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双根树上二阶非齐次马氏链的强大数定律和Shannon-McMillan定理 *

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摘　要

本文首先研究了双根树上转移矩阵为逐点转移的二阶非齐次马氏链的强极限定理, 同时得到双根树上二阶非齐次马氏链的强大数定律. 最后, 给出了双根树上二阶非齐次马氏链几乎处处收敛意义下的Shannon-McMillan定理.

关键词: 双根树, 二阶非齐次马氏链, 强大数定律, Shannon-McMillan定理.
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§1. 引　言

树指标随机过程是近年来发展起来的概率论的研究方向之一. Benjamini和Peres(1994)给出了树指标马氏链的定义并研究了其常返性和射线常返性. Berger和Ye (1990)研究了齐次树图上平稳随机场熵率的存在性. Ye和Berger (1996, 1998)利用Pemantle (1992)中的结果及组合方法, 在依概率收敛意义下研究了齐次树图上PPG不变和遍历随机场的Shannon-McMillan定理. Yang和Liu (2000)研究了齐次树图上马氏链场(这实际上是树指标马氏链和PPG不变随机场的特殊情形)状态发生频率的强大数定律. 近年来, Yang (2003), Yang和Ye (2007)研究了树上马氏链的强大数定律和渐近均分割性, Huang和Yang (2008)研究了一致有界树上马氏链的强大数定律与Shannon-McMillan定理. Dong等(2011)研究了树指标非齐次马氏链的强大数定律和Shannon-McMillan定理. Shi和Yang (2010)研究了m根树m阶非齐次马氏连的强大数定律和Shannon-McMillan定理. Yang和Ye (2007)中研究的树指标马氏链的转移概率矩阵是逐行转移的, 显然逐点转移概率矩阵是逐行转移概率矩阵的更一般形式. 本文首先研究了双根树上转移矩阵为逐点转移的二阶非齐次马氏链的强极限定理, 同时得到双根树上二阶非齐次马氏链的强大数定律. 最后, 给出了双根树上二阶非齐次马氏链几乎处处收敛意义下的Shannon-McMillan定理.

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§2. 基本概念

设T是一个无限树, σ, t $(\sigma \neq t)$是T中任两个顶点, 则存在唯一的以σ到t的路径, $\sigma = z_1, z_2, \ldots, z_m = t$, 其中$z_1, z_2, \ldots, z_m$互不相同且$z_i, z_{i+1}$为相邻两顶点, $m-1$称为σ到t的距离. 为给T中的顶点编号, 我们选定一个顶点作为根顶点(简称根), 并记之为o.

本文中T_o表示任意局部有限无穷树. 同时我们考虑另外一种树 — 双根树T_{-1}, 这种树是由任意局部有限无穷树T_o的根点o与其他任意根点(记为-1)所形成的. 为了更好的理解双根树T_{-1}, 我们以Cayley树$T_{C,N}$为例(Cayley树$T_{C,N}$的根点o与N个支点相连, 其他的支点与$N+1$个支点相连, 这种树是任意局部有限无穷树的特例), 这时以Cayley树$T_{C,N}$的根点o与另外根点-1所形成的双根树如图1所示.

若σ, t为双根树T_{-1}中的不同顶点, 如果一个顶点t位于根-1到顶点σ的唯一路径上, 则记$t \leq \sigma$, 且记$\sigma \wedge t$为同时满足$\sigma \wedge t \leq t$和$\sigma \wedge t \leq \sigma$且离根$-1$最远的顶点. 记$|t|$为顶点$t$到根$o$的距离. 若$|t| = n$, 则$t$位于树的第$n$层, 特别的, 根$-1$位于第$-1$层. 记$T_{-1}^{(n)}$表示双根树从根$-1$到第$n$层所有顶点的子图. L_n表示第n层所有顶点的集合. L_m^n表示含有T_{-1}的从层m到n层所有顶点的集合. 对于任一个顶点t, 从根-1到顶点t的路径上存在唯一一个离顶点t最近的顶点称为t的父代, 记为1_t, 且称t为1_t的子代. 1_t的父代记为2_t, 称为t的祖父代. 令$X^S = \{X_t, t \in S\}$, $S \subset T_{-1}$, 且x^S为X^S的实现, 且记$|S|$表示S中支点的个数.

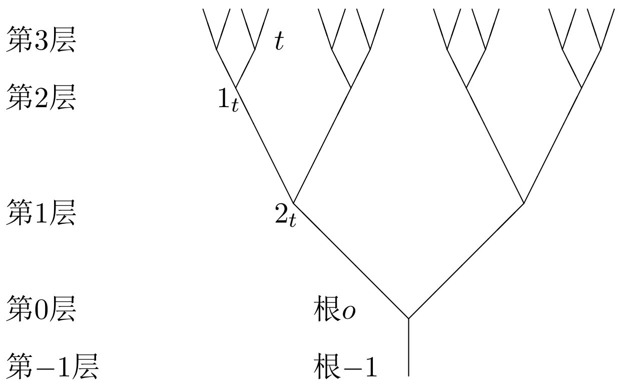

图1 Cayley树$T_{C,2}$形成的双根树

定义 2.1 设$G = \{0, 1, 2, \ldots, b-1\}$且$P(z|y,x)$为$G^3$上的非负函数. 令

$$P = (P(z|y,x)), \qquad P(z|y,x) \geq 0,\ x, y, z \in G.$$

若

$$\sum_{z \in G} P(z|y,x) = 1,$$

则称P为二阶转移矩阵.

定义 2.2　设T_{-1}为双根树, $G=\{0,1,2,\ldots,b-1\}$为有限状态空间, 且$\{X_t,t\in T_{-1}\}$为定义在概率空间$(\Omega,\mathcal{F},\mathsf{P})$上的$G$值变量族. 设

$$p=(p(x,y),x,y\in G), \tag{2.1}$$

$$P_t=(P_t(z|y,x)),\qquad x,y,z\in G,\ t\in T_{-1}\backslash\{o,-1\} \tag{2.2}$$

分别为G^2上的概率分布和G^3上的二阶转移矩阵族. 如果对于任何顶点$t\,(t\neq o,-1)$,

$$\begin{aligned}&\mathsf{P}(X_t=z|X_{1_t}=y,X_{2_t}=x,\ \text{和}X_\sigma\text{满足}\sigma\wedge t\leq 1_t)\\ =&\mathsf{P}(X_t=z|X_{1_t}=y,X_{2_t}=x)=P_t(z|y,x),\qquad\forall\,x,y,z\in G\end{aligned} \tag{2.3}$$

且

$$\mathsf{P}(X_{-1}=x,X_o=y)=p(x,y),\qquad x,y\in G, \tag{2.4}$$

则称$\{X_t,t\in T_{-1}\}$为具有初始分布(2.1)和二阶转移矩阵族(2.2)的树上G值二阶非齐次马氏链.

注记　本文研究树上二阶非齐次马氏链与以往的研究如Dong等(2011), Shi和Yang (2010)是不同的. 以前研究的树上非齐次马氏链是逐行转移, 而本文中是逐点转移. 显然, 逐点转移是逐行转移的推广形式.

§3.　强极限定理

引理 3.1　设T_{-1}为双根数, $\{X_t,t\in T_{-1}\}$为定义2.2中的树T_{-1}上的二阶非齐次马氏链, $\{g_t(y_1,y_2,y_3),t\in T_{-1}\}$是定义在$G^3$上的函数族. 令$L_0=\{o\}$, $L_{-1}=\{-1\}$, $\mathcal{F}_n=\sigma\big(X^{T_{-1}^{(n)}}\big)$,

$$F_n(\omega)=\sum_{t\in T_{-1}^{(n)}\backslash\{o,-1\}}g_t(X_{2_t},X_{1_t},X_t), \tag{3.1}$$

$$t_n(\lambda,\omega)=\frac{\mathrm{e}^{\lambda F_n(\omega)}}{\prod\limits_{t\in T_{-1}^{(n)}\backslash\{o,-1\}}\mathsf{E}[\mathrm{e}^{\lambda g_t(X_{2_t},X_{1_t},X_t)}|X_{1_t},X_{2_t}]}, \tag{3.2}$$

此处λ为实数, 则$\{t_n(\lambda,\omega),\mathcal{F}_n,n\geq 1\}$是非负鞅.

证明:　易知

$$\begin{aligned}P\big(x^{T_{-1}^{(n)}}\big)&=\mathsf{P}\big(X^{T_{-1}^{(n)}}=x^{T_{-1}^{(n)}}\big)\\&=\mathsf{P}(X_{-1}=x_{-1},X_o=x_o)\prod_{t\in T_{-1}^{(n)}\backslash\{o,-1\}}P_t(x_t|X_{1_t},X_{2_t}),\end{aligned} \tag{3.3}$$

该引理的证明方法类似于Shi和Yang (2010)中引理1, 这里省略证明.　□

定理 3.1 设$\{X_t, t\in T_{-1}\}$和$\{g_t(y_1,y_2,y_3), t\in T_{-1}\}$由引理3.1定义, 并且

$$G_n(\omega)=\sum_{t\in T_{-1}^{(n)}\backslash\{o,-1\}}\mathsf{E}[g_t(X_{2_t},X_{1_t},X_t)|X_{1_t},X_{2_t}], \tag{3.4}$$

设$\{a_n, n\geq 1\}$为一列正的随机变量序列. 设$a>0$, $A=\{\lim_{n\to\infty}a_n=\infty\}$且

$$D(a)=\Big\{\limsup_{n\to\infty}\frac{1}{a_n}\sum_{t\in T_{-1}^{(n)}\backslash\{o,-1\}}\mathsf{E}\big[g_t^2(X_{2_t},X_{1_t},X_t)\mathrm{e}^{a|g_t(X_{2_t},X_{1_t},X_t)|}\big|X_{1_t},X_{2_t}\big]=M(\omega)<\infty\Big\}\cap A. \tag{3.5}$$

则

$$\lim_{n\to\infty}\frac{1}{a_n}(F_n(\omega)-G_n(\omega))=0\qquad \text{a.e.}\ \ \omega\in D(a). \tag{3.6}$$

证明: 证明的方法类似于Shi和Yang (2010)中定理1, 这里省略证明. □

推论 3.1 设$\{X_t, t\in T_{-1}\}$为定义2.2中双根树T_{-1}上的二阶非齐次马氏链, 设$\{g_t(y_1, y_2,y_3), t\in T_{-1}\}$为定义在$G^3$上的一致有界函数族, 则

$$\lim_{n\to\infty}\frac{1}{|T_{-1}^{(n)}|}(F_n(\omega)-G_n(\omega))=0\qquad \text{a.e.}. \tag{3.7}$$

证明: 在定理3.1中令$a_n=|T_{-1}^{(n)}|$, 由于$\{g_t(y_1,y_2,y_3), t\in T_{-1}\}$是一致有界函数族, 则$\forall\,\alpha>0$, 有$D(\alpha)=\Omega$. 因此由定理3.1即可得此推论. □

§4. 强大数定律和Shannon-McMillan定理

设$I_k(x)=\begin{cases}1, & x=k\\ 0, & x\neq k\end{cases}$. 设$S_{T_{-1}^{(n)}\backslash\{-1\}}(i_1,i_2)$是$\{(X_{1_t},X_t); t\in T_{-1}^{(n)}\backslash\{-1\}\}$中出现$(i_1,i_2)$的个数, 即

$$S_{T_{-1}^{(n)}\backslash\{-1\}}(i_1,i_2)=\sum_{t\in T_{-1}^{(n)}\backslash\{-1\}}I_{i_1}(X_{1_t})I_{i_2}(X_t), \tag{4.1}$$

设$S_{T_{-1}^{(n)}\backslash\{o,-1\}}(i_1,i_2,i_3)$是$\{(X_{2_t},X_{1_t},X_t); t\in T_{-1}^{(n)}\backslash\{o,-1\}\}$中出现$(i_1,i_2,i_3)$的个数, 即

$$S_{T_{-1}^{(n)}\backslash\{o,-1\}}(i_1,i_2,i_3)=\sum_{t\in T_{-1}^{(n)}\backslash\{o,-1\}}I_{i_1}(X_{2_t})I_{i_2}(X_{1_t})I_{i_3}(X_t). \tag{4.2}$$

记$i^2=(i_1,i_2)$, $j^2=(j_1,j_2)$, $k^2=(k_1,k_2)$.

定义 4.1 设$G=\{0,1,\ldots,b-1\}$是一状态空间, 且

$$P=(P(j|i^2)),\qquad j\in G,\ i^2\in G^2 \tag{4.3}$$

为一二阶转移矩阵. 定义一随机矩阵如下:

$$\overline{P} = (P(j^2|i^2)), \qquad i^2, j^2 \in G^2, \tag{4.4}$$

其中

$$P(j^2|i^2) = \begin{cases} P(j_2|i^2), & \text{当}\ j_1 = i_2; \\ 0, & \text{其它情况}. \end{cases} \tag{4.5}$$

$\overline{P}$称为由二阶转移矩阵P确定的二维随机矩阵.

引理 4.1 (Yang和Liu, 2004)　设$\overline{P}$是由二阶转移矩阵P确定的二维随机矩阵. 如果P中元素均大于0, 即

$$P = (P(j|i^2)), \qquad P(j|i^2) > 0,\ \ \forall\, j \in G,\ i^2 \in G^2, \tag{4.6}$$

则$\overline{P}$是遍历的.

定理 4.1　设T_{-1}为双根树, $\{X_t, t \in T_{-1}\}$为定义2.2中定义的在G中取值的双根树上二阶非齐次马氏链, $S_{T_{-1}^{(n)}\backslash\{-1\}}(i_1,i_2)=S_{T_{-1}^{(n)}\backslash\{-1\}}(i^2)$和$S_{T_{-1}^{(n)}\backslash\{o,-1\}}(i_1,i_2,i_3)=S_{T_{-1}^{(n)}\backslash\{o,-1\}}(i^3)$分别由式(4.1)和(4.2)定义. 设$P = (P(j|i^2))$为另一二阶转移矩阵, 且假设由$P$确定的二维随机矩阵$\overline{P}$是遍历的. 如果

$$\lim_{n\to\infty} \frac{1}{|T_{-1}^{(n)}|} \sum_{t \in T_{-1}^{(n)}\backslash\{o,-1\}} |P_t(j|i^2) - P(j|i^2)| = 0, \qquad \forall\, j \in G,\ i^2 \in G^2, \tag{4.7}$$

则对任意$i_1, i_2, i_3 \in G$, 有

$$\lim_{n\to\infty} \frac{S_{T_{-1}^{(n)}\backslash\{-1\}}(i^2)}{|T_{-1}^{(n)}|} = \pi(i^2) \qquad \text{a.e.}, \tag{4.8}$$

$$\lim_{n\to\infty} \frac{S_{T_{-1}^{(n)}\backslash\{o,-1\}}(i^3)}{|T_{-1}^{(n)}|} = \pi(i^2)P(i_3|i^2) \qquad \text{a.e.}, \tag{4.9}$$

其中$\{\pi(i^2), i^2 \in G^2\}$是由$\overline{P}$确定的平稳分布.

证明:　在推论3.1中令$g_t(y_1, y_2, y_3) = I_{i_1}(y_2)I_{i_2}(y_3)$. 由式(3.1)和(3.4)知

$$\begin{aligned} F_n(\omega) &= \sum_{t \in T_{-1}^{(n)}\backslash\{o,-1\}} g_t(X_{2_t}, X_{1_t}, X_t) \\ &= \sum_{t \in T_{-1}^{(n)}\backslash\{o,-1\}} I_{i_1}(X_{1_t})I_{i_2}(X_t), \end{aligned} \tag{4.10}$$

$$\begin{aligned}G_n(\omega) &= \sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}} \mathsf{E}[g_t(X_{2_t},X_{1_t},X_t)|X_{1_t},X_{2_t}]\\ &= \sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}} \sum_{x_t\in G} g_t(X_{2_t},X_{1_t},x_t)P_t(x_t|X_{1_t},X_{2_t})\\ &= \sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}} \sum_{x_t\in G} I_{i_1}(X_{1_t})I_{i_2}(x_t)P_t(x_t|X_{1_t},X_{2_t})\\ &= \sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}} I_{i_1}(X_{1_t})P_t(i_2|X_{1_t},X_{2_t})\\ &= \sum_{l\in G}\sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}} I_l(X_{2_t})I_{i_1}(X_{1_t})P_t(i_2|i_1,l).\end{aligned}\tag{4.11}$$

显然, $\{g_t(y_1,y_2,y_3),t\in T_{-1}\}$是定义在$G^3$上的一致有界函数, 由推论3.1可得

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{|T_{-1}^{(n)}|}\Big[&\sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}} I_{i_1}(X_{1_t})I_{i_2}(X_t)\\ &-\sum_{l\in G}\sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}} I_l(X_{2_t})I_{i_1}(X_{1_t})P_t(i_2|i_1,l)\Big]=0\qquad \text{a.e.},\end{aligned}\tag{4.12}$$

又由式(4.7), 有

$$\begin{aligned}&\lim_{n\to\infty}\left|\frac{\sum\limits_{l\in G}\sum\limits_{t\in T_{-1}^{(n)}\setminus\{o,-1\}} I_l(X_{2_t})I_{i_1}(X_{1_t})(P_t(i_2|i_1,l)-P(i_2|i_1,l))}{T_{-1}^{(n)}}\right|\\ \le&\sum_{l\in G}\lim_{n\to\infty}\frac{\sum\limits_{t\in T_{-1}^{(n)}\setminus\{o,-1\}}|P_t(i_2|i_1,l)-P(i_2|i_1,l)|}{|T_{-1}^{(n)}|}=0.\end{aligned}\tag{4.13}$$

故由式(4.12)和(4.13)可得

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{|T_{-1}^{(n)}|}\Big[&\sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}} I_{i_1}(X_{1_t})I_{i_2}(X_t)\\ &-\sum_{l\in G}\sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}} I_l(X_{2_t})I_{i_1}(X_{1_t})P(i_2|i_1,l)\Big]=0\qquad \text{a.e.}.\end{aligned}\tag{4.14}$$

注意到

$$\sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}} I_{i_1}(X_{1_t})I_{i_2}(X_t)=S_{T_{-1}^{(n)}\setminus\{-1\}}(i_1,i_2)-I_{i_1}(X_{-1})I_{i_2}(X_o),\tag{4.15}$$

$$\sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}} I_l(X_{2_t})I_{i_1}(X_{1_t})=N\sum_{t\in T_{-1}^{(n-1)}\setminus\{-1\}} I_l(X_{1_t})I_{i_1}(X_t),\tag{4.16}$$

$$\frac{N}{|T_{-1}^n|}=\frac{1}{|T_{-1}^{n-1}|}.\tag{4.17}$$

由式(4.14)-(4.17)和(4.5)有

$$\lim_{n\to\infty}\Big\{\frac{S_{T_{-1}^{(n)}\backslash\{-1\}}(i^2)}{|T_{-1}^{(n)}|}-\frac{1}{|T_{-1}^{(n-1)}|}\sum_{k^2\in G^2}S_{T_{-1}^{(n-1)}\backslash\{-1\}}(k^2)P(i^2|k^2)\Big\}=0\qquad \text{a.e.},\tag{4.18}$$

类似于Shi和Yang (2010)中的定理, 由式(4.18)及数学归纳法, 可得

$$\lim_{n\to\infty}\Big\{\frac{S_{T_{-1}^{(n+N)}\backslash\{-1\}}(j^2)}{|T_{-1}^{(n+N)}|}-\frac{1}{|T_{-1}^{(n-1)}|}\sum_{k^2\in G^2}S_{T_{-1}^{(n-1)}\backslash\{-1\}}(k^2)P^{(N+1)}(j^2|k^2)\Big\}=0\qquad \text{a.e.},\tag{4.19}$$

其中$P^{(h)}(j^2|k^2)$是由$\overline{P}$确定的h步转移概率. 由遍历性知

$$\lim_{N\to\infty}P^{(N+1)}(j^2|k^2)=\pi(j^2),\qquad \forall\, k^2\in G^2,\tag{4.20}$$

$$\sum_{k^2\in G^2}S_{T_{-1}^{(n-1)}\backslash\{-1\}}(k^2)=|T_{-1}^{(n-1)}|-1,\tag{4.21}$$

由式(4.19), (4.20)和(4.21)可得式(4.8). 类似地, 在推论3.1中令

$$g_t(y_1,y_2,y_3)=I_{i_1}(y_1)I_{i_2}(y_2)I_{i_3}(y_3),$$

由式(4.8)可得式(4.9). □

设T_{-1}为双根树, $\{X_t,t\in T_{-1}\}$为在G中取值的树指标随机过程, $x^{T_{-1}^n}$为$X^{T_{-1}^n}$的实现. 记

$$P\big(x^{T_{-1}^{(n)}}\big)=P\big(X^{T_{-1}^{(n)}}=x^{T_{-1}^{(n)}}\big).$$

令

$$f_n(\omega)=-\frac{1}{|T_{-1}^{(n)}|}\ln P\big(X^{T_{-1}^{(n)}}\big),$$

称$f_n(\omega)$为$X^{T_{-1}^{(n)}}$的熵密度. 如果$\{X_t,t\in T_{-1}\}$为定义2.2中定义的双根树上二阶非齐次马氏链, 由式(3.3)为

$$f_n(\omega)=-\frac{1}{|T_{-1}^{(n)}|}\Big[\ln P(X_{-1},X_o)+\sum_{t\in T_{-1}^{(n)}\backslash\{o,-1\}}\ln P_t(X_t|X_{1_t},X_{2_t})\Big].\tag{4.22}$$

$f_n(\omega)$在某种意义下(L_1收敛, 依概率收敛, 几乎处处收敛)收敛于常数, 称为Shannon-McMillan定理, 或信源的渐近均分割性(AEP)以及熵定理.

下面研究在状态空间G下双根树上二阶非齐次马氏链的Shannon-McMillan定理. 在证明定理之前, 首先给出一个引理.

引理 4.2 (Dong等, 2011)　设T_{-1}为双根树, 设$\phi(x)$是定义在区间Δ上的有界函数, 且在点$x=a$处连续. 设$\{a_t,t\in T_{-1}\}$是一实数列. 如果

$$\lim_{n\to\infty}\frac{1}{|T_{-1}^{(n)}|}\sum_{t\in T_{-1}^{(n)}\backslash\{o,-1\}}|a_t-a|=0,$$

则

$$\lim_{n\to\infty}\frac{1}{|T_{-1}^{(n)}|}\sum_{t\in T_{-1}^{(n)}\backslash\{o,-1\}}|\phi(a_t)-\phi(a)|=0. \tag{4.23}$$

定理 4.2 设$\{X_t,t\in T_{-1}\}$为G中取值的双根树上二阶非齐次马氏链, $P=(P(j|i^2))$是另一二阶转移矩阵, 并且假定由P确定的二维随机矩阵$\overline{P}$是遍历的. 设$f_n(\omega)$由式(4.22)给出. 若式(4.7)成立, 则

$$\lim_{n\to\infty}f_n(\omega)=-\sum_{i^2\in G^2}\pi(i^2)\sum_{j\in G}P(j|i^2)\ln P(j|i^2)\qquad\text{a.e.}. \tag{4.24}$$

证明: 在引理4.2中令$\phi(x)=x\log x$, $(\phi(0)=0)$, 由式(4.7)有

$$\lim_{n\to\infty}\frac{1}{|T_{-1}^{(n)}|}\sum_{t\in T_{-1}^{(n)}\backslash\{o,-1\}}|P_t(j|i^2)\ln P_t(j|i^2)-P(j|i^2)\ln P(j|i^2)|=0,\qquad\forall j\in G,\ i^2\in G^2. \tag{4.25}$$

在定理3.1中令$a=1/2$, $t\in T_{-1}$,

$$a_n=|T_{-1}^{(n)}|,\qquad g_t(y_1,y_2,y_3)=-\ln P_t(y_3|y_1,y_2),$$

由式(3.1)和(3.4)有

$$F_n(\omega)=\sum_{t\in T_{-1}^{(n)}\backslash\{o,-1\}}[-\ln P_t(X_t|X_{1_t},X_{2_t})], \tag{4.26}$$

$$G_n(\omega)=\sum_{t\in T_{-1}^{(n)}\backslash\{o,-1\}}\sum_{j\in G}[-P_t(j|X_{1_t},X_{2_t})\ln P_t(j|X_{1_t},X_{2_t})]. \tag{4.27}$$

$$\begin{aligned}&\mathsf{E}\big[g_t^2(X_{2_t},X_{1_t},X_t)\mathrm{e}^{a|g_t(X_{2_t},X_{1_t},X_t)|}\big|X_{1_t},X_{2_t}\big]\\&=\sum_{x_t\in G}(\ln P_t(x_t|X_{1_t},X_{2_t}))^2\mathrm{e}^{-(1/2)\ln P_t(x_t|X_{1_t},X_{2_t})}P_t(x_t|X_{1_t},X_{2_t})\\&=\sum_{x_t\in G}(\ln P_t(x_t|X_{1_t},X_{2_t}))^2(P_t(x_t|X_{1_t},X_{2_t}))^{1/2}\\&\le 16b\mathrm{e}^{-2},\end{aligned}$$

所以

$$\limsup_{n\to\infty}\frac{1}{|T_{-1}^{(n)}|}\sum_{t\in T_{-1}^{(n)}\backslash\{o,-1\}}\mathsf{E}\big[g_t^2(X_{2_t},X_{1_t},X_t)\mathrm{e}^{a|g_t(X_{2_t},X_{1_t},X_t)|}\big|X_{1_t},X_{2_t}\big]\le 16b\mathrm{e}^{-2}. \tag{4.28}$$

故$D(1/2)=\Omega$. 因此由式(4.26)-(4.28)和定理3.1有

$$\lim_{n\to\infty}\left\{\frac{\sum_{t\in T_{-1}^{(n)}\backslash\{o,-1\}}[-\ln P_t(X_t|X_{1_t},X_{2_t})]}{|T_{-1}^{(n)}|}\right.\\\left.+\sum_{j\in G}\frac{\sum_{t\in T_{-1}^{(n)}\backslash\{o,-1\}}[P_t(j|X_{1_t},X_{2_t})\ln P_t(j|X_{1_t},X_{2_t})]}{|T_{-1}^{(n)}|}\right\}=0\qquad\text{a.e.}, \tag{4.29}$$

由于

$$\begin{aligned}
&\left|\frac{\sum\limits_{j\in G}\sum\limits_{t\in T_{-1}^{(n)}\setminus\{o,-1\}}[P_t(j|X_{1_t},X_{2_t})\ln P_t(j|X_{1_t},X_{2_t})]}{|T_{-1}^{(n)}|}-\sum_{i^2\in G^2}\pi(i^2)\sum_{j\in G}P(j|i^2)\ln P(j|i^2)\right|\\
\le&\left|\frac{1}{|T_{-1}^{(n)}|}\sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}}\sum_{j\in G}\sum_{i^2\in G^2}I_{i_1}(X_{2_t})I_{i_2}(X_{1_t})P_t(j|i^2)\ln P_t(j|i^2)\right.\\
&\left.-\frac{1}{|T_{-1}^{(n)}|}\sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}}\sum_{j\in G}\sum_{i^2\in G^2}I_{i_1}(X_{2_t})I_{i_2}(X_{1_t})P(j|i^2)\ln P(j|i^2)\right|\\
&+\left|\frac{1}{|T_{-1}^{(n)}|}\sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}}\sum_{j\in G}\sum_{i^2\in G^2}I_{i_1}(X_{2_t})I_{i_2}(X_{1_t})P(j|i^2)\ln P(j|i^2)\right.\\
&\left.-\sum_{i^2\in G^2}\pi(i^2)\sum_{j\in G}P(j|i^2)\ln P(j|i^2)\right|\\
\le&\sum_{j\in G}\sum_{i^2\in G^2}\frac{1}{|T_{-1}^{(n)}|}\sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}}|P_t(j|i^2)\ln P_t(j|i^2)-P(j|i^2)\ln P(j|i^2)|\\
&+\sum_{j\in G}\sum_{i^2\in G^2}P(j|i^2)\ln P(j|i^2)\left|\frac{1}{|T_{-1}^{(n)}|}\sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}}I_{i_1}(X_{2_t})I_{i_2}(X_{1_t})-\pi(i^2)\right|\\
\le&\sum_{j\in G}\sum_{i^2\in G^2}\frac{1}{|T_{-1}^{(n)}|}\sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}}|P_t(j|i^2)\ln P_t(j|i^2)-P(j|i^2)\ln P(j|i^2)|\\
&+\sum_{j\in G}\sum_{i^2\in G^2}P(j|i^2)\ln P(j|i^2)\left|\frac{1}{|T_{-1}^{(n-1)}|}S_{T_{-1}^{(n-1)}\setminus\{-1\}}(i^2)-\pi(i^2)\right|,
\end{aligned}$$

由式(4.8), (4.25)和(4.29)有

$$\lim_{n\to\infty}\frac{1}{|T_{-1}^{(n)}|}\sum_{t\in T_{-1}^{(n)}\setminus\{o,-1\}}[-\ln P_t(X_t|X_{1_t},X_{2_t})]=-\sum_{i^2\in G^2}\pi(i^2)\sum_{j\in G}P(j|i^2)\ln P(j|i^2)\qquad \text{a.e..}\tag{4.30}$$

由式(4.22), (4.29)和(4.30)可得式(4.24).　　□

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The Strong Law of Large Numbers and the Shannon-McMillan Theorem for the Two-Order Nonhomogeneous Markov Chains Indexed by an Two Rooted Cayley Tree

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In this paper, we first study the strong convergence theorem for finite two-order nonhomogeneous Markov chains indexed by an two rooted Cayley tree, then we obtain the strong law of large numbers for this Markov chains. Finally, we obtain the Shannon-McMillan theorem with a.e. convergence for an two-order nonhomogeneous Markov chain indexed by an two rooted Cayley tree.

Keywords: Two rooted Cayley tree, two-order nonhomogeneous Markov chain, strong law of large numbers, Shannon-McMillan theorem.

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