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Optimal Dividend and Capital Injection Strategies with Transaction Costs and Exponentially Distributed Observation Time *

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Abstract

In this paper, we consider the optimal joint dividend and capital injection strategy with proportional and fixed costs. It supposes that capitals can be injected whenever they are profitable, but dividends can only be paid at the arrival times of a Poisson process with intensity $\gamma > 0$. Our objective is to determine an optimal strategy of maximizing the expected cumulative discounted dividends minus the expected discounted costs of capital injections before bankruptcy. By solving some impulse problems, we get the closed-form solutions depending on the parameters of model. Some known results in Løkka and Zervos (2008) can be viewed as limiting cases when $\gamma \to \infty$.

Keywords: Dividend payment, capital injection, transaction cost, optimal strategy, exponentially distributed observation time.

AMS Subject Classification: 93E20, 62P05, 49L20.

§1. Introduction

Dividend payment and capital injection are two common approaches to control the company's surplus. The expected cumulative discounted dividends minus the expected discounted costs of capital injections before bankruptcy can be regarded as the company's value, the management seeks the optimal join dividend payment and capital injection strategies that maximize this value. Løkka and Zervos (2008) make a good contribution to this topic, which consider a diffusion model with proportion costs for capital injections. The optimal strategy happens to be either a dividend barrier strategy without capital injections, or another dividend barrier strategy with forced injections when surplus is null

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to prevent bankruptcy. The tradeoff between two choices depends on the parameters of the model. By adopting their technique, some extended results are obtained in other models. See, He and Liang (2009), Yao et al. (2010) and Meng and Siu (2011).

Note that above all literature assume the dividends can be paid at any time. In practice, it is more reasonable for the board of the company to check the balance on a periodic basis and then decide whether to pay dividends to shareholders, resulting in lump sum dividend payments at such discrete time points rather than continuous payment streams. So Albrecher et al. (2011a) suppose that the dividends can only be paid to shareholders at the arrival times of a Poisson process with rate $\gamma > 0$. A few extended results have been obtained in other papers. For instance, Albrecher et al. (2011b, 2011c), Wei et al. (2012) and Peng et al. (2013). As far as we know, there is no papers concerning with optimal joint dividend and capital injection strategy with discrete observation time. In this paper we suppose the company is bankrupt and has to go out of the business whenever the surplus is negative. To prevent the bankruptcy, capital injection is allowed if it is profitable. However, the dividends can only be paid to shareholders at the arrival times of a Poisson process $M(t)$ with rate $\gamma > 0$. To be more realistic, we take into account the proportional and fixed transaction costs for the dividend payment process and capital injection process in the model. Correspondingly, the appearance of cost makes thing more complex, some impulse problems arise. For some related research, we can see Paulsen (2008), Bai et al. (2010), Meng and Siu (2011) and Yao et al. (2011).

Here is a brief outline of this paper. Section 2 introduces the framework of this paper and formulates the general optimization problems concerning with dividend payments and capital injections. In Section 3, we solve a suboptimal problem without capital injections. In Section 4, we solve a suboptimal problem that arises when the admissible strategies are constrained to allow for no bankruptcy. Finally, by comparing the solutions of above two suboptimal problems, we identify the closed-form solutions to the general optimal problems in Section 5, which depend on the relationships among the parameters of risk model. Some known results in Løkka and Zervos (2008) are extended.

§2. Formulation of the General Optimal Control Problem

Suppose that the surplus of a large company at time t follows

$$X_t = x + \mu t + \sigma B_t,$$

where $x \geq 0$ represents the reserves at time zero, parameters μ and $\sigma > 0$ are fixed and $\{B_t\}_{t\geq 0}$ is a standard Brownian motion adapted to information filtration $\{\mathcal{F}_t\}_{t\geq 0}$ in the probability space $(\Omega, \mathcal{F}, \mathsf{P})$. It is worth mentioning that we allow for the possibility $\mu \leq 0$

in this paper. Let $\{M(t)\}_{t\geq 0}$ be a Poisson process with intensity $\gamma > 0$ which is assumed to be independent of $\{B_t\}_{t\geq 0}$. Suppose that the process $\{\eta_t\}_{t\geq 0}$ denotes the amount of dividends, which can only be paid at the jump times of the Poisson process $\{M(t)\}_{t\geq 0}$. Let L_t denote the cumulative amount of dividends paid from time zero up to time t, then we can write that

$$L_t = \int_0^t \eta_s \mathrm{d}M(s). \tag{2.1}$$

The capital injection process $\Big\{G_t = \sum\limits_{n=1}^{\infty} I_{\{\tau_n \leq t\}} \xi_n\Big\}$ is described by a sequence of increasing stopping times $\{\tau_n, n = 1, 2, \ldots\}$ and a sequence of random variables $\{\xi_n, n = 1, 2, \ldots\}$, which represent the times and the sizes of capital injections, respectively. A control policy π is described by $\pi = \{L^\pi; G^\pi\} = \{L^\pi; \tau_1^\pi, \ldots, \tau_n^\pi, \ldots; \xi_1^\pi, \ldots, \xi_n^\pi, \ldots\}$. The controlled surplus process associated with π follows that

$$X_t^\pi = x + \mu t + \sigma B_t - L_t^\pi + \sum_{n=1}^{\infty} I_{\{\tau_n^\pi \leq t\}} \xi_n^\pi. \tag{2.2}$$

Definition 2.1　A strategy π is said to be admissible if

(i) $L_t^\pi = \displaystyle\int_0^t \eta_s^\pi \mathrm{d}M(s)$ is an increasing, $\{\mathcal{F}_t\}$-adapted càdlàg process.

(ii) τ_n^π is a stopping time w.r.t. $\{\mathcal{F}_t\}$, and $0 \leq \tau_1^\pi < \cdots < \tau_n^\pi < \cdots$, a.s..

(iii) $\xi_n^\pi > 0$ is measurable w.r.t. $\mathcal{F}_{\tau_n^\pi}$.

(iv) $\mathsf{P}\big(\lim\limits_{n\to\infty} \tau_n^\pi < T\big) = 0$, $\forall T > 0$.

Denote the set of all admissible strategies by Π. Define the time of bankruptcy by

$$\tau := \tau^\pi = \inf\{t \geq 0 : X_t^\pi < 0\},$$

which is an $\{\mathcal{F}_t\}$-stopping time. The company needs to keep its surplus non-negative, or else, the bankruptcy happens. The company's value associated with $\pi \in \Pi$ is described by the following performance function

$$V(x,\pi) = \mathsf{E}^x\Big(\beta_1 \int_0^{\tau^\pi} e^{-\delta s} \eta_s^\pi \mathrm{d}M(s) - \sum_{n=1}^{\infty} e^{-\delta \tau_n^\pi} (\beta_2 \xi_n^\pi + K) I_{\{\tau_n^\pi \leq \tau^\pi\}}\Big), \tag{2.3}$$

which is the expected present value of dividends less the discount costs of capital injections until bankruptcy (may be infinity). Here E^x is the mathematical expectation corresponding to $X_0^\pi = x$; $\delta > 0$ denotes the discount rate reflecting the time preference of shareholders. We assume that the shareholders need to pay $\beta_2\xi + K$, $\beta_2 > 1$, to meet the capital injection of ξ. $(\beta_2 - 1)\xi$ is the proportional transaction costs, $K > 0$ is the fixed transaction costs. Proportional costs on dividend transaction are taken into account through the value of β_1, with $0 < \beta_1 \leq 1$ representing the net proportion of leakages from the surplus received by investors after transaction costs have been paid. We are interested in finding the value function

$$V(x) = \sup_{\pi\in\Pi} V(x,\pi), \tag{2.4}$$

and the associated optimal strategy $\widehat{\pi} \in \Pi$ such that $V(x) = V(x,\widehat{\pi})$.

To tackle this optimal control problem, we need to solve the associated HJB (Hamilton-Jacobi-Bellman) equation satisfied by value function. It supposes that all value functions appearing in this paper are sufficiently smooth and regular to perform. Since the derivations of the HJB equations are standard in the theory of stochastic control, they are omitted in the rest of the paper. To develop our result, for any function $\omega(x) \in \mathbb{C}^2$, we define the capital injection operator $\mathscr{M}$ by

$$\mathscr{M}\omega(x) = \max_{y\geq 0}\{\omega(x+y) - \beta_2 y - K\}, \tag{2.5}$$

and the infinitesimal operator $\mathscr{L}^a$ by

$$\mathscr{L}^a\omega(x) = \frac{1}{2}\sigma^2\omega''(x) + \mu\omega'(x) - \delta\omega(x) + \gamma[\omega(x-a) + \beta_1 a - \omega(x)]. \tag{2.6}$$

Next, we need to consider two categories of suboptimal models in the following two sections. By comparing the solutions for suboptimal problems, the general optimal control problem are solved.

§3. Suboptimal Problem without Capital Injection

In this section, we consider a suboptimal problem without considering capital injections. Denote $\pi_p = \{L^{\pi_p}; G^{\pi_p}\} = \{L^{\pi_p}; 0\} \in \Pi$ stand for the control process in which capital injection is not allowed. Then the performance function associated with π_p becomes

$$V(x,\pi_p) = \mathsf{E}^x\Big(\beta_1\int_0^{\tau^{\pi_p}} e^{-\delta s}\eta_s^{\pi_p}\mathrm{d}M(s)\Big). \tag{3.1}$$

Our objective is to find the value function

$$V_p(x) = \sup_{\pi_p\in\Pi} V(x,\pi_p), \tag{3.2}$$

and the associated optimal strategy $\pi_p^* = \{L^{\pi_p^*}; 0\} \in \Pi$ such that $V_p(x) = V(x,\pi_p^*)$.

With reference to the theory of optimal control, $V_p(x)$ should satisfy the HJB equation

$$\max_{a\geq 0}\{\mathscr{L}^a V_p(x)\} = 0 \tag{3.3}$$

with the boundary condition

$$V_p(0) = 0. \tag{3.4}$$

Theorem 3.1 Let $f(x) \in \mathbb{C}^2$ be an increasing, concave solution of (3.3)-(3.4) and the derivative $f'(x)$ is bounded, then we have the following statements:

(i) $f(x) \geq V_p(x)$ for all $x \geq 0$.

(ii) If there exists some strategy $\pi_p^* = \{L^{\pi_p^*}; 0\} \in \Pi$ such that $f(x) = V(x, \pi_p^*)$, then $f(x) = V_p(x)$ and π_p^* is optimal.

Proof The proof of (i) is similar to Appendix A, it is omitted here. The result of (ii) comes from the optimality of $V_p(x)$. □

Theorem 3.2 The value function $V_p(x)$ coincides with

$$f(x) = \begin{cases} A_0(b_p^*)e^{r_0 x} + A_1(b_p^*)e^{r_1 x}, & \mu > 0,\ \gamma > \delta^2\sigma^2/(2\mu^2) \ \text{and}\ 0 \leq x \leq b_p^*; \\ B_0(b_p^*)e^{s_0 x} + Cx + E(b_p^*), & \mu > 0,\ \gamma > \delta^2\sigma^2/(2\mu^2) \ \text{and}\ x \geq b_p^*; \\ \widetilde{B}_0 e^{s_0 x} + \widetilde{C}x + \widetilde{E}, & \mu > 0,\ \gamma \leq \delta^2\sigma^2/(2\mu^2) \ \text{and}\ x \geq 0; \\ \widetilde{B}_0 e^{s_0 x} + \widetilde{C}x + \widetilde{E}, & \mu \leq 0 \ \text{and}\ x \geq 0, \end{cases} \tag{3.5}$$

in which the parameters $A_0(b_p^*), A_1(b_p^*), B_0(b_p^*), C, E(b_p^*), \widetilde{B}_0, \widetilde{C}, \widetilde{E}$ and b_p^* are determined in the following proof process. The associated optimal strategy $\pi_p^* = \{L^{\pi_p^*}; 0\}$ is given by

$$L_t^{\pi_p^*} = \int_0^t (X_s^{\pi_p^*} - b_p^*)_+ \mathrm{d}M(s), \tag{3.6}$$

which is a modified barrier strategy with the level $b_p^* \geq 0$. If the surplus exceeds the barrier b_p^* at the moment when the board of the company is checking the balance, the excess $\eta_s^{\pi_p^*} = (X_s^{\pi_p^*} - b_p^*)_+$ is paid out immediately as dividends.

Proof We try to find a concave solution $f(x)$ to (3.3) and (3.4). Thereby the barrier $b_p^* > 0$ such that $f'(b_p^*) = \beta_1$ exists if and only if $f'(0) > \beta_1$ holds. Let us consider the first case with $f'(0) > \beta_1$, then

$$\max_{a \geq 0}\{\mathscr{L}^a f(x)\} = \mathscr{L}^0 f(x) = 0, \qquad 0 < x \leq b_p^*, \tag{3.7}$$

$$\max_{a \geq 0}\{\mathscr{L}^a f(x)\} = \mathscr{L}^{x - b_p^*} f(x) = 0, \qquad x \geq b_p^*. \tag{3.8}$$

Specifically,

$$\frac{1}{2}\sigma^2 f''(x) + \mu f'(x) - \delta f(x) = 0, \qquad 0 < x \leq b_p^*, \tag{3.9}$$

$$\frac{1}{2}\sigma^2 f''(x) + \mu f'(x) - \delta f(x) + \gamma[f(b_p^*) + \beta_1(x - b_p^*) - f(x)] = 0, \qquad x \geq b_p^*. \tag{3.10}$$

By observing the structure of above system of equations, we give the general solution as

$$f(x) = A_0(b_p^*)e^{r_0 x} + A_1(b_p^*)e^{r_1 x}, \qquad 0 \leq x \leq b_p^*, \tag{3.11}$$

$$f(x) = B_0(b_p^*)e^{s_0 x} + B_1(b_p^*)e^{s_1 x} + C(b_p^*)x + D(b_p^*), \qquad x \geq b_p^*. \tag{3.12}$$

Substituting (3.12) into (3.10) yields

$$C(b_p^*) := C := \frac{\beta_1\gamma}{\gamma+\delta}, \tag{3.13}$$

$$D(b_p^*) = \frac{1}{\delta+\gamma}\Big(\frac{\beta_1\gamma\mu}{\delta+\gamma} + \gamma(f(b_p^*) - \beta_1 b_p^*)\Big), \tag{3.14}$$

$$B_0(b_p^*) = \frac{\beta_1\delta}{(\gamma+\delta)s_0}e^{-s_0 b_p^*}, \tag{3.15}$$

$$s_{1,0} = \frac{1}{\sigma^2}\big(-\mu \pm \sqrt{\mu^2 + 2\sigma^2(\gamma+\delta)}\big), \tag{3.16}$$

and $B_1(b_p^*) = 0$ holds since $s_1 > 0$ and $f'(x)$ is bounded. Similarly, putting (3.11) into (3.9) gives

$$r_{1,0} = \frac{1}{\sigma^2}\big(-\mu \pm \sqrt{\mu^2 + 2\sigma^2\delta}\big). \tag{3.17}$$

We write the smooth pasting conditions as

$$f(b_p^*-) = f(b_p^*+), \tag{3.18}$$

$$f'(b_p^*-) = f'(b_p^*+), \tag{3.19}$$

$$f''(b_p^*-) = f''(b_p^*+). \tag{3.20}$$

Then (3.18)-(3.20) lead to

$$A_0(b_p^*) = \frac{\beta_1[\delta s_0 - r_1(\gamma+\delta)]}{r_0(r_0-r_1)(\gamma+\delta)}e^{-r_0 b_p^*} < 0, \tag{3.21}$$

$$A_1(b_p^*) = \frac{\beta_1[\delta s_0 - r_0(\gamma+\delta)]}{r_1(r_1-r_0)(\gamma+\delta)}e^{-r_1 b_p^*} > 0. \tag{3.22}$$

Finally, we rewrite the boundary condition (3.4) as

$$\psi(b_p^*) := \frac{\beta_1[\delta s_0 - r_1(\gamma+\delta)]}{r_0(r_0-r_1)(\gamma+\delta)}e^{-r_0 b_p^*} + \frac{\beta_1[\delta s_0 - r_0(\gamma+\delta)]}{r_1(r_1-r_0)(\gamma+\delta)}e^{-r_1 b_p^*} = 0, \tag{3.23}$$

which gives

$$b_p^* = \frac{1}{r_1 - r_0}\ln\Big(\frac{r_0(r_0(\delta+\gamma) - \delta s_0)}{r_1(r_1(\delta+\gamma) - \delta s_0)}\Big). \tag{3.24}$$

To ensure that $b_p^* > 0$, the condition

$$\frac{r_0(r_0(\delta+\gamma) - \delta s_0)}{r_1(r_1(\delta+\gamma) - \delta s_0)} > 1 \tag{3.25}$$

is required. Under the assumption of $\mu > 0$, some calculations show that (3.25) equals to

$$\gamma > \frac{\delta^2\sigma^2}{2\mu^2}. \tag{3.26}$$

In such case, we can verify that

$$f'(0) > \beta_1 \tag{3.27}$$

is true. Next let us consider the opposite case with $f'(0) \le \beta_1$, which leads to

$$\max_{a\ge 0}\{\mathscr{L}^a f(x)\} = \mathscr{L}^x f(x) = 0, \qquad x \ge 0. \tag{3.28}$$

Explicitly,

$$\frac{1}{2}\sigma^2 f''(x) + \mu f'(x) - \delta f(x) + \gamma[f(0) + \beta_1 x - f(x)] = 0, \qquad x \ge 0. \tag{3.29}$$

The candidate solution $f(x)$ takes the form as

$$f(x) = \widetilde{B}_0 e^{s_0 x} + \widetilde{C}x + \widetilde{E}, \qquad x \ge 0. \tag{3.30}$$

By plugging (3.30) in (3.29) and using condition $f(0) = 0$, we get

$$\widetilde{C} = C = \frac{\beta_1 \gamma}{\gamma + \delta}, \tag{3.31}$$

$$\widetilde{B}_0 = -\widetilde{E} = -\frac{\beta_1 \mu \gamma}{(\gamma + \delta)^2}. \tag{3.32}$$

Now, we can calculate that $f'(0) \le \beta_1$ holds if and only if either

$$0 < \gamma \le \frac{\delta^2 \sigma^2}{2\mu^2} \qquad \text{or} \qquad \mu \le 0 \tag{3.33}$$

happens. Finally, following the argument as that of Theorem 3.2 in Yao et al. (2010), we can prove that $f(x)$ is indeed a twice continuously differentiable, increasing, concave solution of (3.3) and (3.4) and $f'(x)$ is bounded. The optimality of π_p^* can be verified as doing in Appendix B. The proof procedure is omitted. $\square$

§4. Suboptimal Problem without Bankruptcy

In this section we require that the company survives forever by forced capital injections. Denote $\pi_r = \{L^{\pi_r}; G^{\pi_r}\} \in \Pi$ as the control process such that the company never goes bankrupt, so $\tau^{\pi_r} = \infty$. For each admissible strategy π_r, the performance function becomes

$$V(x, \pi_r) = \mathsf{E}^x\Big(\beta_1 \int_0^\infty e^{-\delta s}\eta_s^{\pi_r}\mathrm{d}M(s) - \sum_{n=1}^{\infty} e^{-\delta \tau_n^{\pi_r}}(\beta_2 \xi_n^{\pi_r} + K) I_{\{\tau_n^{\pi_r} < \infty\}}\Big). \tag{4.1}$$

Our objective is to find the value function

$$V_r(x) = \sup_{\pi_r \in \Pi} V(x, \pi_r), \tag{4.2}$$

and the associated optimal strategy $\pi_r^* = \{L^{\pi_r^*}; G^{\pi_r^*}\} \in \Pi$ such that $V_r(x) = V(x, \pi_r^*)$.

With reference to the theory of optimal control, $V_r(x)$ should satisfy the HJB equation

$$\max\left\{\max_{a\geq 0}\{\mathscr{L}^a V_r(x)\}, \mathscr{M}V_r(x) - V_r(x)\right\} = 0, \tag{4.3}$$

and the boundary condition

$$\mathscr{M}V_r(0) - V_r(0) = 0. \tag{4.4}$$

Because the time value of money, we know the optimal timing of capital injection can only come at the moments when the surplus process hits the barrier 0. Mathematically, the equation $\mathscr{M}V_r(x) = V_r(x)$ has a unique solution $x = 0$, the inequality $\mathscr{M}V_r(x) < V_r(x)$ holds strictly for $x > 0$. Actually, when the surplus reaches 0, we have to inject new capitals to prevent from bankruptcy, then the surplus jumps to some appropriate level $\xi^* > 0$ immediately. Hence the corresponding boundary condition is $V_r(0) = \mathscr{M}V_r(0) = V_r(\xi^*) - \beta_2\xi^* - K$. By the definition of operator $\mathscr{M}$, it has $\xi^* = \inf\{x : V_r'(x) = \beta_2\}$. We can construct a injection strategy $G^{\pi_r^*}$ by letting

$$\tau_1^{\pi_r^*} = \inf\{t \geq 0 : X_{t-}^{\pi_r^*} = 0\}, \tag{4.5}$$

$$\tau_n^{\pi_r^*} = \inf\{t > \tau_{n-1}^{\pi_r^*} : X_{t-}^{\pi_r^*} = 0\}, \qquad n = 2, 3, \ldots, \tag{4.6}$$

$$\xi_n^{\pi_r^*} \equiv \xi^*, \qquad n = 1, 2, 3, \ldots. \tag{4.7}$$

In addition, if we further assume that $V_r(x)$ is concave and there exists some number $b_r^* = \inf\{x : V_r'(x) = \beta_1\} > 0$, then the optimal dividend strategy should be a modified barrier strategy with the barrier b_r^*. Mathematically, $L_t^{\pi_r^*}$ satisfies

$$L_t^{\pi_r^*} = \int_0^t (X_s^{\pi_r^*} - b_r^*)_+ \mathrm{d}M(s), \tag{4.8}$$

i.e, $\eta_s^{\pi_r^*} = (X_s^{\pi_r^*} - b_r^*)_+$. The optimality of $\pi_r^* = \{L^{\pi_r^*}; G^{\pi_r^*}\} \in \Pi$ will be confirmed later.

Theorem 4.1 Let $g(x) \in \mathbb{C}^2$ be an increasing, concave solution of (4.3) and (4.4) and $g'(x)$ is bounded, then we have the following statements:

(i) $g(x) \geq V_r(x)$ for all $x \geq 0$.

(ii) If there exists some strategy $\pi_r^* = \{L^{\pi_r^*}; G^{\pi_r^*}\} \in \Pi$ such that $g(x) = V(x, \pi_r^*)$, then $g(x) = V_r(x)$ and π_r^* is optimal.

Proof The proof of (i) is similar to Appendix A, it is omitted here. The result of (ii) comes from the optimality of $V_r(x)$. □

Theorem 4.2 The value function $V_r(x)$ coincides with

$$g(x) = \begin{cases} A_0(b_r^*)e^{r_0x} + A_1(b_r^*)e^{r_1x}, & 0 \leq x \leq b_r^*; \\ B_0(b_r^*)e^{s_0x} + Cx + E(b_r^*), & x \geq b_r^*. \end{cases} \tag{4.9}$$

Correspondingly, the optimal strategy $\pi_r^* = \{L^{\pi_r^*}; G^{\pi_r^*}\}$ is given by (4.5)-(4.8). The unique pair (ξ^*, b_r^*) is determined by (4.10) and (4.11).

Proof Note that $V_r(x)$ and $V_p(x)$ satisfy the same HJB equation (see (3.3) and (4.3)) but with different boundary conditions (see (3.4) and (4.4)). Therefore $V_r(x)$ takes the same form as $g(x)$ in (4.9). Furthermore, by employing boundary condition (4.4), we will determine the barrier b_r^* and the optimal amount of injection $\xi^* \in (0, b_r^*)$ by the following equations

$$g'(\xi^*) = \beta_2, \tag{4.10}$$

$$g(0) = g(\xi^*) - \beta_2\xi^* - K. \tag{4.11}$$

For convenience, let us define an increasing function

$$\phi(z) := \frac{\beta_1[\delta s_0 - r_1(\gamma+\delta)]}{(r_0 - r_1)(\gamma+\delta)} e^{-r_0 z} + \frac{\beta_1[\delta s_0 - r_0(\gamma+\delta)]}{(r_1 - r_0)(\gamma+\delta)} e^{-r_1 z}, \qquad z \geq 0. \tag{4.12}$$

Then (4.10) can be written as

$$\phi(b_r^* - \xi^*) = \frac{\beta_1[\delta s_0 - r_1(\gamma+\delta)]}{(r_0 - r_1)(\gamma+\delta)} e^{-r_0(b_r^* - \xi^*)} + \frac{\beta_1[\delta s_0 - r_0(\gamma+\delta)]}{(r_1 - r_0)(\gamma+\delta)} e^{-r_1(b_r^* - \xi^*)} = \beta_2. \tag{4.13}$$

It is not difficult to see that

$$\phi(0) = \beta_1 < \beta_2, \qquad \phi'(0) > 0, \qquad \phi''(z) > 0, \qquad \lim_{z\to\infty} \phi(z) = \infty.$$

Above properties admit a unique solution $\Delta = b_r^* - \xi^* > 0$ satisfying (4.13). Recalling (3.23), we represent (4.11) as

$$\psi(b_r^*) - \psi(\Delta) + \beta_2(b_r^* - \Delta) + K = 0. \tag{4.14}$$

Denote that

$$Q(b) = \psi(b) - \psi(\Delta) + \beta_2(b - \Delta) + K, \qquad b \geq \Delta. \tag{4.15}$$

Noting that the function $\phi(z)$ is increasing and $\psi'(z) = -\phi(z)$, we calculate that

$$Q(\Delta) = K > 0, \tag{4.16}$$

$$\lim_{b\to\infty} Q(b) = -\infty, \tag{4.17}$$

$$Q'(b) = \psi'(b) + \beta_2 = -\phi(b) + \beta_2 \leq -\phi(\Delta) + \beta_2 = 0. \tag{4.18}$$

So we conclude that there exists a unique solution $b_r^* > \Delta$ to (4.14), thus $\xi^* = b_r^* - \Delta$ can also be determined. That is to say, there exists a unique pair (ξ^*, b_r^*) satisfying (4.10) and (4.11). According to Theorem 4.1, the result follows. □

Remark 1 According to above analysis, we see that the difference $\Delta = b_r^* - \xi^*$ and b_r^* can viewed as decreasing functions in β_1 or increasing functions in β_2. In addition, larger K results in higher level b_r^*, but it is independent of $\Delta = b_r^* - \xi^*$. However, Theorem 3.2 shows that b_p^* is independent of the transaction factor β_1, β_2 and K.

§5. The Solution to the General Optimal Problem

We now address the problem of maximizing the performance criterion $V(x,\pi)$ over all admissible strategies. According to the stochastic control theory, $V(x)$ should satisfy the following HJB equation

$$\max\Big\{\max_{a\geq 0}\{\mathscr{L}^a V(x)\}, \mathscr{M}V(x)-V(x)\Big\}=0 \tag{5.1}$$

with boundary condition

$$\max\{\mathscr{M}V(0)-V(0), -V(0)\}=0. \tag{5.2}$$

Theorem 5.1 Let $v(x)\in\mathbb{C}^2$ be an increasing, concave solution of equations (5.1) and (5.2) and $v'(x)$ is bounded, then we have the followings:

(i) For each $\pi\in\Pi$, it has $v(x)\geq V(x,\pi)$. So $v(x)\geq V(x)$ for all $x\geq 0$.

(ii) If there exists some strategy $\widehat{\pi}=\{L^{\widehat{\pi}};G^{\widehat{\pi}}\}\in\Pi$ such that $v(x)=V(x,\widehat{\pi})$, then $v(x)=V(x)$ and $\widehat{\pi}$ is optimal.

Proof Please see the proof of (i) in Appendix A. The result of (ii) is obvious. $\square$

Lemma 5.1 For future use, we give the following statements:

(i) When $0\leq b_p^*\leq b_r^*$ holds, it has $f(0)=0$ and $\mathscr{M}f(0)-f(0)\leq 0$.

(ii) When $0<b_r^*<b_p^*$ holds, it has $g(0)>0$ and $\mathscr{M}g(0)-g(0)=0$.

Proof (i) The equality $f(0)=0$ is known, we shall prove that $\mathscr{M}f(0)-f(0)\leq 0$ in different cases.

Case 1: If $b_p^*=0$, then the inequalities $b_p^*<b_r^*$ and $f'(0)\leq\beta_1$ holds. Thus,

$$\begin{aligned}\mathscr{M}f(0)-f(0) &= \max_{y\geq 0}(f(y)-\beta_2 y)-f(0)-K \qquad (5.3)\\ &= (f(0)-\beta_2\cdot 0)-f(0)-K=-K<0. \qquad (5.4)\end{aligned}$$

Case 2: When $0<b_p^*<b_r^*-\xi^*<b_r^*$, we derive that

$$\beta_1<f'(0)=\phi(b_p^*)\leq\phi(b_r^*-\xi^*)=\beta_2, \tag{5.5}$$

which suggests that

$$\mathscr{M}f(0)-f(0)=(f(0)-\beta_2 0)-f(0)-K=-K<0. \tag{5.6}$$

Case 3: When $0<b_r^*-\xi^*<b_p^*<b_r^*$, we get that $f'(0)=\phi(b_p^*)>\phi(b_r^*-\xi^*)=\beta_2$. Because of the concavity of $f(x)$, there exists some number $\overline{\xi}\in(0,b_p^*)$ such that $f'(\overline{\xi})=$

$\beta_2 \Leftrightarrow \phi(b_p^* - \bar{\xi}) = \beta_2 \Leftrightarrow b_p^* - \bar{\xi} = b_r^* - \xi^* = \Delta$. Then

$$\begin{aligned} \mathscr{M} f(0) - f(0) &= f(\bar{\xi}) - \beta_2 \bar{\xi} - f(0) - K = \psi(b_p^* - \bar{\xi}) - \beta_2 \bar{\xi} - \psi(b_p^*) - K \\ &= \psi(\Delta) - \beta_2 (b_p^* - \Delta) - \psi(b_p^*) - K = -Q(b_p^*). \end{aligned} \tag{5.7}$$

Recalling that the function $Q(b)$ is decreasing and $Q(b_r^*) = 0$, we obtain

$$\mathscr{M} f(0) - f(0) \le 0 \Leftrightarrow Q(b_p^*) \ge 0 = Q(b_r^*) \Leftrightarrow b_p^* \le b_r^*. \tag{5.8}$$

(ii) The property of $\mathscr{M} g(0) - g(0) = 0$ is known. Then we note that

$$\begin{aligned} & g(0) = A_0(b_r^*) + A_1(b_r^*) = \psi(b_r^*), \\ & \psi(b_p^*) = A_0(b_p^*) + A_1(b_p^*) = 0, \\ & \psi'(b) = -\phi(b) < 0, \qquad b \ge 0. \end{aligned}$$

Thus $g(0) \ge 0$ equals to $\psi(b_r^*) \ge \psi(b_p^*)$, which happens if and only if $b_r^* \le b_p^*$. □

Theorem 5.2 We can identify the solution to the general control problem as follows:

(i) When $f(0) = 0$ and $\mathscr{M} f(0) - f(0) < 0$, or equivalently, $0 \le b_p^* \le b_r^*$ holds, then $V(x) = V_p(x) = f(x)$, and the associated optimal strategy $\widehat{\pi}$ is consistent with π_p^*.

(ii) When $\mathscr{M} g(0) = g(0)$ and $g(0) \ge 0$, or equivalently, $b_p^* > b_r^*$ holds, then $V(x) = V_r(x) = g(x)$, and the associated optimal strategy $\widehat{\pi}$ is consistent with π_r^*.

Proof Together with Lemma 5.1 and Appendix B, the results are proved. □

Remark 2 Above theorem shows, in our model, that the decision to declare bankruptcy or to collect new capitals depends on the model parameters, which is consistent with the results and idea in Løkka and Zervos (2008). By letting $K \to 0$, $\beta_1 \to 1$ and $\gamma \to \infty$, all results there can be obtained.

Appendix

A. Proof of Theorem 5.1

(i) Consider an arbitrage admissible strategy $\pi \in \Pi$, applying Itô formula yields

$$\begin{aligned} & e^{-\delta(t\wedge\tau^\pi)} v(X^\pi_{t\wedge\tau^\pi}) \\ = \; & v(x) + \int_0^{t\wedge\tau^\pi} e^{-\delta s} \mathscr{L}^a v(X_s^\pi) \mathrm{d}s + Z_1(t\wedge\tau^\pi) + Z_2(t\wedge\tau^\pi) \\ & - \beta_1 \int_0^{t\wedge\tau^\pi} e^{-\delta s} \eta_s^\pi \mathrm{d}M(s) + \sum_{n=1}^{\infty} e^{-\delta\tau_n^\pi} [v(X_{s-}^\pi + \xi_n^\pi) - v(X_{s-}^\pi)] I_{\{\tau_n^\pi \le t\wedge\tau^\pi\}} \end{aligned} \tag{A.1}$$

with

$$Z_1(t\wedge\tau^\pi)=\int_0^{t\wedge\tau^\pi}e^{-\delta s}\sigma v'(X_s^\pi)\mathrm{d}B_s,$$

$$Z_2(t\wedge\tau^\pi)=\int_0^{t\wedge\tau^\pi}e^{-\delta s}[v(X_{s-}^\pi-\eta_s^\pi)+\beta_1\eta_s^\pi-v(X_{s-}^\pi)](\mathrm{d}M(s)-\gamma\mathrm{d}s).$$

Note that the stopping processes $Z_1(t\wedge\tau^\pi)$ and $Z_2(t\wedge\tau^\pi)$ are both martingales. Moreover, suggested by (5.1), we know

$$\sum_{n=1}^{\infty}e^{-\delta\tau_n^\pi}[v(X_{s-}^\pi+\xi_n^\pi)-v(X_{s-}^\pi)]I_{\{\tau_n^\pi\le t\wedge\tau^\pi\}}\le\sum_{n=1}^{\infty}e^{-\delta\tau_n^\pi}(\beta_2\xi_n^\pi+K)I_{\{\tau_n^\pi\le t\wedge\tau^\pi\}}\tag{A.2}$$

and

$$\int_0^{t\wedge\tau^\pi}e^{-\delta s}\mathscr{L}^a v(X_s^\pi)\mathrm{d}s\le 0.\tag{A.3}$$

Thereby, taking conditional expectations on both side of (A.1) yields:

$$\begin{aligned}&\mathsf{E}^x\left(e^{-\delta(t\wedge\tau^\pi)}v(X_{t\wedge\tau^\pi}^\pi)\right)\\ \le\;& v(x)-\mathsf{E}^x\Big(\beta_1\int_0^{t\wedge\tau^\pi}e^{-\delta s}\eta_s^\pi\mathrm{d}M(s)-\sum_{n=1}^{\infty}e^{-\delta\tau_n^\pi}(\beta_2\xi_n^\pi+K)I_{\{\tau_n^\pi\le t\wedge\tau^\pi\}}\Big).\end{aligned}\tag{A.4}$$

Then, letting $t\to\infty$ and using dominated convergence theorem, we have

$$v(x)\ge\mathsf{E}^x\Big(\beta_1\int_0^{\tau^\pi}e^{-\delta s}\eta_s^\pi\mathrm{d}M(s)-\sum_{n=1}^{\infty}e^{-\delta\tau_n^\pi}(\beta_2\xi_n^\pi+K)I_{\{\tau_n^\pi\le\tau^\pi\}}\Big)=V(x,\pi).\tag{A.5}$$

Then the arbitrariness of π and the definition of $V(x)$ suggest that $v(x)\ge V(x)$.

(ii) The proof of (ii) is obvious from (i) and the definition of $V(x)$.

B. Proof of Theorem 5.2

(i) If $f(0)=0$ and $\mathscr{M}f(0)<f(0)$, then $f(x)$ satisfies conditions of Theorem 5.1, so $f(x)\ge V(x)$. Under the strategy π_p^*, we can get the following result from (3.3)

$$\int_0^{t\wedge\tau^{\pi_p^*}}e^{-\delta s}\mathscr{L}^{\eta_s^{\pi_p^*}}f(X_s^{\pi_p^*})\mathrm{d}s=0.\tag{A.6}$$

Moreover, the capital injection never occurs since $\pi_p^*=\{L^{\pi_p^*};0\}$, so

$$\sum_{n=1}^{\infty}e^{-\delta\tau_n^{\pi_p^*}}\left[v(X_{s-}^{\pi_p^*}+\xi_n^{\pi_p^*})-v(X_{s-}^{\pi_p^*})\right]I_{\{\tau_n^{\pi_p^*}\le t\wedge\tau^{\pi_p^*}\}}\equiv 0.\tag{A.7}$$

Replacing π,τ,v by $\pi_p^*,\tau^{\pi_p^*},f$, respectively, in Itô formula (A.1) and taking expectations, we have

$$f(x)=\mathsf{E}^x\Big(\beta_1\int_0^{t\wedge\tau^{\pi_p^*}}e^{-\delta s}\mathrm{d}L_s^{\pi_p^*}+f(X_{t\wedge\tau^{\pi_p^*}}^{\pi_p^*})e^{-\delta(t\wedge\tau^{\pi_p^*})}\Big).\tag{A.8}$$

Letting $t \to \infty$ yields

$$f(x) = \mathsf{E}^x\Big(\beta_1 \int_0^{\tau^{\pi_p^*}} e^{-\delta s}\eta_s^{\pi_p^*}\mathrm{d}M(s)\Big) = V(x, \pi_p^*), \tag{A.9}$$

which, together with $f(x) \geq V(x)$, implies that $f(x) = V(x) = V(x, \pi_p^*)$ and π_p^* is the associated optimal strategy.

(ii) If $\mathscr{M}g(0) = g(0)$ and $g(0) \geq 0$, then $g(x)$ satisfies conditions of Theorem 5.1, $g(x) \geq V(x)$. From (4.3) and (4.4) we know, under the strategy π_r^*,

$$\int_0^{t\wedge\tau^{\pi_r^*}} e^{-\delta s}\mathscr{L}^{\eta_s^{\pi_r^*}} g(X_s^{\pi_r^*})\mathrm{d}s = 0. \tag{A.10}$$

Furthermore, (4.5)-(4.8) indicate that

$$\begin{aligned}&\sum_{n=1}^{\infty} e^{-\delta\tau_n^{\pi_r^*}}\big[g(X_{s-}^{\pi_r^*} + \xi_n^{\pi_r^*}) - g(X_{s-}^{\pi_r^*})\big]I_{\{\tau_n^{\pi_r^*}\leq t\wedge\tau^{\pi_r^*}\}}\\ =&\sum_{n=1}^{\infty} e^{-\delta\tau_n^{\pi_r^*}}\big[g(\xi_n^{\pi_r^*}) - g(0)\big]I_{\{\tau_n^{\pi_r^*}\leq t\wedge\tau^{\pi_r^*}\}} = \sum_{n=1}^{\infty} e^{-\delta\tau_n^{\pi_r^*}}(\beta_2\xi_n^{\pi_r^*} + K)I_{\{\tau_n^{\pi_r^*}\leq t\wedge\tau^{\pi_r^*}\}}.\end{aligned} \tag{A.11}$$

Replacing π, τ^π, v by $\pi_r^*, \tau^{\pi_r^*} = \infty, g$ in Itô formula (A.1) and taking expectations, we have

$$\begin{aligned}g(x) &= \mathsf{E}^x(e^{-\delta t}g(X_t^{\pi_r^*}))\\ &\quad+\mathsf{E}^x\Big(\beta_1\int_0^t e^{-\delta s}\eta_s^{\pi_r^*}\mathrm{d}M(s) - \sum_{n=1}^{\infty} e^{-\delta\tau_n^{\pi_r^*}}(\beta_2\xi_n^{\pi_r^*} + K)I_{\{\tau_n^{\pi_r^*}\leq t\}}\Big).\end{aligned} \tag{A.12}$$

Letting $t \to \infty$, the first term on the right hand side vanishes. Then we obtain

$$g(x) = \mathsf{E}^x\Big(\beta_1\int_0^t e^{-\delta s}\eta_s^{\pi_r^*}\mathrm{d}M(s) - \sum_{n=1}^{\infty} e^{-\delta\tau_n^{\pi_r^*}}(\beta_2\xi_n^{\pi_r^*} + K)I_{\{\tau_n^{\pi_r^*}\leq t\}}\Big) = V(x, \pi_r^*), \tag{A.13}$$

which, together with $g(x) \geq V(x)$, implies that $g(x) = V(x) = V(x, \pi_r^*)$ and $\pi_r^* = \{L^{\pi_r^*}; G^{\pi_r^*}\}$ is associated optimal strategy.

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带交易费用和指数观察时间间隔的最优分红注资策略

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本文中我们考虑比例交易费用和固定交易费用影响下的最优分红与注资策略问题. 我们假设如有必要公司随时可以得到注资以避免破产, 但是只有在参数为$\gamma > 0$的泊松过程的跳跃时刻才可能分红. 为了最大化破产前分红现值与注资现值之差, 我们寻找最优的分红和注资策略. 通过求解相应的脉冲控制问题, 我们找到了依赖于模型参数的显示解. Løkka和Zervos (2008)中的已知结果可以看成是本文结果在$\gamma \to \infty$时的极限情形.

关键词: 分红, 注资, 交易费用, 最优策略, 指数观察时间.

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