应用概率统计 第二十九卷
第三期 2013年6月

Chinese Journal of Applied Probability
and Statistics Vol.29 No.3 Jun. 2013

Complete Convergence of Pairwise NQD Random Sequences *

Huang Haiwu[1] Wang Dingcheng[2,3] Wu Qunying[1] Peng Jiangyan[2]

([1] *College of Science, Guilin University of Technology, Guilin, 541004*)

([2] *School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, 610054*)

([3] *Center of Financial Engineering, Nanjing Audit University, Nanjing, 211815*)

Abstract

In this paper, we will further study the complete convergence of pairwise NQD random sequences. Some results for pairwise NQD random sequences are obtained under some simple and weak conditions. The results obtained not only extend and generalize the results of Liu (2004) and the corresponding result of Gan and Chen (2008), but also improve them.

Keywords: Pairwise NQD random sequences, complete convergence, strong law of large numbers.

AMS Subject Classification: 60F15.

§1. Introduction

The concept of complete convergence was introduced by Hsu and Robbins (1947) as follows. A sequence $\{X_n; n \geq 1\}$ of random variables is said to converge completely to a constant θ if $\sum\limits_{n=1}^{\infty} \mathsf{P}(|X_n - \theta| > \varepsilon) < \infty$ for $\forall\, \varepsilon > 0$. In view of the Borel-Cantelli lemma, this implies that $X_n \to \theta$ almost surely (a.s., in short). The converse is true if $\{X_n; n \geq 1\}$ are independent random variables. Hsu and Robbins (1947) proved that if the sequence of arithmetic means of independent and identically distributed (i.i.d., in short) random variables converges completely to the expected value if the variance of the summands is finite. Since then, many researchers studied the complete convergence for partial sums of random variables. The main purpose of this paper is to provide the complete convergence results of pairwise negative quadrant dependent (NQD, in short) random sequences.

The concept of NQD random variables was introduced by Lehmann (1966).

*This paper is supported by the Project supported by Program to Sponsor Teams for Innovation in the Construction of Talent Highlands in Guangxi Institutions of Higher Learning ([2011]47), the National Nature Science Foundation of China (11061012, 71271042), Guangxi Provincial Scientific Research Projects (201204LX157), the Fundamental Research Funds for the Central Universities (ZYGX2012J119).

Received March 30, 2012. Revised January 4, 2013.

Definition 1.1 (Lehmann, 1966) Two random variables X and Y are said to be NQD random variables if for $\forall\, x, y \in \mathbb{R}$

$$\mathsf{P}(X < x, Y < y) \leq \mathsf{P}(X < x)\mathsf{P}(Y < y). \tag{1.1}$$

A sequence $\{X_n; n \geq 1\}$ of random variables is said to be pairwise NQD random variables if for $\forall\, i, j \in \mathbb{N}$, $i \neq j$, X_i and X_j are NQD.

Obviously, a sequence of pairwise NQD random variables is a family of very wide scope, which includes pairwise independent random variable sequences. Many known types of negative dependence such as negatively orthant dependent random variables and negatively associated random variables have developed on the basis of this notion. So, it is very significant to study probabilistic properties of pairwise NQD random sequences. Many limit theorems for pairwise NQD random variables have been established by many scholars. For example, Matula (1992) for the Kolmogorov strong law of large numbers for pairwise NQD random variable sequences with the same distribution; Wang et al. (1998) for the Marcinkiewicz type strong law of large numbers under the additional condition; Wang et al. (2001) for the strong stability for Jamison type weighted product sums; Wu (2002) for the Three series theorem, the Marcinkiewicz type strong law of large numbers; Liu (2004), Gan and Chen (2008) for the strong law of large numbers, respectively; Chen (2008), Li and Yang (2008) for the Kolmogorov-Chung type strong law of large numbers for the non-identically distributed, and so on. The main purpose of this paper is to further study the complete convergence of pairwise NQD random sequences. Some strong laws of large numbers for pairwise NQD random sequences are obtained under some simple and weak conditions. Our results obtained extend and improve the results of Liu (2004), Gan and Chen (2008). For this goal, we will give the results of Liu (2004), Gan and Chen (2008) in the following, respectively.

Theorem 1.1 (Liu, 2004) Let $\{X_n; n \geq 1\}$ be a sequence of pairwise NQD random variables with $\mathsf{E}X_n = 0$. Let $\{a_n; n \geq 1\}$ be a sequence of positive real numbers such that $0 \leq a_n \uparrow \infty$. If one of the following two statements can be satisfied:

$$\sum_{n=1}^{\infty} \log^2 n \mathsf{E}\Big(\frac{|X_n|^\beta}{(Ma_n)^\beta + |X_n|^\beta}\Big) < \infty, \qquad 0 < \beta \leq 1, \tag{1.2}$$

$$\sum_{n=1}^{\infty} \log^2 n \mathsf{E}\Big(\frac{|X_n|^\beta}{Ma_n|X_n|^{\beta-1} + |Ma_n|^\beta}\Big) < \infty, \qquad 1 \leq \beta \leq 2, \tag{1.3}$$

where $M > 0$, then

$$\frac{1}{a_n}\sum_{i=1}^{n} X_i \to 0 \ \text{ a.s.}, \qquad \text{as } \ n \to \infty. \tag{1.4}$$

Theorem 1.2 (Gan and Chen, 2008) Let $\{X_n; n \geq 1\}$ be a sequence of pairwise NQD random variables with $\mathsf{E}X_n = 0$. Let $\{a_n; n \geq 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$ and $\{\psi_n(t); n \geq 1\}$ be a sequence of positive, even functions such that for each $n \geq 1$, $\psi_n(t) > 0$ for $t > 0$ and

$$\frac{\psi_n(|t|)}{|t|} \uparrow \quad \text{and} \quad \frac{\psi_n(|t|)}{t^2} \downarrow \quad \text{as} \quad |t| \uparrow . \tag{1.5}$$

If

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \mathsf{E}\frac{\psi_i(X_i)}{\psi_i(a_n)} < \infty, \tag{1.6}$$

then

$$\sum_{n=1}^{\infty} \mathsf{P}\Big(\Big|\frac{1}{a_n} \sum_{i=1}^{n} X_i\Big| > \varepsilon\Big) < \infty \qquad \text{for} \ \ \forall \varepsilon > 0. \tag{1.7}$$

§2. Main Results

Throughout this paper, the symbol C will stand for a generic positive constant which may differ from one place to another, $a_n = O(b_n)$ will mean $a_n \leq C(b_n)$.

To prove our main results, we need the following lemmas.

Lemma 2.1 (Lehmann, 1966) Let X and Y be NQD, then

(1) $\mathsf{E}XY \leq \mathsf{E}X\mathsf{E}Y$;

(2) If f and g are both nondecreasing (or nonincreasing) functions, then $f(X)$ and $g(Y)$ are NQD.

Lemma 2.2 (Matula, 1992) Let $\{X_n; n \geq 1\}$ be a sequence of pairwise NQD random variables with $\mathsf{E}X_n = 0$ and $\mathsf{E}X_n^2 < \infty$ for all $n \geq 1$. Then,

$$\mathsf{E}\Big(\sum_{i=1}^{n} X_i\Big)^2 \leq \sum_{i=1}^{n} \mathsf{E}X_i^2 \qquad \text{for} \ \ \forall n \geq 1.$$

Now, we state and prove our main results.

Theorem 2.1 Let $\{X_n; n \geq 1\}$ be a sequence of pairwise NQD random variables and $\{a_n; n \geq 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $\{g_n(t); n \geq 1\}$ be a sequence of positive, even functions such that $g_n(|t|)$ is an increasing function of $|t|$ and $g_n(|t|)/|t|$ is a decreasing function of $|t|$ for every $n \geq 1$, If

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \mathsf{E}\frac{g_i(|X_i|)}{g_i(a_n)} < \infty, \tag{2.1}$$

then

$$\sum_{n=1}^{\infty} \mathsf{P}\Big(\Big|\frac{1}{a_n} \sum_{i=1}^{n} X_i\Big| > \varepsilon\Big) < \infty \qquad \text{for} \ \ \forall \varepsilon > 0. \tag{2.2}$$

Proof of Theorem 2.1 For all $i \geq 1$, define

$$X_i^{(a_n)} = X_i I(|X_i| \leq a_n) + a_n I(X_i > a_n) - a_n I(X_i < -a_n);$$
$$T_k^{(a_n)} = \frac{1}{a_n}\sum_{i=1}^{k}(X_i^{(a_n)} - \mathsf{E}X_i^{(a_n)}), \qquad k = 1, 2, \ldots, n.$$

It follows from Lemma 2.1 that $\{X_i^{(a_n)}; i \geq 1\}$ is still a sequence of pairwise NQD random variables. It is easy to check that for $\forall\,\varepsilon > 0$,

$$\Big(\Big|\frac{1}{a_n}\sum_{i=1}^{n} X_i\Big| > \varepsilon\Big) \subset \Big(\max_{1\leq i\leq n}|X_i| > a_n\Big) \cup \Big(\Big|\frac{1}{a_n}\sum_{i=1}^{n} X_i^{(a_n)}\Big| > \varepsilon\Big), \tag{2.3}$$

which implies that

$$\begin{aligned} \mathsf{P}\Big(\Big|\frac{1}{a_n}\sum_{i=1}^{n} X_i\Big| > \varepsilon\Big) &\leq \mathsf{P}\Big(\max_{1\leq i\leq n}|X_i| > a_n\Big) + \mathsf{P}\Big(\Big|\frac{1}{a_n}\sum_{i=1}^{n} X_i^{(a_n)}\Big| > \varepsilon\Big) \\ &\leq \sum_{i=1}^{n}\mathsf{P}(|X_i| > a_n) + \mathsf{P}\Big(|T_n^{(a_n)}| > \varepsilon - \Big|\frac{1}{a_n}\sum_{i=1}^{n}\mathsf{E}X_i^{(a_n)}\Big|\Big). \end{aligned} \tag{2.4}$$

Firstly, we will prove that

$$\Big|\frac{1}{a_n}\sum_{i=1}^{n}\mathsf{E}X_i^{(a_n)}\Big| \to 0, \qquad \text{as } n \to \infty. \tag{2.5}$$

By the conditions of $g_n(|t|)$ and (2.1), we have that

$$\begin{aligned} \Big|\frac{1}{a_n}\sum_{i=1}^{n}\mathsf{E}X_i^{(a_n)}\Big| &\leq \sum_{i=1}^{n}\mathsf{P}(|X_i| > a_n) + \Big|\frac{1}{a_n}\sum_{i=1}^{n}\mathsf{E}X_i I(|X_i| \leq a_n)\Big| \\ &\leq \sum_{i=1}^{n}\frac{\mathsf{E}g_i(|X_i|)}{g_i(a_n)} + \frac{1}{a_n}\sum_{i=1}^{n}\mathsf{E}|X_i| I(|X_i| \leq a_n) \\ &\leq \sum_{i=1}^{n}\frac{\mathsf{E}g_i(|X_i|)}{g_i(a_n)} + \sum_{i=1}^{n}\frac{\mathsf{E}g_i(|X_i| I(|X_i| \leq a_n))}{g_i(a_n)} \\ &\leq 2\sum_{i=1}^{n}\frac{\mathsf{E}g_i(|X_i|)}{g_i(a_n)} \to 0, \qquad \text{as } n \to \infty, \end{aligned} \tag{2.6}$$

which implies (2.5). It follows from (2.4) and (2.5) that for n large enough,

$$\mathsf{P}\Big(\Big|\frac{1}{a_n}\sum_{i=1}^{n} X_i\Big| > \varepsilon\Big) \leq \sum_{i=1}^{n}\mathsf{P}(|X_i| > a_n) + \mathsf{P}\Big(|T_n^{(a_n)}| > \frac{\varepsilon}{2}\Big). \tag{2.7}$$

Therefore, to prove (2.2), we need only to prove that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathsf{P}(|X_i| > a_n) < \infty; \tag{2.8}$$
$$\sum_{n=1}^{\infty}\mathsf{P}\Big(|T_n^{(a_n)}| > \frac{\varepsilon}{2}\Big) < \infty. \tag{2.9}$$

The conditions $g_n(|t|)\uparrow$ as $|t|\uparrow$ and (2.1) yield that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathsf{P}(|X_i|>a_n)\le\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{\mathsf{E}g_i(|X_i|)}{g_i(a_n)}<\infty, \tag{2.10}$$

which implies (2.8).

Actually, by the conditions of $g_n(|t|)/|t|\downarrow$ as $|t|\uparrow$, (2.1), Lemma 2.2 and Markov's inequality, we can see that

$$\begin{aligned}\sum_{n=1}^{\infty}\mathsf{P}\Big(|T_n^{(a_n)}|>\frac{\varepsilon}{2}\Big) &\le C\sum_{n=1}^{\infty}\mathsf{E}(|T_n^{(a_n)}|^2)\\ &\le C\sum_{n=1}^{\infty}\frac{1}{a_n^2}\mathsf{E}\Big(\sum_{i=1}^{n}|X_i^{(a_n)}|^2\Big)\le C\sum_{n=1}^{\infty}\frac{1}{a_n^2}\sum_{i=1}^{n}\mathsf{E}|X_i^{(a_n)}|^2\\ &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathsf{P}(|X_i|>a_n)+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{\mathsf{E}|X_i|^2I(|X_i|\le a_n)}{a_n^2}\\ &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{\mathsf{E}g_i(|X_i|)}{g_i(a_n)}+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{\mathsf{E}|X_i|I(|X_i|\le a_n)}{a_n}\\ &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{\mathsf{E}g_i(|X_i|)}{g_i(a_n)}+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{\mathsf{E}g_i(|X_i|)I(|X_i|\le a_n)}{g_i(a_n)}\\ &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{\mathsf{E}g_i(|X_i|)}{g_i(a_n)}<\infty,\end{aligned}$$

which implies (2.9). The proof of Theorem 2.1 is completed. □

Corollary 2.1　Under the conditions of Theorem 2.1,

$$\frac{1}{a_n}\sum_{i=1}^{n}X_i\to 0\ \text{ a.s.,}\qquad \text{as }\ n\to\infty. \tag{2.11}$$

Theorem 2.2　Let $\{X_n;n\ge1\}$ be a sequence of pairwise NQD random variables and $\{a_n;n\ge1\}$ be a sequence of positive real numbers such that $0<a_n\uparrow\infty$. Let $\{g_n(t);n\ge1\}$ be a sequence of positive, even functions such that $g_n(|t|)$ is an increasing function of $|t|$ for all $n\ge1$. Assume that there exists a constant $\delta>0$ such that $g_n(t)\ge\delta t$ for $0<t\le1$. If

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big)<\infty, \tag{2.12}$$

then for $\forall\,\varepsilon>0$, (2.2) holds true.

Proof of Theorem 2.2　We use the same notations as that in Theorem 2.1. The proof is similar to that of Theorem 2.1.

Firstly, we shall show that (2.5) holds true. Actually, by $g_n(t)\ge\delta t$ for $0<t\le1$ and

(2.12), we can see that

$$\begin{aligned}\Big|\frac{1}{a_n}\sum_{i=1}^{n}\mathsf{E}X_i^{(a_n)}\Big| &\le \sum_{i=1}^{n}\mathsf{P}(|X_i|>a_n)+\Big|\frac{1}{a_n}\sum_{i=1}^{n}\mathsf{E}X_iI(|X_i|\le a_n)\Big|\\ &\le \frac{1}{\delta}\sum_{i=1}^{n}\mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big)+\frac{1}{\delta}\sum_{i=1}^{n}\mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big)I(|X_i|\le a_n)\\ &\le C\sum_{i=1}^{n}\mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big)\to 0,\qquad \text{as } n\to\infty,\end{aligned}\tag{2.13}$$

which implies (2.5).

According to the proof of Theorem 2.1, we need only to prove that (2.8) and (2.9) hold true.

When $|X_i|>a_n>0$, we know that $g_n(|X_i|/a_n)\ge g_n(1)\ge\delta$, which yields that

$$\mathsf{P}(|X_i|>a_n)=\mathsf{E}I(|X_i|>a_n)\le\frac{1}{\delta}\mathsf{E}g_n\Big(\frac{|X_i|}{a_n}\Big).\tag{2.14}$$

Hence,

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathsf{P}(|X_i|>a_n)\le\frac{1}{\delta}\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big)<\infty,\tag{2.15}$$

which implies (2.8).

In fact, by $g_n(t)\ge\delta t$ for $0<t\le 1$, (2.12), Lemma 2.2 and Markov's inequality, we can obtain that

$$\begin{aligned}\sum_{n=1}^{\infty}\mathsf{P}\Big(|T_n^{(a_n)}|>\frac{\varepsilon}{2}\Big) &\le C\sum_{n=1}^{\infty}\mathsf{E}|T_n^{(a_n)}|^2\le C\sum_{n=1}^{\infty}\frac{1}{a_n^2}\mathsf{E}\Big(\sum_{i=1}^{n}|X_i^{(a_n)}|^2\Big)\\ &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathsf{P}(|X_i|>a_n)+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{\mathsf{E}|X_i|^2I(|X_i|\le a_n)}{a_n^2}\\ &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big)+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{\mathsf{E}|X_i|I(|X_i|\le a_n)}{a_n}\\ &\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big)+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big)I(|X_i|\le a_n)\\ &\le C+C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big)<\infty,\end{aligned}$$

which implies (2.9). The proof of Theorem 2.2 is completed. □

Corollary 2.2 Under the conditions of Theorem 2.2,

$$\frac{1}{a_n}\sum_{i=1}^{n}X_{ni}\to 0\ \text{ a.s.},\qquad \text{as } n\to\infty.\tag{2.16}$$

Corollary 2.3 Let $\{X_n;n\ge 1\}$ be a sequence of pairwise NQD random variables and $\{a_n;n\ge 1\}$ be a sequence of positive real numbers such that $0<a_n\uparrow\infty$. If there

exists some constant $\beta \in (0,1]$ such that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathsf{E}\Big(\frac{|X_i|^{\beta}}{|a_n|^{\beta}+|X_i|^{\beta}}\Big)<\infty, \tag{2.17}$$

Then for $\forall\,\varepsilon>0$, (2.2) holds true.

Proof of Corollary 2.3 In Theorem 2.2, we take

$$g_n(t)=\frac{|t|^{\beta}}{K+|t|^{\beta}},\qquad 0<\beta\le 1,\ K>0,\ n\ge 1.$$

It is easy to show that $\{g_n(t);n\ge 1\}$ is a sequence of positive, even functions such that $g_n(|t|)$ is an increasing function of $|t|$ for all $n\ge 1$. In addition

$$g_n(t)\ge\frac{1}{2}|t|^{\beta}\ge\frac{1}{2}t,\qquad 0<t\le 1,\ 0<\beta\le 1.$$

Then, by the proof of Theorem 2.2, we can easily obtain (2.2). The proof of Corollary 2.3 is completed. □

Theorem 2.3 Let $\{X_n;n\ge 1\}$ be a sequence of pairwise NQD random variables with $\mathsf{E}X_i=0$ and $\{a_n;n\ge 1\}$ be a sequence of positive real numbers such that $0<a_n\uparrow\infty$. Let $\{g_n(t);n\ge 1\}$ be a sequence of positive, even functions. Assume that there exists some constant $\beta\in(1,2]$ and a constant $\delta>0$ such that $g_n(t)\ge\delta t^{\beta}$ for $0<t\le 1$ and there exists $\delta>0$ such that $g_n(t)\ge\delta t$ for $t>1$. If (2.12) satisfies, then for $\forall\,\varepsilon>0$, (2.2) holds still true.

Proof of Theorem 2.3 The proof is similar to that of Theorem 2.1.

Firstly, we shall show that (2.5) holds true. Actually, by the conditions of $\mathsf{E}X_i=0$, $g_n(t)\ge\delta t$ for $t>1$ and (2.12), we can have that

$$\begin{aligned}\Big|\frac{1}{a_n}\sum_{i=1}^{n}\mathsf{E}X_i^{(a_n)}\Big| &\le \sum_{i=1}^{n}\mathsf{P}(|X_i|>a_n)+\Big|\frac{1}{a_n}\sum_{i=1}^{n}\mathsf{E}X_iI(|X_i|>a_n)\Big|\\ &\le \frac{1}{\delta}\sum_{i=1}^{n}\mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big)+\frac{1}{\delta}\sum_{i=1}^{n}\mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big)I(|X_i|>a_n)\\ &\le \frac{2}{\delta}\sum_{i=1}^{n}\mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big)\to 0,\qquad \text{as}\ \ n\to\infty,\end{aligned} \tag{2.18}$$

which implies (2.5). Hence, to prove (2.2), we need only to prove that (2.8) and (2.9) hold true.

The conditions $g_n(t) \ge \delta t$ for $t > 1$ and (2.12) yield that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{P}(|X_i| > a_n) &= \sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{E} I(|X_i| > a_n) \\
&\le \sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{E}\Big(\frac{|X_i|}{a_n} I(|X_i| > a_n)\Big) \\
&\le \frac{1}{\delta}\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{E} g_i\Big(\frac{|X_i|}{a_n}\Big) I(|X_i| > a_n) \\
&\le \frac{1}{\delta}\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{E} g_i\Big(\frac{|X_i|}{a_n}\Big) < \infty, \qquad (2.19)
\end{aligned}
$$

which implies (2.8).

By the conditions of $g_n(t) \ge \delta t^{\beta}$ for $1 < \beta \le 2$, $0 < t \le 1$, (2.12), Lemma 2.2 and Markov's inequality, we can see that

$$
\begin{aligned}
\sum_{n=1}^{\infty} \mathsf{P}\Big(|T_n^{(a_n)}| > \frac{\varepsilon}{2}\Big) &\le C\sum_{n=1}^{\infty} \mathsf{E}(|T_n^{(a_n)}|^2) \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{P}(|X_i| > a_n) + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{\mathsf{E}|X_i|^2 I(|X_i| \le a_n)}{a_n^2} \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{E} g_i\Big(\frac{|X_i|}{a_n}\Big) + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{E} g_i\Big(\frac{|X_i|}{a_n}\Big) I(|X_i| \le a_n) \\
&\le C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{E} g_i\Big(\frac{|X_i|}{a_n}\Big) < \infty, \qquad (2.20)
\end{aligned}
$$

which implies (2.9). The proof of Theorem 2.3 is completed. □

Corollary 2.4 Let $\{X_n; n \ge 1\}$ be a sequence of pairwise NQD random variables with $\mathsf{E}X_i = 0$ and $\{a_n; n \ge 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. If there exists some constant $\beta \in (1, 2]$ such that

$$
\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{E}\Big(\frac{|X_i|^{\beta}}{a_n|X_i|^{\beta-1} + a_n^{\beta}}\Big) < \infty, \qquad (2.21)
$$

then for $\forall\, \varepsilon > 0$, (2.2) holds true.

Proof of Corollary 2.4 In Theorem 2.3, we take

$$
g_n(t) = \frac{|t|^{\beta}}{K + |t|^{\beta-1}}, \qquad 1 < \beta \le 2,\ K > 0,\ n \ge 1.
$$

It is easy to show that $\{g_n(t); n \ge 1\}$ is a sequence of positive, even functions such that

$$
g_n(t) \ge \frac{1}{2}|t|^{\beta} \ \text{ for } \ 0 < t \le 1,\ 1 < \beta \le 2 \qquad \text{and} \qquad g_n(t) \ge \frac{1}{2}|t| \ \text{ for } \ t > 1.
$$

Then, by the proof of Theorem 2.3, we can easily obtain (2.2). The proof of Corollary 2.4 is completed. □

From Corollary 2.3 and Corollary 2.4, we can summarize the following strong law of large numbers for pairwise NQD random sequences.

Corollary 2.5 Let $\{X_n; n \geq 1\}$ be a sequence of pairwise NQD random variables and $\{a_n; n \geq 1\}$ be a sequence of positive real numbers. If there exists some constant $\beta \in (0, 2]$ such that

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{\mathsf{E}|X_i|^{\beta}}{a_n^{\beta}} < \infty, \tag{2.22}$$

and $\mathsf{E}X_n = 0$, $n \geq 1$ if $\beta \in (1, 2]$, then (2.2) holds true and $(1/a_n) \sum_{i=1}^{n} X_i \to 0$ a.s..

Theorem 2.4 Let $\{X_n; n \geq 1\}$ be a sequence of pairwise NQD random variables and $\{a_n; n \geq 1\}$ be a sequence of positive real numbers such that $0 < a_n \uparrow \infty$. Let $\{g_n(t); n \geq 1\}$ be a sequence of positive, even functions. Assume that there exists some constant$\beta \in [2, \infty)$ and a constant $\delta > 0$ such that $g_n(t) \geq \delta t^{\beta}$ for $t > 0$. If

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \Big(\mathsf{E}g_i\Big(\frac{X_i}{a_n}\Big)\Big)^{1/\beta} < \infty, \tag{2.23}$$

then for $\forall\, \varepsilon > 0$, (2.2) holds still true.

Proof of Theorem 2.4 We shall use the same notations as that in Theorem 2.1. The proof is as follows.

It follows from (2.23) that

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \mathsf{E}g_i\Big(\frac{X_i}{a_n}\Big) < \infty; \tag{2.24}$$

and

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n} \Big(\mathsf{E}g_i\Big(\frac{X_i}{a_n}\Big)\Big)^{2/\beta} < \infty. \tag{2.25}$$

Firstly, we shall show that (2.5) holds true. Actually, by the conditions of $g_n(t) \geq \delta t^{\beta}$ for $t > 0$, (2.23) and (2.24), we can get that

$$\begin{aligned}
\Big|\frac{1}{a_n} \sum_{i=1}^{n} \mathsf{E}X_i^{(a_n)}\Big| &\leq \sum_{i=1}^{n} \mathsf{P}(|X_i| > a_n) + \Big|\frac{1}{a_n} \sum_{i=1}^{n} \mathsf{E}X_i I(|X_i| \leq a_n)\Big| \\
&\leq \sum_{i=1}^{n} \mathsf{E}\Big(\frac{|X_i|^{\beta}}{a_n^{\beta}} I(|X_i| > a_n)\Big) + \sum_{i=1}^{n} \mathsf{E}\Big(\frac{|X_i|^{\beta}}{a_n^{\beta}} I(|X_i| \leq a_n)\Big)^{1/\beta} \\
&\leq C \sum_{i=1}^{n} \mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big) + C \sum_{i=1}^{n} \Big(\mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big) I(|X_i| \leq a_n)\Big)^{1/\beta} \\
&\leq C \sum_{i=1}^{n} \mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big) + C \sum_{i=1}^{n} \Big(\mathsf{E}g_i\Big(\frac{|X_i|}{a_n}\Big)\Big)^{1/\beta} \to 0, \qquad \text{as } n \to \infty,
\end{aligned} \tag{2.26}$$

which implies (2.5). Hence, to prove (2.2), we need only to prove that (2.8) and (2.9) hold true.

The conditions of $g_n(t) \geq \delta t^\beta$ for $t>0$ and (2.24) yield that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{P}(|X_i|>a_n) &= \sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{E} I(|X_i|>a_n)\\
&\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{E}\Big(\frac{|X_i|^\beta}{a_n^\beta} I(|X_i|>a_n)\Big)\\
&\leq \frac{1}{\delta}\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{E} g_i\Big(\frac{|X_i|}{a_n}\Big) I(|X_i|>a_n)\\
&\leq \frac{1}{\delta}\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{E} g_i\Big(\frac{|X_i|}{a_n}\Big) < \infty, \qquad (2.27)
\end{aligned}
$$

which implies (2.8).

In fact, by the conditions of $g_n(t)\geq \delta t^\beta$ for $t>0$, (2.25), Lemma 2.2 and Markov's inequality, we can obtain that

$$
\begin{aligned}
\sum_{n=1}^{\infty} \mathsf{P}\Big(|T_n^{(a_n)}| > \frac{\varepsilon}{2}\Big) &\leq C\sum_{n=1}^{\infty} \mathsf{E}(|T_n^{(a_n)}|^2)\\
&\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{P}(|X_i|>a_n) + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{\mathsf{E}|X_i|^2 I(|X_i|\leq a_n)}{a_n^2}\\
&\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{P}(|X_i|>a_n) + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \Big(\mathsf{E}\frac{|X_i|^\beta}{a_n^\beta} I(|X_i|\leq a_n)\Big)^{2/\beta}\\
&\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathsf{E} g_i\Big(\frac{|X_i|}{a_n}\Big) + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \Big(\mathsf{E} g_i\Big(\frac{|X_i|}{a_n}\Big) I(|X_i|\leq a_n)\Big)^{2/\beta}\\
&\leq C + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} \Big(\mathsf{E} g_i\Big(\frac{|X_i|}{a_n}\Big)\Big)^{2/\beta} < \infty, \qquad (2.28)
\end{aligned}
$$

which implies (2.9). The proof of Theorem 2.4 is completed. □

Corollary 2.6 Under the conditions of Theorem 2.4,

$$
\frac{1}{a_n}\sum_{i=1}^{n} X_i \to 0 \ \text{ a.s.,} \qquad \text{as } \ n\to\infty.
$$

Remark 1 In Theorem 2.1; Theorem 2.2 and Theorem 2.4, if the positive even functions $g_n(|t|)$ satisfy the corresponding assumptions, then the results are still true for arbitrary random sequence by using the other method. In deed, by the proof of (2.3) in Theorem 2.1, we have

$$
\begin{aligned}
\mathsf{P}\Big(\Big|\frac{1}{a_n}\sum_{i=1}^{n} X_i\Big| > \varepsilon\Big) &\leq \mathsf{P}\Big(\max_{1\leq i\leq n}|X_i|>a_n\Big) + \mathsf{P}\Big(\Big|\frac{1}{a_n}\sum_{i=1}^{n} X_i^{(a_n)}\Big| > \varepsilon\Big)\\
&\leq \sum_{i=1}^{n} \mathsf{P}(|X_i|>a_n) + \mathsf{P}\Big(\Big|\frac{1}{a_n}\sum_{i=1}^{n} X_i^{(a_n)}\Big| > \varepsilon\Big).
\end{aligned}
$$

It follows from Markov's inequality that

$$\begin{aligned}
\mathsf{P}\Big(\Big|\frac{1}{a_n}\sum_{i=1}^{n}X_i^{(a_n)}\Big|>\varepsilon\Big) &\leq C\mathsf{E}\Big|\frac{1}{a_n}\sum_{i=1}^{n}X_i^{(a_n)}\Big| \\
&\leq C\sum_{i=1}^{n}\mathsf{P}(|X_i|>a_n)+C\frac{1}{a_n}\sum_{i=1}^{n}\mathsf{E}|X_i|I(|X_i|\leq a_n) \\
&\leq C\sum_{i=1}^{n}\frac{\mathsf{E}g_i(|X_i|)}{g_i(a_n)}.
\end{aligned}$$

Hence, the result is follows by the condition (2.1).

Remark 2 In Theorem 2.1; Theorem 2.2; Theorem 2.3 and Theorem 2.4, the positive even function $g_n(|t|)$ are different from the condition (1.5) of Theorem 1.2 mentioned in Section 1. In addition, the conditions of (2.12) and (2.23) differ from the condition (1.6), respectively. So, the results obtained extend and generalize the corresponding result of Gan and Chen (2008).

Remark 3 In Corollary 2.3 and Corollary 2.4, the conditions of (2.17) and (2.21) are weaker than the condition (1.2) and condition (1.3) of Theorem 1.1 mentioned in Section 1, however, the obtained results are stronger than the corresponding result of Liu (2004).

References

[1] Hsu, P.L. and Robbins, H., Complete convergence and the law of large numbers, *Proceedings of the National Academy of Sciences of the United States of America*, **33**(1947), 25–31.

[2] Katz, M., The probability in the tail of a distribution, *The Annals of Mathematical Statistics*, **34(1)**(1963), 312–318.

[3] Lehmann, E.L., Some concepts of dependence, *The Annals of Mathematical Statistics*, **37(5)**(1966), 1137–1153.

[4] Matula, P., A note on the almost sure convergence of sums of negatively dependent random variables. *Statistics and Probability Letters*, **15(3)**(1992), 209–213.

[5] Wang, Y.B., Su, C. and Liu, X.G., On some limit properties for pairwise NQD sequences, *Acta Mathematicae Applicatae Sinica*, **21(3)**(1998), 404–414. (in Chinese)

[6] Wang, Y.B., Yan, J.G., Cheng, F.Y. and Cai, X.Z., On the strong stability for Jamison type weighted product sums of pairwise NQD series with different distribution, *Chinese Annals of Mathematics (Series A)*, **22A(6)**(2001), 701–706. (in Chinese)

[7] Wu, Q.Y., Convergence properties of pairwise NQD sequences, *Acta Mathematica Sinica*, **45(3)** (2002), 617–624. (in Chinese)

[8] Liu, L., A strong law of large numbers for pairwise NQD random sequences, *Chinese Journal of Applied Probability and Statistics*, **20(2)**(2004), 184–188. (in Chinese)

[9] Gan, S.X. and Chen, P.Y., Some limit theorems for sequences of pairwise NQD random variables, *Acta Mathematica Scientia*, **28(2)**(2008), 269–281.

[10] Chen, P.Y., On the strong law of large numbers for pairwise NQD random variables, *Acta Mathematica Scientia*, **25A(3)**(2005), 386–392. (in Chinese)

[11] Li, R. and Yang, W.G., Strong convergence of pairwise NQD random sequences, *Journal of Mathematical Analysis and Applications*, **344(2)**(2008), 741–747.

[12] Zhang, L.X. and Wang, J.F., A note on complete convergence of pairwise NQD random equences, *Applied Mathematics - A Journal of Chinese Universities* (*Series A*), **19(2)**(2004), 203–208. (in Chinese)

[13] Wan, C.G., Law of large numbers and complete convergence for pairwise NQD random sequences, *Acta Mathematicae Applicatae Sinica*, **28(2)**(2005), 253–261. (in Chinese)

[14] Chen, P.Y., L^r convergence for pairwise NQD random variables, *Acta Mathematica Scientia*, **28A(3)**(2008), 447–453. (in Chinese)

[15] Huang, H.W., Wang, D.C., Wu, Q.Y. and Zhang, Q.X., A note on the complete convergence for sequences of pairwise NQD random variables, *Journal of Inequalities and Applications*, 2011, 2011:92, doi:10.1186/1029-242X-2011-92.

两两NQD随机序列的完全收敛性

黄海午[1] 王定成[2,3] 吴群英[1] 彭江艳[2]

([1]桂林理工大学理学院, 桂林, 541004; [2]电子科技大学数学科学学院, 成都, 610054)

([3]南京审计学院金融工程中心, 南京, 211815)

本文进一步研究了两两NQD随机序列的完全收敛性. 在一些简单和弱的条件下获得了两两NQD随机序列的一些结果. 所获结果不仅推广了Liu (2004)以及Gan和Chen (2008)的相应结果, 而且还改进了他们的结论.

关 键 词: 两两NQD随机序列, 完全收敛性, 强大数定律.

学科分类号: O211.4.