\ & \mathsf{E}|X_0-u_\infty|^2 + \frac{\gamma}{\lambda_1}\int_0^t e^{\gamma s}\mathsf{E}\|X_s-u_\infty\|^2\mathrm{d}s - 2\nu\int_0^t e^{\gamma s}\mathsf{E}\|X_s-u_\infty\|^2\mathrm{d}s \\
& +2\beta\int_0^t e^{\gamma s}\mathsf{E}\|X_s-u_\infty\|^2\mathrm{d}s + \int_0^t C_g^2 e^{\gamma s}\mathsf{E}\|X_s-u_\infty\|^2\mathrm{d}s \\
& +\int_0^t \frac{K_2}{\lambda_1}e^{\gamma s}\mathsf{E}\|X_s-u_\infty\|\mathrm{d}s + \int_0^t \frac{2K_1}{\sqrt{\lambda_1}}\|u_\infty\|e^{\gamma s}\mathsf{E}\|X_s-u_\infty\|^2\mathrm{d}s \\
=\ & \mathsf{E}|X_0-u_\infty|^2 + C(\gamma)\int_0^t e^{\gamma s}\mathsf{E}\|X_s-u_\infty\|^2\mathrm{d}s,
\end{aligned}\tag{2.6}$$

where

$$C(\gamma)=\lambda_1^{-1}\gamma-2\nu+2\beta+C_g^2+2\lambda_1^{-1/2}K_1\|u_\infty\|+\lambda_1^{-1}K_2.$$

Since $2\beta-2\nu+C_g^2+2\lambda_1^{-1/2}K_1\|u_\infty\|+\lambda_1^{-1}K_2<0$, we get that there exists a real number $\gamma>0$ such that $C(\gamma)<0$. Which completes the proof of the first part of the theorem.

(ii) Let $N$ be a natural number. By the Itô formula, it follows for any $t \geq N$,

$$\begin{aligned}|X_t - u_\infty|^2 \leq\ & |X_N - u_\infty|^2 + \int_N^t \Big[-2\nu + 2\beta + C_g^2 + \frac{2K_1}{\sqrt{\lambda_1}}\|u_\infty\|\Big]\|X_s - u_\infty\|^2 \mathrm{d}s \\ & + 2\int_N^t \langle X_s - u_\infty, g(s, X_s)\mathrm{d}W_s\rangle + \int_N^t\int_U |f(X_{s-}, z)|^2 N(\mathrm{d}z, \mathrm{d}s) \\ & + 2\int_N^t\int_U \langle X_{s-} - u_\infty, f(X_{s-}, z)\rangle \widetilde{N}(\mathrm{d}z, \mathrm{d}s). \end{aligned} \tag{2.7}$$

$2\beta - 2\nu + C_g^2 + (2K_1/\sqrt{\lambda_1})\|u_\infty\| + K_2/\lambda_1 < 0$, $\langle x - y, (g(s,x) - g(s,y))\widetilde{g}\rangle = 0$ for $x, y \in \mathbb{V}$, $\widetilde{g} \in K$, $s \geq 0$ and (2.7) show

$$\begin{aligned}|X_t - u_\infty|^2 \leq\ & |X_N - u_\infty|^2 + \int_N^t\int_U |f(X_{s-}, z)|^2 N(\mathrm{d}z, \mathrm{d}s) \\ & + 2\int_N^t\int_U \langle X_{s-} - u_\infty, f(X_{s-}, z)\rangle \widetilde{N}(\mathrm{d}z, \mathrm{d}s).\end{aligned}$$

As the proof of (2.4) and (2.5), we have

$$\frac{1}{2}\mathsf{E}\Big[\sup_{N \leq t \leq N+1} |X_t - u_\infty|^2\Big] \leq \mathsf{E}|X_N - u_\infty|^2 + 13K_2 \int_N^{N+1} \mathsf{E}|X_t - u_\infty|^2 \mathrm{d}t.$$

This yields, thanks to Theorem 2.3 (i), that there exists $M_2 = M_2(X_0) \geq 1$ such that

$$\mathsf{E}\Big[\sup_{N \leq t \leq N+1} |X_s - u_\infty|^2\Big] \leq M_2 e^{-\gamma N}.$$

By the Borel-Cantelli lemma one can finish the proof. □

In the final of this section we consider the case where the external force $h$ can depend on time, that is, $h(\cdot,\cdot) : [0,\infty) \times \mathbb{V} \to \mathbb{V}^*$. In this case, we give the assumption:

($\mathrm{H}_5$) There exist integrable functions $\alpha(t), \beta(t)$ and real numbers $d > 0$, $\theta > 0$, $M_\alpha \geq 1$ and $M_\beta \geq 1$ such that

$$\langle h(t,x), x\rangle \leq \alpha(t) + (d + \beta(t))|x|^2$$

and

$$\alpha(t) \leq M_\alpha e^{-\theta t}, \quad \beta(t) \leq M_\beta e^{-\theta t}, \qquad t \geq 0.$$

**Theorem 2.4** Suppose that ($\mathrm{H}_1$) and ($\mathrm{H}_5$) are satisfied and there exists a constant $\zeta > 0$ such that $\|g(t,u)\|_{L_2^0}^2 \leq \gamma(t) + (\zeta + \delta(t))|u|^2$, where the functions $\gamma(t), \delta(t)$ satisfy the same conditions as that in ($\mathrm{H}_4$). Furthermore, suppose $2\lambda_1\nu > 2d + \zeta + K_2$ and $f(0,u) \equiv 0$ for $u \in U$. Then the weak solution $X_t$ of (1.1) converges to zero almost surely exponentially.

**Proof** By the hypotheses we have that there is a positive number $a \in (0, \theta)$ such that $2\lambda_1\nu > 2d + \zeta + K_2 + a$. As the above proof, we can get

$$\begin{aligned} e^{at}\mathsf{E}|X_t|^2 \quad \le \quad & \mathsf{E}|X_0|^2 + \int_0^t [2\alpha(s) + \gamma(s)]e^{as}\mathrm{d}s + \int_0^t [2\beta(s) + \delta(s)]e^{as}\mathsf{E}|X_s|^2\mathrm{d}s \\ & + \int_0^t [a - 2\lambda_1\nu + 2d + \zeta + K_2]\mathsf{E}|X_s|^2\mathrm{d}s. \end{aligned}$$

Since $a - 2\lambda_1\nu + 2d + \zeta + K_2 < 0$, the Gronwall inequality yields

$$e^{at}\mathsf{E}|X_t|^2 \le \Big[\mathsf{E}|X_0|^2 + \frac{2M_\alpha + M_\gamma}{\theta - a}\Big]e^{(2M_\beta + M_\delta)/\theta}.$$

Put $M_3 = [\mathsf{E}|X_0|^2 + (2M_\alpha + M_\gamma)/(\theta - a)]e^{(2M_\beta + M_\delta)/\theta}$. We have

$$\mathsf{E}|X_t|^2 \le M_3 e^{-at}. \tag{2.8}$$

Let $N$ be a natural number. By the Itô formula, it follows that, for $t \ge N$,

$$\begin{aligned} |X_t|^2 \quad \le \quad & |X_N|^2 + 2\int_N^t (\alpha(s) + \gamma(s))\mathrm{d}s + \int_N^t (2\beta(s) + \delta(s))|X_s|^2\mathrm{d}s \\ & + 2\int_N^t \int_U \langle X_{s-}, f(X_{s-}, z)\rangle \widetilde{N}(\mathrm{d}z, \mathrm{d}s) + 2\int_N^t \langle X_s, g(s, X_s)\mathrm{d}W_s\rangle \\ & + \int_N^t \int_U |f(X_{s-}, z)|^2 N(\mathrm{d}z, \mathrm{d}s). \end{aligned} \tag{2.9}$$

Furthermore, by the Burkholds-Davis inequality

$$\begin{aligned} & \mathsf{E}\Big[\sup_{N \le t \le N+1} \int_N^t \langle X_s, g(s, X_s)\mathrm{d}W_s\rangle\Big] \\ \le \quad & 48\int_N^{N+1} \mathsf{E}\|g(s, X_s)\|_{L_2^0}^2\mathrm{d}s + \frac{1}{8}\mathsf{E}\Big[\sup_{N \le t \le N+1} |X_t|^2\Big] \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} & \mathsf{E}\Big[\sup_{N \le t \le N+1} \int_N^t \int_U \langle X_{s-}, f(X_{s-} z)\rangle \widetilde{N}(\mathrm{d}z, \mathrm{d}s)\Big] \\ \le \quad & 48\int_N^{N+1} K_2\mathsf{E}|X_s|^2\mathrm{d}s + \frac{1}{8}\mathsf{E}\Big[\sup_{N \le t \le N+1} |X_t|^2\Big]. \end{aligned} \tag{2.11}$$

By (2.5) and (2.9)-(2.11), we obtain

$$\begin{aligned} & \mathsf{E}\Big[\sup_{N \le t \le N+1} |X_t|^2\Big] \\ \le \quad & \mathsf{E}|X_N|^2 + 2\int_N^{N+1} (\alpha(s) + \gamma(s))\mathrm{d}s + \int_N^{N+1} (2\beta(s) + \delta(s))\mathsf{E}|X_s|^2\mathrm{d}s \\ & + 96\int_N^{N+1} \mathsf{E}\|g(s, X_s)\|_{L_2^0}^2\mathrm{d}s + \frac{1}{2}\mathsf{E}\Big[\sup_{N \le t \le N+1} |X_t|^2\Big] + 97K_2\int_N^{N+1} \mathsf{E}|X_s|^2\mathrm{d}s. \end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{1}{2}\mathsf{E}\Big[\sup_{N\le t\le N+1}|X_t|^2\Big]\\
\le\ &\mathsf{E}|X_N|^2+2\int_N^{N+1}(\alpha(s)+\gamma(s))\mathrm{d}s+\int_N^{N+1}(2\beta(s)+\delta(s))\mathsf{E}|X_s|^2\mathrm{d}s\\
&+96\int_N^{N+1}[\gamma(s)+(\zeta+\delta(s))\mathsf{E}|X_s|^2]\mathrm{d}s+97K_2\int_N^{N+1}\mathsf{E}|X_s|^2\mathrm{d}s.
\end{aligned}$$

The conditions $\alpha(t)\le M_\alpha e^{-\theta t}$, $\beta(t)\le M_\beta e^{-\theta t}$, $\gamma(t)\le M_\gamma e^{-\theta t}$, $\delta(t)\le M_\delta e^{-\theta t}$ $(0<a<\theta)$ and $M_\alpha\wedge M_\beta\wedge M_\gamma\wedge M_\delta\ge 1$ imply that there exists $M_4=M_4(X_0)>0$ such that

$$\mathsf{E}\Big[\sup_{N\le t\le N+1}|X_t|^2\Big]\le M_4e^{-aN}.$$

Finally, using the Borel-Cantelli lemma, we have that

$$\limsup_{t\to\infty}\frac{1}{t}\ln|X_t|\le -\frac{a}{4}.$$

Which completes the proof. $\square$

## §3. Stabilizability and Stabilization of Solutions

In this section, we shall analyze some aspects related to the problem of stabilizability and stabilization of our Navier-Stokes model. First, notice that the pathwise stability. However, it may happen that a solution of a stochastic equation can be pathwise exponentially stable and not exponentially stable in the mean square.

Indeed, let us consider the following scalar ordinary differential equation to illustrate this fact,

$$\mathrm{d}x_t=ax_t\mathrm{d}t+bx_t\mathrm{d}W_t+cx_{t-}\mathrm{d}N_t, \tag{3.1}$$

where $a,b,c$ are real numbers and $c>-1$, independent processes $W_t$ and $N_t$ are one dimensional Wiener process and one dimensional Poisson process with intensity $\lambda>0$, respectively. It's easy to get the solution of (3.1)

$$x_t=x_0(c+1)^{N_t}\exp\{(a-b^2/2)t+bW_t\}.$$

Thus, it is obvious that the zero solution of (3.1) is pathwise exponentially stable with probability one if and only if $\lambda\ln(c+1)+a-b^2/2<0$.

Also, one can prove that

$$\mathsf{E}|x_t|^2=\mathsf{E}|x_0|^2\exp\{[\lambda(c^2+2c)+2a+b^2]t\}.$$

Hence, the zero solution of (3.1) is exponentially stable in the mean square if and only if $\lambda(c^2+2c)+2a+b^2<0$. So, we observe that there exist many possibilities of being the zero solution pathwise exponentially stable and, at the same time, exponentially unstable in the mean square.

Consequently, it would be very interesting to obtain pathwise exponential stability results by avoiding the method of using the mean square stability as previous step. This will be one of the aims of this section. However, it is worth pointing out that to get some results in this direction, we will need to assume some additional hypotheses on the stochastic perturbation so that we can obtain better stability criteria but for mire specific situations. In particular, in some of our situations, the noise is so special that one can perform a time change, a substitution that transforms the stochastic equation into a deterministic one. For example, the Itô formula for the logarithm in the proof of Theorem 3.1 in this section is one way to perform this transformation; another is to multiply by the exponential to the noise (see Crauel and Flandoli (1994), p.382).

To this end let us first state the following condition:

($H_6$) $h(\cdot):\mathbb{H}\to\mathbb{H}$ and $g(t,\cdot):\mathbb{H}\to L(K,\mathbb{H})$ satisfy that there exist constants $c>0$ and $C_g>0$ such that

$$|h(u)-h(v)|\le c|u-v|,\qquad u,v\in\mathbb{H}$$

and

$$\|g(t,u)-g(t,v)\|_{L_2^0}\le C_g|u-v|,\qquad t>0,\ u,v\in\mathbb{H}.$$

**Theorem 3.1** In addition to ($H_6$), assume following conditions hold:

(1) There exists a function $\widetilde{f}(\cdot):U\to K$ such that $|f(x,z)|\le|\widetilde{f}(z)||x|$ and $\displaystyle\int_U|\widetilde{f}(z)|^4$ $\cdot\lambda(\mathrm{d}z)<\infty$;

(2) $h(0)=0$, $g(t,0)=0$ for $t\ge0$, $f(0,z)=0$ for $z\in U$ and $\lambda(U)<+\infty$;

(3) Put $\Psi(x)=|x|^2$. Suppose that there exists $\rho>0$ such that

$$\widetilde{Q}\Psi(s,x):=\mathrm{Tr}[(\Psi_x(x)\otimes\Psi_x(x))(g(s,x)Qg(s,x)^*)]\ge\rho^2|x|^4,$$

where $(\Psi_x(x)\otimes\Psi_x(x))(\widetilde{h})=\Psi_x(x)(\Psi_x(x),\widetilde{h})$ for $x$, $\widetilde{h}\in\mathbb{H}$.

Then, there exists $\Omega_0\subset\Omega$ with $\mathsf{P}(\Omega_0)=0$ such that for $\omega\notin\Omega_0$ there exists $T(\omega)>0$ such that any weak solution $X(t)$ of (1.1) satisfies

$$|X_t|^2(\omega)\le|X_0(\omega)|^2e^{-\gamma t}\qquad\text{for all}\ \ t\ge T(\omega),$$

where

$$\gamma=\lambda_1\nu-2c-C_g^2+\frac{1}{2}\rho^2-\lambda(U)-\int_U[4|\widetilde{f}(z)|+\widetilde{f}(z)|^2]\lambda(\mathrm{d}z).$$

In particular, exponential stability of sample paths with probability one holds if $\gamma > 0$.

**Proof** Using Itô formula to the function $\ln|X_t|^2$ and taking into account the hypotheses, it follows

$$\begin{aligned}
\ln|X_t|^2 \le\ & \ln|X_0|^2 + 2\int_0^t |X_s|^{-2}[c-\lambda_1\nu + C_g^2/2]|X_s|^2 \mathrm{d}s - \frac{1}{2}\rho^2 t \\
&+2\int_0^t |X_s|^{-2}\langle X_s, g(s,X_s)\mathrm{d}W_s\rangle + 2\int_0^t\int_U \langle X_{s-}, f(X_{s-},z)\rangle |X_{s-}|^{-2}\widetilde{N}(\mathrm{d}z,\mathrm{d}s) \\
&+\int_0^t\int_U |X_{s-}+f(X_{s-},z)|^2 - 2\langle X_{s-}, f(X_{s-},z)\rangle |X_{s-}|^{-2} N(\mathrm{d}z,\mathrm{d}s) \\
\le\ & \ln|X_0|^2 + [2c-2\lambda_1\nu + C_g^2 - \rho^2/2]t + 2\int_0^t |X_s|^{-2}\langle X_s, g(s,X_s)\mathrm{d}W_s\rangle \\
&+2\int_0^t\int_U |\langle X_{s-}, f(X_{s-},z)\rangle||X_{s-}|^{-2}[N(\mathrm{d}z,\mathrm{d}s)+\lambda(\mathrm{d}z)\mathrm{d}s] \\
&+\int_0^t\int_U |f(X_{s-},z)|^2|X_{s-}|^{-2}N(\mathrm{d}z,\mathrm{d}s) + N([0,t]\times U) \\
\le\ & \ln|X_0|^2 + \Big(2c - 2\lambda_1\nu + C_g^2 - 1/2\rho^2 + 2\int_U |\widetilde{f}(z)|\lambda(\mathrm{d}z)\Big)t + N([0,t]\times U) \\
&+2\int_0^t |X_s|^{-2}\langle X_s, g(s,X_s)\mathrm{d}W_s\rangle + 2\int_U [|\widetilde{f}(z)| + |\widetilde{f}(z)|^2] N([0,t],\mathrm{d}z).
\end{aligned} \tag{3.2}$$

Now, due to our assumptions, $\displaystyle\int_0^t |X_s|^{-2}\langle X_s, g(s,X_s)\mathrm{d}W_s\rangle$ is a real martingale and it is not difficult to prove, by means of the law of iterated logarithm,

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t |X_s|^{-2}\langle X_s, g(s,X_s)\mathrm{d}W_s\rangle = 0, \qquad \text{a.s.}. \tag{3.3}$$

By using the Kolmogorov strong law of large numbers, we have

$$\lim_{t\to+\infty}\frac{N([0,t]\times U)}{t} = \lambda(U), \qquad \text{a.s.}, \tag{3.4}$$

$$\lim_{t\to+\infty}\frac{1}{t}\int_U |\widetilde{f}(z)|N([0,t],\mathrm{d}z) = \int_U |\widetilde{f}(z)|\lambda(\mathrm{d}z), \qquad \text{a.s.} \tag{3.5}$$

and

$$\lim_{t\to+\infty}\frac{1}{t}\int_U |\widetilde{f}(z)|^2 N([0,t],\mathrm{d}z) = \int_U |\widetilde{f}(z)|^2\lambda(\mathrm{d}z), \qquad \text{a.s.}. \tag{3.6}$$

Then, (3.3)-(3.6) imply that there exists a set $\Omega_0 \subset \Omega$ with $\mathsf{P}(\Omega_0) = 0$ such that, for every $\omega \notin \Omega_0$, there exists $T(\omega) > 0$ such that, for all $t \ge T(\omega)$,

$$\frac{1}{t}\int_0^t |X_s|^{-2}\langle X_s, g(s,X_s)\mathrm{d}W_s\rangle \le \frac{1}{4}\lambda_1\nu, \tag{3.7}$$

$$\frac{N([0,t]\times U)}{t} \le \lambda(U) + \frac{1}{4}\lambda_1\nu, \tag{3.8}$$

$$\frac{1}{t}\int_U |\widetilde{f}(z)|N([0,t],\mathrm{d}z) \le \int_U |\widetilde{f}(z)|\lambda(\mathrm{d}z) + \frac{1}{8}\lambda_1\nu \tag{3.9}$$

and

$$\frac{1}{t}\int_U |\widetilde{f}(z)|^2 N([0,t],\mathrm{d}z) \le \int_U |\widetilde{f}(z)|^2\lambda(\mathrm{d}z) + \frac{1}{4}\lambda_1\nu. \tag{3.10}$$

Therefore, for any $t \ge T(\omega)$, (3.2) and (3.7)-(3.10) show

$$\ln|X_t|^2 \le \ln|X_0|^2 - \Big[\lambda_1\nu - 2c - C_g^2 - \lambda(U) - 4\int_U |\widetilde{f}(z)|\lambda(\mathrm{d}z) - \int_U |\widetilde{f}(z)|^2\lambda(\mathrm{d}z) + \frac{1}{2}\rho^2\Big]t.$$

Put

$$\gamma = \lambda_1\nu - 2c - C_g^2 - \lambda(U) - 4\int_U |\widetilde{f}(z)|\lambda(\mathrm{d}z) - \int_U |\widetilde{f}(z)|^2\lambda(\mathrm{d}z) + \frac{1}{2}\rho^2.$$

We obtain the first result.

Furthermore, if $\gamma > 0$, it is easy to get that

$$\limsup_{t\to+\infty}\frac{1}{t}\ln|X_t| \le -\frac{\gamma}{2}, \qquad \text{a.s..}$$

The proof is now completed. $\square$

Next we consider the following deterministic equation

$$\mathrm{d}X_t = [-\nu AX_t - B(X_t) + h(X_t)]\mathrm{d}t. \tag{3.11}$$

It obvious that the stationary solution of (3.11) may be not exponentially stable. However, Theorem 10.2 in [14] (p.69) shows that, if the Lipschitz constant $c$ of the external force field $h$ is sufficiently small, that is, if $\lambda_1\nu > K_1\sqrt{\lambda_1}\|u_\infty\| + c$, then the stationary solution to (3.11) is exponentially stable.

But if the Lipschitz constant $c$ is sufficiently large, that is, $\lambda_1\nu \le K_1\sqrt{\lambda_1}\|u_\infty\|+c$, then we do not know whether $u_\infty$ is exponentially stable or not. However, by adding Wiener noise or Lévy noise, the system can reach to stabilization, that is, $u_\infty$ is exponentially stable.

In the following theorem, we suppose $K = \mathbb{R}$, $Q = 1$, $W_t$ and $N_t$ are one dimensional Wiener process and one dimensional Possion process with the parameter $\lambda > 0$, respectively.

**Theorem 3.2** Suppose that the function $h$ satisfies condition (H$_6$), $c_0 \widehat{=} \lambda_1\nu - K_1\sqrt{\lambda_1}\|u_\infty\| > 0$, $\lambda_1\nu \le K_1\sqrt{\lambda_1}\|u_\infty\| + c$ and $2\lambda_1\nu - 2K_1\sqrt{\lambda_1}\|u_\infty\| + \sigma^2 > 2c$ for some $\sigma \in \mathbb{R}$, then there exists $\Omega_0 \subset \Omega$ with $\mathsf{P}(\Omega_0) = 0$ such that, for all $\omega \notin \Omega_0$, there is $T(\omega) > 0$ such that

$$|X_t(\omega) - u_\infty|^2 \le |X_0(\omega) - u_\infty|^2 e^{-\gamma t}, \qquad t \ge T(\omega),$$

where $\gamma > 0$ is a constant, $X_t$ is a weak solution of (1.1), functions $g$ and $f$ are given by $g(t,x) = \sigma(x - u_\infty)$ and $f(x) = \tau(x - u_\infty)$ with $\tau > 0$, respectively.

**Proof** Since

$$b(X_s - u_\infty, u_\infty, X_s - u_\infty) \le \frac{K_1}{\sqrt{\lambda_1}}\|u_\infty\|\|X_s - u_\infty\|^2,$$

we have

$$\begin{aligned} & -2\nu\|X_s - u_\infty\|^2 + 2b(X_s - u_\infty, u_\infty, X_s - u_\infty) \\ \le\ & \Big( -2\nu + \frac{2K_1}{\sqrt{\lambda_1}}\|u_\infty\|\Big)\|X_s - u_\infty\|^2 \\ \le\ & (-2\lambda_1\nu + 2K_1\sqrt{\lambda_1}\|u_\infty\|)|X_s - u_\infty|^2. \end{aligned}$$

This implies

$$\begin{aligned} & \ln|X_t - u_\infty|^2 \\ =\ & \ln|X_0 - u_\infty|^2 + \int_0^t \frac{1}{|X_s - u_\infty|^2}[-2\nu\|X_s - u_\infty\|^2 - 2b(X_s - u_\infty, u_\infty, X_s - u_\infty) \\ & + \sigma^2|X_s - u_\infty|^2 + 2\langle h(X_s) - h(u_\infty), X_s - u_\infty\rangle]\mathrm{d}s \\ & + 2\int_0^t \frac{\tau|X_{s-} - u_\infty|^2}{|X_{s-} - u_\infty|^2}\mathrm{d}\widetilde{N}_s + 2\int_0^t \frac{\sigma|X_s - u_\infty|^2}{|X_s - u_\infty|^2}\mathrm{d}W_s - \frac{1}{2}\int_0^t \frac{4\sigma^2|X_s - u_\infty|^4}{|X_s - u_\infty|^4}\mathrm{d}s \\ & + \int_0^t \Big[\ln|X_{s-} + \tau(X_{s-} - u_\infty) - u_\infty|^2 - \ln|X_{s-} - u_\infty|^2 \\ & - 2\Big\langle \frac{X_{s-} - u_\infty}{|X_{s-} - u_\infty|^2}, \tau(X_{s-} - u_\infty)\Big\rangle\Big]\mathrm{d}N_s \\ \le\ & \ln|X_0 - u_\infty|^2 + 2\sigma W_t + 2\tau\widetilde{N}_t + 2(\ln(\tau+1) - \tau)N_t \\ & + (2c - \sigma^2 - 2\lambda_1\nu + 2K_1\sqrt{\lambda_1}\|u_\infty\|)t \\ \le\ & \ln|X_0 - u_\infty|^2 + (2c - 2c_0 - \sigma^2 - 2\tau\lambda)t + 2\sigma W_t + 2N_t\ln(\tau+1). \end{aligned} \tag{3.12}$$

Since

$$\lim_{t\to\infty}\frac{W_t}{t} = 0, \quad \text{a.s.}, \qquad \lim_{t\to\infty}\frac{N_t}{t} = \lambda, \quad \text{a.s.}.$$

So there exists $\Omega_0 \subset \Omega$ with $\mathsf{P}(\Omega_0) = 0$ such that, for $\omega \notin \Omega_0$, there exists $T(\omega) > 0$ such that, for $t \ge T(\omega)$,

$$\frac{N_t(\omega)}{t} \le \lambda + \sigma^2 + 2c_0 - 2c \tag{3.13}$$

and

$$\frac{|W_t(\omega)|}{t} \le \sigma^2 + 2c_0 - 2c. \tag{3.14}$$

(3.12)-(3.14) yield that, for $t \ge T(\omega)$ ($\omega \notin \Omega_0$),

$$\ln|X_t(\omega) - u_\infty|^2 \le \ln|X_0(\omega) - u_\infty|^2 - \gamma(\lambda)t, \tag{3.15}$$

where $-\gamma(\lambda) = 2c - 2c_0 - \sigma^2 + 2[\ln(\tau+1) - \tau]\lambda + 2[|\sigma| + \ln(\tau+1)](\sigma^2 + 2c_0 - 2c)$.

Since $\ln(\tau+1) - \tau < 0$ for $\tau > 0$, we have that there is a large enough $\lambda_0 > 0$ such that $\gamma(\lambda_0) > 0$. Taking $\gamma = \gamma(\lambda_0)$. Which completes the proof. □

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# Lévy过程驱动的随机二维Navier-Stokes方程解的指数性态

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本文研究了Lévy过程驱动的随机二维Navier-Stokes方程弱解的指数性态. 给出了不同条件下解的长时间形态, 获得了一些特殊情形下解的样本轨道的指数稳定性.

**关键词:** 2维Navier-Stokes方程, Lévy过程, 稳定性, 指数稳定性.

**学科分类号:** O211.63, O175.2.