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Inequalities for Independent Random Variables on Upper Expectation Space *

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Abstract

In this paper, we establish maximal inequalities, exponential inequalities and Marcinkiewicz-Zygmund inequality for partial sum of random variables which are independent on an upper expectation space. As applications, we give the complete convergence for the partial sum of independent random variables on upper expectation space.

Keywords: Upper expectation, maximal inequality, exponential inequality, Marcinkiewicz-Zygmund inequality.

AMS Subject Classification: 60E15, 60G50.

§1. Introduction

The inequalities of partial sum of independent random variables paly an important role in classical probability theory. In this paper, we will consider the inequalities for sum of random variables which are independent on an upper expectation space, where the independence is used the definition introduced by Peng (2007). The idea of this paper is inspired by the inequalities of demimartingales and N-demimartingales (see Christofides and Hadjikyriakou, 2009; Hadjikyriakou, 2011; Lin and Bai, 2010; Newman and Wright, 1982; Rio, 2009; Sung, 2011). Next we introduce the notations and lemmas which will be useful in this paper.

Let $(\Omega, \mathcal{F})$ be a measurable space, and $\mathcal{M}$ be the set of all probability measures on $(\Omega, \mathcal{F})$. Every non-empty subset $\mathcal{P} \subseteq \mathcal{M}$ defines an upper probability

$$V(A) := \sup_{\mathsf{P} \in \mathcal{P}} \mathsf{P}(A), \qquad A \in \mathcal{F},$$

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and a lower probability

$$v(A) := \inf_{\mathsf{P}\in\mathcal{P}} \mathsf{P}(A), \qquad A \in \mathcal{F}.$$

Obviously $V(\cdot)$ and $v(\cdot)$ are conjugate to each other, that is

$$V(A) + v(A^c) = 1,$$

where A^c is the complement set of A.

Definition 1.1 (quasi-surely, see Denis et al., 2011) A set D is a polar set if $V(D) = 0$ and a property holds "quasi-surely" (q.s. for short) if it holds outside a polar set.

For every non-empty subset $\mathcal{P} \subseteq \mathcal{M}$, we set

$$L^0(\Omega) := \{X : X \text{ is } \mathcal{F}\text{-mearsurable and for all } \mathsf{P} \in \mathcal{P},\ \mathsf{E}_{\mathsf{P}}[X] \text{ exists}\}.$$

Now we define the upper expectation $\mathsf{E}_u[\cdot]$ (following Huber and Strassen, 1973) and the lower expectation $\mathsf{E}_l[\cdot]$ on $(\Omega, \mathcal{F})$ generated by $\mathcal{P}$. For each $X \in L^0(\Omega)$, define

$$\mathsf{E}_u[X] := \sup_{\mathsf{P}\in\mathcal{P}} \mathsf{E}_{\mathsf{P}}[X], \qquad \mathsf{E}_l[X] := \inf_{\mathsf{P}\in\mathcal{P}} \mathsf{E}_{\mathsf{P}}[X].$$

$(\Omega, \mathcal{F}, \mathcal{P}, \mathsf{E}_u)$ is called an upper expectation space and $(\Omega, \mathcal{F}, \mathcal{P}, \mathsf{E}_l)$ is called a lower expectation space.

It is easy to check that $\mathsf{E}_l[X] = -\mathsf{E}_u[-X]$ and $\mathsf{E}_u[\cdot]$ is a sub-linear expectation (more details can see Peng, 2010) on $(\Omega, \mathcal{F})$, in other words, $\mathsf{E}_u[\cdot]$ satisfies the following properties (1)-(4): for all $X, Y \in L^0(\Omega)$,

(1) Monotonicity: $X \geq Y$ implies $\mathsf{E}_u[X] \geq \mathsf{E}_u[Y]$;

(2) Constant preserving: $\mathsf{E}_u[c] = c$, $\forall\, c \in \mathbb{R}$;

(3) Positive homogeneity: $\mathsf{E}_u[\lambda X] = \lambda \mathsf{E}_u[X]$, $\forall\, \lambda \geq 0$;

(4) Sub-additivity: $\mathsf{E}_u[X+Y] \leq \mathsf{E}_u[X] + \mathsf{E}_u[Y]$.

By properties (2) and (4), it is easy to check that $\mathsf{E}_u[\cdot]$ satisfies translation invariance, that is, for any constant c, $\mathsf{E}_u[X+c] = \mathsf{E}_u[X] + c$.

For $p > 0$, we set

$$\mathcal{L}^p := \{X \in L^0(\Omega) : \mathsf{E}_u[|X|^p] < \infty\};$$
$$\mathcal{N}^p := \{X \in L^0(\Omega) : \mathsf{E}_u[|X|^p] = 0\};$$
$$\mathcal{N} := \{X \in L^0(\Omega) : X = 0,\ \text{q.s.}\}.$$

From Denis et al. (2011) we know that $\mathcal{N}^p = \mathcal{N}$, and $L^p := \mathcal{L}^p/\mathcal{N}$ is a Banach space under norm $\|X\|_p := (\mathsf{E}_u[|X|^p])^{1/p}$ for $p \geq 1$, as well $L^p := \mathcal{L}^p/\mathcal{N}$ is a complete metric space under the distance $d(X,Y) := \mathsf{E}_u[|X-Y|^p]$ for $0 < p < 1$.

Definition 1.2 (independence under upper expectation, see Peng, 2007) Let $(\Omega, \mathcal{F}, \mathcal{P}, \mathsf{E}_u)$ be an upper expectation space, $X_1, X_2, \ldots, X_n$ be a sequence of random variables such that $X_i \in L^1$, $i = 1, \ldots, n$. Random variable X_n is said to be independent to random vector $Y := (X_1, \ldots, X_{n-1})$ under $\mathsf{E}_u[\cdot]$ (or $\mathcal{P}$), if for each measurable function φ on $\mathbb{R}^n$ with $\varphi(Y, X_n) \in L^1$ and $\varphi(y, X_n) \in L^1$ for all $y \in \mathbb{R}^{n-1}$, we have

$$\mathsf{E}_u[\varphi(Y, X_n)] = \mathsf{E}_u[\mathsf{E}_u[\varphi(y, X_n)]_{y=Y}].$$

A sequence of random variables $\{X_i\}_{i=1}^{\infty}$ is said to be an independent sequence if X_{i+1} is independent to $Y := (X_1, \ldots, X_i)$ for each $i \in \mathbb{N}^*$.

The following lemma which has been proved in Peng (2010) is very useful in upper expectation theory.

Lemma 1.1 Let $(\Omega, \mathcal{F}, \mathcal{P}, \mathsf{E}_u)$ be an upper expectation space and X, Y be two random variables such that $\mathsf{E}_u[Y] = \mathsf{E}_l[Y]$, then

$$\mathsf{E}_u[X + \alpha Y] = \mathsf{E}_u[X] + \alpha \mathsf{E}_u[Y], \qquad \forall\, \alpha \in \mathbb{R}.$$

In particular, if $\mathsf{E}_u[Y] = \mathsf{E}_l[Y] = 0$, then $\mathsf{E}_u[X + \alpha Y] = \mathsf{E}_u[X]$.

Lemma 1.2 Let $(\Omega, \mathcal{F}, \mathcal{P}, \mathsf{E}_u)$ be an upper expectation space and random variable X is independent to random vector $Y := (X_1, \ldots, X_n)$ and $\mathsf{E}_u[X] = \mathsf{E}_l[X] = 0$. Then for each measurable function φ on $\mathbb{R}^n$ such that $X\varphi(Y) \in L^1$, we have

$$\mathsf{E}_u[X\varphi(Y)] = 0.$$

It is easy to check Lemma 1.2 hold, so we omit the proof.

Remark 1 In the remainder of this paper, we always suppose that $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent random variables on upper expectation space $(\Omega, \mathcal{F}, \mathcal{P}, \mathsf{E}_u)$ with $\mathsf{E}_u[X_i] = \mathsf{E}_l[X_i] = 0$ for $i \in \mathbb{N}^*$. The partial sum sequence of $\{X_i\}_{i=1}^{\infty}$ is denoted by $\{S_n\}_{n=1}^{\infty}$, that is $S_n := \sum\limits_{i=1}^{n} X_i$.

The rest of this paper is organized as follows. In Section 2, we establish the maximal inequalities. In Section 3, we derive the exponential inequalities and their applications. The Marcinkiewicz-Zygmund inequality and its applications are given in Section 4.

§2. Maximal Inequalities

For $n \in \mathbb{N}^*$, we set

$$M_n := \max(S_1, \ldots, S_n), \qquad m_n := \min(S_1, \ldots, S_n),$$

and the more general rank orders

$$S_{n,j} := \begin{cases} j\text{-th largest of } (S_1, \ldots, S_n), & \text{if } j \le n, \\ \min(S_1, \ldots, S_n), & \text{if } j > n. \end{cases}$$

It is obvious that $S_{n,1} = M_n$.

Theorem 2.1 Let $f(\cdot)$ be a nondecreasing function on $\mathbb{R}$ with $f(0) = 0$, then for any $n, j \in \mathbb{N}^*$,

$$\mathsf{E}_u\Big[\int_0^{S_{n,j}} u \,\mathrm{d}f(u)\Big] \le \mathsf{E}_u[S_n f(S_{n,j})], \tag{2.1}$$

in particular for any $\lambda > 0$,

$$V(S_{n,j} \ge \lambda) \le \lambda^{-1}\mathsf{E}_u[I_{\{S_{n,j}\ge\lambda\}} S_n]. \tag{2.2}$$

Proof For fixed $n, j \in \mathbb{N}^*$, let $Y_k = S_{k,j}$ and $Y_0 = 0$, then

$$S_n f(Y_n) = \sum_{k=0}^{n-1} S_{k+1}(f(Y_{k+1}) - f(Y_k)) + \sum_{k=1}^{n-1} (S_{k+1} - S_k) f(Y_k). \tag{2.3}$$

It can be deduced from the definition of $S_{n,j}$ that

$$\text{for } k < j, \text{ either } Y_k = Y_{k+1} \text{ or } S_{k+1} = Y_{k+1},$$
$$\text{for } k \ge j, \text{ either } Y_k = Y_{k+1} \text{ or } S_{k+1} \ge Y_{k+1}.$$

Thus for any k,

$$S_{k+1}(f(Y_{k+1}) - f(Y_k)) \ge Y_{k+1}(f(Y_{k+1}) - f(Y_k)) \ge \int_{Y_k}^{Y_{k+1}} u \,\mathrm{d}f(u).$$

So equality (2.3) yields

$$S_n f(Y_n) \ge \int_0^{Y_n} u \,\mathrm{d}f(u) + \sum_{k=1}^{n-1} (S_{k+1} - S_k) f(Y_k).$$

Taking $\mathsf{E}_u[\cdot]$ on both sides of the above inequality, we have

$$\begin{aligned} \mathsf{E}_u[S_n f(S_{n,j})] = \mathsf{E}_u[S_n f(Y_n)] &\ge \mathsf{E}_u\Big[\int_0^{Y_n} u \,\mathrm{d}f(u) + \sum_{k=1}^{n-1} X_{k+1} f(Y_k)\Big] \\ &\ge \mathsf{E}_u\Big[\int_0^{S_{n,j}} u \,\mathrm{d}f(u)\Big] - \sum_{k=1}^{n-1} \mathsf{E}_u[-X_{k+1} f(Y_k)] \\ &= \mathsf{E}_u\Big[\int_0^{S_{n,j}} u \,\mathrm{d}f(u)\Big], \end{aligned}$$

where the last equality follows from Lemma 1.2. Therefore, inequality (2.1) is proved. Taking $f(u)$ be the indicator function $I_{u\geq\lambda}$, we can obtain inequality (2.2) from (2.1). $\square$

Corollary 2.1 For any $\lambda>0$, $t<0$, we have

$$V(M_n\geq\lambda)\leq\lambda^{-1}\mathsf{E}_u[I_{\{M_n\geq\lambda\}}S_n], \tag{2.4}$$

and

$$V(m_n\leq t)\leq t^{-1}\mathsf{E}_l[I_{\{m_n\leq t\}}S_n]. \tag{2.5}$$

Proof Inequality (2.4) can be obtained easily from Theorem 2.1 since $M_n=S_{n,1}$. To prove inequality (2.5), define $Z_i:=-X_i$, then $\{Z_i\}_{i=1}^{\infty}$ is a sequence of independent random variables on upper expectation space $(\Omega,\mathcal{F},\mathcal{P},\mathsf{E}_u)$ with $\mathsf{E}_u[Z_i]=\mathsf{E}_l[Z_i]=0$ and $\sum\limits_{i=1}^{n}Z_i=-S_n$. Note that

$$\{m_n\leq t\}=\Big\{\max_{1\leq i\leq n}(-S_i)\geq -t\Big\}.$$

It follows from Theorem 2.1 that

$$\begin{aligned}V(m_n\leq t)&=V\Big(\max_{1\leq i\leq n}(-S_i)\geq -t\Big)\\&\leq -t^{-1}\mathsf{E}_u\Big[-I_{\big\{\max\limits_{1\leq i\leq n}(-S_i)\geq -t\big\}}S_n\Big]\\&=t^{-1}\mathsf{E}_l\big[I_{\{m_n\leq t\}}S_n\big].\qquad\square\end{aligned}$$

Corollary 2.2 For any $p>1$, if further assume $\{X_i\}_{i=1}^{\infty}\subseteq L^p$, then for any $0\leq\lambda_1<\lambda_2$,

$$V(M_n\geq\lambda_2)\leq(\lambda_2-\lambda_1)^{-1}\mathsf{E}_u^{1/p}[|S_n|^p]V(S_n\geq\lambda_1)^{1/q}, \tag{2.6}$$

where $1/p+1/q=1$. In particular, when $p=2$, we have

$$V(M_n\geq\lambda_2)\leq(\lambda_2-\lambda_1)^{-1}\Big(\sum_{i=1}^{n}\sigma_i^2\Big)^{1/2}V(S_n\geq\lambda_1)^{1/2}, \tag{2.7}$$

where $\sigma_i^2=\mathsf{E}_u[X_i^2]$.

Proof Taking $\lambda=\lambda_2$ in (2.4), we have

$$\begin{aligned}V(M_n\geq\lambda_2)&\leq\lambda_2^{-1}\mathsf{E}_u\big[I_{\{M_n\geq\lambda_2\}}S_n\big]\\&\leq\lambda_2^{-1}\big(\mathsf{E}_u\big[I_{\{S_n\geq\lambda_1\}}S_n\big]+\mathsf{E}_u\big[I_{\{M_n\geq\lambda_2,S_n<\lambda_1\}}S_n\big]\big)\\&\leq\lambda_2^{-1}\big(\mathsf{E}_u\big[I_{\{S_n\geq\lambda_1\}}S_n\big]+\lambda_1V(M_n\geq\lambda_2)\big).\end{aligned}$$

Therefore

$$V(M_n \geq \lambda_2) \leq (\lambda_2 - \lambda_1)^{-1}\mathsf{E}_u[I_{\{S_n\geq\lambda_1\}}S_n] \leq (\lambda_2 - \lambda_1)^{-1}\mathsf{E}_u^{1/p}[|S_n|^p]V(S_n \geq \lambda_1)^{1/q},$$

where the last inequality is obtained by Hölder inequality under upper expectation space (see Chen et al., 2013). Hence inequality (2.6) is proved. Since $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent random variables with $\mathsf{E}_u[X_i] = \mathsf{E}_l[X_i] = 0$, from Lemma 1.1 and Lemma 1.2, we can get $\mathsf{E}_u[|S_n|^2] = \sum\limits_{i=1}^{n} \mathsf{E}_u[X_i^2]$. Then we can obtain inequality (2.7) from inequality (2.6). □

§3. Exponential Inequalities and Their Applications

Theorem 3.1 Let $\{c_i\}_{i=1}^{\infty}$ be positive real numbers, and random variables $|X_i| \leq c_i$, $\forall\, i \in \mathbb{N}^*$. Then for any $\varepsilon \geq 0$,

$$V(S_n \geq n\varepsilon) \leq \exp\left(\frac{-n^2\varepsilon^2}{2\sum\limits_{i=1}^{n} c_i^2}\right) \tag{3.1}$$

and

$$V(|S_n| \geq n\varepsilon) \leq 2\exp\left(\frac{-n^2\varepsilon^2}{2\sum\limits_{i=1}^{n} c_i^2}\right). \tag{3.2}$$

Proof Let $t \in \mathbb{R}^+$ and $x \in [-c_i, c_i]$. We can write

$$tx = \Big(\frac{1}{2} + \frac{x}{2c_i}\Big)c_i t + \Big(\frac{1}{2} - \frac{x}{2c_i}\Big)(-c_i t).$$

By the convexity of the exponential function we have

$$\mathrm{e}^{tx} \leq \cosh(c_i t) + \frac{x}{c_i}\sinh(c_i t).$$

Thus

$$\mathsf{E}_u[\mathrm{e}^{tS_n}] = \mathsf{E}_u\Big[\prod_{i=1}^{n} \mathrm{e}^{tX_i}\Big] \leq \mathsf{E}_u\Big[\prod_{i=1}^{n}\Big(\cosh(c_i t) + X_i\frac{\sinh(c_i t)}{c_i}\Big)\Big].$$

Now, we want to show the following inequality

$$\mathsf{E}_u[\mathrm{e}^{tS_n}] \leq \prod_{i=1}^{n}\cosh(c_i t), \qquad \forall\, t \geq 0. \tag{3.3}$$

Firstly, we notice that

$$\begin{aligned}\mathsf{E}_u[\mathrm{e}^{tS_1}] &\leq \mathsf{E}_u\Big[\cosh(c_1 t) + X_1\frac{\sinh(c_1 t)}{c_1}\Big]\\ &= \cosh(c_1 t) + \frac{\sinh(c_1 t)}{c_1}\mathsf{E}_u[X_1] = \cosh(c_1 t),\end{aligned}$$

where the last equality follows from $\mathsf{E}_u[X_1] = 0$. Therefore inequality (3.3) is true for $n = 1$. Assume that inequality (3.3) holds for $n = k$. We consider the case $n = k + 1$.

$$\begin{aligned}
\mathsf{E}_u[\mathrm{e}^{tS_{k+1}}] &= \mathsf{E}_u[\mathrm{e}^{tX_{k+1}} \cdot \mathrm{e}^{tS_k}] \\
&\le \mathsf{E}_u\Big[\Big(\cosh(c_{k+1}t) + X_{k+1}\frac{\sinh(c_{k+1}t)}{c_{k+1}}\Big) \cdot \mathrm{e}^{tS_k}\Big] \\
&\le \cosh(c_{k+1}t)\mathsf{E}_u[\mathrm{e}^{tS_k}] + \frac{\sinh(c_{k+1}t)}{c_{k+1}}\mathsf{E}_u[X_{k+1}\mathrm{e}^{tS_k}] \\
&\le \prod_{i=1}^{k+1} \cosh(c_i t),
\end{aligned}$$

where the last inequality follows from Lemma 1.2 and the induction hypothesis. Thus inequality (3.3) is established.

Since $\cosh(c_i t) \le \mathrm{e}^{c_i^2 t^2/2}$, by inequality (3.3) we have

$$\mathsf{E}_u[\mathrm{e}^{tS_n}] \le \exp\left(\frac{t^2 \sum\limits_{i=1}^{n} c_i^2}{2}\right).$$

For any $\varepsilon, t > 0$

$$V(S_n \ge n\varepsilon) = V(\mathrm{e}^{tS_n} \ge \mathrm{e}^{tn\varepsilon}) \le \mathrm{e}^{-tn\varepsilon}\mathsf{E}_u[\mathrm{e}^{tS_n}] \le \exp\left(-tn\varepsilon + \frac{t^2 \sum\limits_{i=1}^{n} c_i^2}{2}\right).$$

The above upper bound is minimized by choosing $t = n\varepsilon / \sum\limits_{i=1}^{n} c_i^2$. Hence, the inequality (3.1) is established.

To prove inequality (3.2), note that

$$V(|S_n| \ge n\varepsilon) \le V(S_n \ge n\varepsilon) + V(-S_n \ge n\varepsilon)$$

and $\{-X_i\}_{i=1}^{\infty}$ is also a sequence of independent random variables on upper expectation space $(\Omega, \mathcal{F}, \mathcal{P}, \mathsf{E}_u)$ with $\mathsf{E}_u[-X_i] = \mathsf{E}_l[-X_i] = 0$ for $i \in \mathbb{N}^*$. Then we can get inequality (3.2) by applying inequality (3.1) twice. $\square$

Corollary 3.1 If $\{X_i\}_{i=1}^{\infty}$ is uniformly bounded, in other words, there exists $c > 0$ such that for any $i \in \mathbb{N}^*$, $|X_i| \le c$. Then we have

$$V(S_n \ge n\varepsilon) \le \exp\left(\frac{-n\varepsilon^2}{2c^2}\right) \qquad \text{and} \qquad V(|S_n| \ge n\varepsilon) \le 2\exp\left(\frac{-n\varepsilon^2}{2c^2}\right).$$

The next two asymptotic results concern the complete convergence for the partial sum of bounded independent random variables on upper expectation space $(\Omega, \mathcal{F}, \mathcal{P}, \mathsf{E}_u)$ with $\mathsf{E}_u[X_i] = \mathsf{E}_l[X_i] = 0$ for $i \in \mathbb{N}^*$.

Theorem 3.2 If $|X_i| \le c < \infty$ for $i \in \mathbb{N}^*$, then for $r > 1/2$, $n^{-r}S_n \to 0$ q.s.

Proof From Corollary 3.1 we get

$$\sum_{n=1}^{\infty} V(|S_n| \ge n^r \varepsilon) \le 2\sum_{n=1}^{\infty} \exp\left(\frac{-n^{2r-1}\varepsilon^2}{2c^2}\right) < \infty.$$

By Borel-Cantelli lemma on upper expectation space (see Chen et al., 2013), we complete the proof of this theorem. □

Theorem 3.3 Let $\{c_i\}_{i=1}^{\infty}$ be positive real numbers and $|X_i| \le c_i$, $i \in \mathbb{N}^*$. Assume that $\sum_{i=1}^{\infty} c_i^2 < \infty$. Then for any $r > 0$, we have $n^{-r}S_n \to 0$ q.s.

Proof From Theorem 3.1 we get

$$\sum_{n=1}^{\infty} V(|S_n| \ge n^r \varepsilon) \le 2\sum_{n=1}^{\infty} \exp\left(\frac{-n^2 n^{2r-2}\varepsilon^2}{2\sum_{i=1}^{n} c_i^2}\right) = 2\sum_{n=1}^{\infty} \exp\left(\frac{-n^{2r}\varepsilon^2}{2\sum_{i=1}^{n} c_i^2}\right)$$
$$\le 2\sum_{n=1}^{\infty} \exp\left(\frac{-n^{2r}\varepsilon^2}{2\sum_{i=1}^{\infty} c_i^2}\right) < \infty.$$

By Borel-Cantelli lemma on upper expectation space (see Chen et al., 2013), we complete the proof of this theorem. □

§4. Marcinkiewicz-Zygmund Inequality and Its Applications

The following lemma will be useful in the proof of Marcinkiewicz-Zygmund inequality. The proof follows from standard arguments and it is therefore omitted.

Lemma 4.1 Let a, b be real numbers and $p \in (1, 2]$, then

$$|a+b|^p \le |a|^p + p|a|^{p-1}\mathrm{sign}(a)b + 2^{2-p}|b|^p.$$

The following theorem gives a Marcinkiewicz-Zygmund inequality for $\{S_n\}_{n=1}^{\infty}$.

Theorem 4.1 For $p \in (1, 2]$, we have

$$\|S_n\|_p^p \le \|X_1\|_p^p + 2^{2-p}\sum_{j=1}^{n-1}\|X_{j+1}\|_p^p \le 2^{2-p}\sum_{j=1}^{n}\|X_j\|_p^p.$$

Proof By applying Lemma 4.1 for $a = S_j$, $b = X_{j+1}$, we have

$$\begin{aligned}\mathsf{E}_u[|S_{j+1}|^p] &\le \mathsf{E}_u[|S_j|^p] + p\mathsf{E}_u[|S_j|^{p-1}\mathrm{sign}(S_j)X_{j+1}] + 2^{2-p}\mathsf{E}_u[|X_{j+1}|^p] \\ &= \mathsf{E}_u[|S_j|^p] + 2^{2-p}\mathsf{E}_u[|X_{j+1}|^p],\end{aligned}$$

where the last equality follows from Lemma 1.2 for X_{j+1} being independent to $(X_1, \dots, X_j)$ and $\mathsf{E}_u[X_{j+1}] = \mathsf{E}_l[X_{j+1}] = 0$. By induction we have

$$\begin{aligned}\mathsf{E}_u[|S_n|^p] &\le \mathsf{E}_u[|X_1|^p] + 2^{2-p}\sum_{j=2}^{n} \mathsf{E}_u[|X_j|^p] = \|X_1\|_p^p + 2^{2-p}\sum_{j=2}^{n}\|X_j\|_p^p \\ &\le 2^{2-p}\sum_{j=1}^{n}\|X_j\|_p^p. \qquad \square\end{aligned}$$

Theorem 4.2 Let $p \in (1, 2]$, $\|X_j\|_p < c_j < \infty$ for $j \in \mathbb{N}^*$. Then for any $\varepsilon > 0$, we have

$$V(|S_n| > n\varepsilon) \le \frac{4}{(2n\varepsilon)^p}\sum_{j=1}^{n} c_j^p.$$

Proof By Theorem 4.1 and $\|X_j\|_p < c_j$, we have

$$\mathsf{E}_u[|S_n|^p] \le 2^{2-p}\sum_{j=1}^{n}\|X_j\|_p^p \le 2^{2-p}\sum_{j=1}^{n} c_j^p.$$

Thus for any $\varepsilon > 0$,

$$V(|S_n| > n\varepsilon) \le \frac{\mathsf{E}_u[|S_n|^p]}{n^p\varepsilon^p} \le \frac{4}{(2n\varepsilon)^p}\sum_{j=1}^{n} c_j^p. \qquad \square$$

Theorem 4.3 Let $\|X_j\|_p < c_j < \infty$, for $j \in \mathbb{N}^*$. Consider the following conditions for $p \in (1, 2]$:

(i) $\sum\limits_{j=1}^{\infty} c_j^p < \infty$ and let r be a positive number such that $pr > 1$;

(ii) $\sum\limits_{j=1}^{n} c_j^p = O(n^\alpha)$ where α is a positive number and let r be a positive number such that $pr - 1 > \alpha$.

If any one of the above conditions is true, then $n^{-r}S_n \to 0$, q.s.

Proof From the Borel-Cantelli lemma under upper expectation space (see Chen et al., 2013), we only need to prove $\sum\limits_{n=1}^{\infty} V(|S_n| > n^r\varepsilon) < \infty$.

Assume that (i) is true. From Theorem 4.2, we have

$$\sum_{n=1}^{\infty} V(|S_n| > n^r\varepsilon) \le \frac{4}{(2\varepsilon)^p}\sum_{n=1}^{\infty}\frac{\sum_{j=1}^{n} c_j^p}{n^{pr}} \le \frac{4}{(2\varepsilon)^p}\sum_{n=1}^{\infty}\frac{\sum_{j=1}^{\infty} c_j^p}{n^{pr}} < \infty.$$

Now assume that condition (ii) is valid. Using Theorem 4.2, for $pr - 1 > \alpha$, we have

$$\sum_{n=1}^{\infty} V(|S_n| > n^r\varepsilon) \le \sum_{n=1}^{\infty}\frac{4}{(2\varepsilon)^p}\frac{1}{n^{pr}}O(n^\alpha) \le \frac{4}{(2\varepsilon)^p}\sum_{n=1}^{\infty} O(n^{\alpha-pr}) < \infty. \qquad \square$$

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上期望空间中独立随机变量的不等式

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本文给出了上期望空间中独立随机变量部分和的最大不等式、指数不等式、Marcinkiewicz-Zygmund不等式. 并且应用指数不等式和Marcinkiewicz-Zygmund不等式研究了随机变量部分和序列完备收敛的性质.

关键词: 上期望, 最大不等式, 指数不等式, Marcinkiewicz-Zygmund不等式.

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