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The Periods of States for Markov Chains in a Random Environment *

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Abstract

The periods of states for Markov chains in a random environment are introduced and some properties about periods are investigated. An open problem (Orey, 1991; Problem 1.3.3) is studied under the assumption that states have periods.

Keywords: Markov chains in a random environment, period, classification.

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§1. Introduction

Let $(\mathscr{X}, \mathscr{A})$ be a denumerable state space and $(\Theta, \mathscr{B})$ be an environment space. $\{P(\theta), \theta \in \Theta\}$ is a family of stochastic matrices acting on $(\mathscr{X}, \mathscr{A})$. Let $p(\theta; x, y)$ be the (x, y) entry of $P(\theta)$ such that $p(\cdot\,; x, y)$ is $\mathscr{B}$ measurable for each $x, y \in \mathscr{X}$. Let $\overrightarrow{\Theta} = \Theta^Z$ be the product space of doubly infinite sequences $\{\theta_n\}$ and let $\overrightarrow{\mathscr{B}} = \mathscr{B}^Z$ be its product σ-field. $T: \overrightarrow{\Theta} \to \overrightarrow{\Theta}$ is a shift operator on $\overrightarrow{\Theta}$ defined by $(T\overrightarrow{\theta})_n = \theta_{n+1}$ for every $\overrightarrow{\theta} \in \overrightarrow{\Theta}$ and π is its distribution on $\overrightarrow{\mathscr{B}}$. We assume π is a shift invariant probability on $(\overrightarrow{\Theta}, \overrightarrow{\mathscr{B}})$, i.e., $\pi = \pi \circ T$. Consider a doubly infinite stochastic environment sequence $\overrightarrow{\xi} = \{\xi_n, n = 0, \pm1, \pm2, \ldots\}$ taking values in Θ and a stochastic sequence $\overrightarrow{X} = \{X_n, n = 0, 1, 2, \ldots\}$ taking values in $\mathscr{X}$, if the following condition is satisfied:

$$P(X_{n+1} \in A, \xi_{n+1} \in B | \overrightarrow{X}_0^n, \overrightarrow{\xi}_{-\infty}^n) = P(\xi_n; X_n, A) P(\xi_{n+1} \in B | \overrightarrow{\xi}_{-\infty}^n) \qquad \text{a.s.}$$

for every $A \in \mathscr{A}$, $B \in \mathscr{B}$ and $n = 0, 1, 2, \ldots$, then $(\overrightarrow{X}, \overrightarrow{\xi})$ is called a Markov chain in a random environment (MCRE).

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The study of MCRE has been pursued for some time, often for special cases and examples such as random walks (Kozlov, 1974; Solomon, 1975; Kalikow, 1981; Andjel,1988; Kesten et al., 1975) and birth and death chains (Smith and Wilkinson, 1969; Athreya and Karlin, 1971) in a random environment. Nawrotzki (1981, 1982) introduced a general theory. Then Cogburn (1980, 1984) developed such a theory in a wider context making the connection with the well-developed theory of Hopf Markov chains. Orey (1991) gave an introductory exposing some basic results of Nawrotzki and Cogburn and presented some open problems. In this article the periods of states for Markov chains in a random environment are introduced and some properties about periods are investigated. An open problem (Orey, 1991; Problem 1.3.3) is studied under the assumption that states have periods.

§2. Period

Let $F \in \mathscr{A} \times \overrightarrow{\mathscr{B}}$, $(F)_y = \{\overrightarrow{\theta} : (y, \overrightarrow{\theta}) \in F\}$, $\eta_n = (X_n, T^n \overrightarrow{\xi})$, $n \geq 0$,

$$P(\theta_m, \theta_{m+1}, \ldots, \theta_n) = P(\theta_m)P(\theta_{m+1}) \cdots P(\theta_n) = (p(\theta_m, \theta_{m+1}, \ldots, \theta_n, x, y), x, y \in \mathscr{X}),$$

$$P^{(n)}(\overrightarrow{\theta}; x, F) = P_{(x, \overrightarrow{\theta})}(\eta_n \in F) = \sum_{y \in X} p(\theta_0, \theta_1, \ldots, \theta_{n-1}, x, y) I_{F_y}(T^n \overrightarrow{\theta}),$$

$$P^{(n)}(\overrightarrow{\theta}; x, y) = P^{(n)}(\overrightarrow{\theta}; x, \{y\} \times \overrightarrow{\Theta}),$$

$$G(\overrightarrow{\theta}; x, F) = \sum_{n=1}^{\infty} P_{(x, \overrightarrow{\theta})}(\eta_n \in F) = \sum_{n=1}^{\infty} P^{(n)}(\overrightarrow{\theta}; x, F),$$

$$G(\overrightarrow{\theta}; x, y) = G(\overrightarrow{\theta}; x, \{y\} \times \overrightarrow{\Theta}).$$

Definition 2.1 If the set $\{n \geq 1 : \pi(\{\overrightarrow{\theta} : P^{(n)}(\overrightarrow{\theta}; x, x) > 0\}) = 1\}$ is nonempty, we define the period of x as follows:

$$d_x = G \cdot C \cdot D \cdot \{n \geq 1 : \pi(\{\overrightarrow{\theta} : P^{(n)}(\overrightarrow{\theta}; x, x) > 0\}) = 1\},$$

where $G \cdot C \cdot D \cdot$ denotes the greatest common divisor.

Definition 2.2 We say x strongly leads to y if there exists a positive integer n such that

$$\pi(\{\overrightarrow{\theta} : P^{(n)}(\overrightarrow{\theta}; x, y) > 0\}) = 1,$$

we say that x and y strongly communicate iff x strongly leads to y and y strongly leads to x.

Theorem 2.1 If x and y strongly communicate, then the periods of x and y exist and $d_x = d_y$.

Proof Suppose x and y strongly communicate. Then there are positive m and n such that

$$\pi(\{\overrightarrow{\theta}:P^{(m)}(\overrightarrow{\theta};x,y)>0\})=1,\qquad \pi(\{\overrightarrow{\theta}:P^{(n)}(\overrightarrow{\theta};y,x)>0\})=1.$$

Hence, by $\pi\circ T^{-1}=\pi$,

$$\pi(\{\overrightarrow{\theta}:P^{(n)}(T^m\overrightarrow{\theta};y,x)>0\})=1.$$

Let

$$\overrightarrow{\Theta}=(\{\overrightarrow{\theta}:P^{(m)}(\overrightarrow{\theta};x,y)>0,P^{(n)}(T^m\overrightarrow{\theta};y,x)>0\}).$$

Then $\pi(\overrightarrow{\Theta})=1$. To every $\overrightarrow{\theta}\in\overrightarrow{\Theta}$,

$$P^{(m+n)}(\overrightarrow{\theta};x,x)\geq P^{(m)}(\overrightarrow{\theta};x,y)P^{(n)}(T^m\overrightarrow{\theta};y,x)>0.$$

So by Definition 2.2 the period of x exists. Since states x and y are interchangeable in the argument above, it follows that the period of y exists. This proves the first assertion.

Let

$$\pi(\{\overrightarrow{\theta}:P^{(s)}(\overrightarrow{\theta};x,x)>0\})=1,$$

$$\overrightarrow{\Theta_1}=\{\overrightarrow{\theta}:P^{(n)}(\overrightarrow{\theta};y,x)>0,P^{(s)}(T^n\overrightarrow{\theta};x,x)>0,P^{(m)}(T^{n+s}\overrightarrow{\theta};x,y)>0\},$$

$$\overrightarrow{\Theta_2}=\{\overrightarrow{\theta}:P^{(n)}(\overrightarrow{\theta};y,x)>0,P^{(2s)}(T^n\overrightarrow{\theta};x,x)>0,P^{(m)}(T^{n+2s}\overrightarrow{\theta};x,y)>0\},$$

$$\overrightarrow{\Theta_3}=\overrightarrow{\Theta_1}\cap\overrightarrow{\Theta_2}.$$

Then $\pi(\overrightarrow{\Theta_3})=1$. To every $\overrightarrow{\theta}\in\overrightarrow{\Theta_3}$,

$$P^{(n+s+m)}(\overrightarrow{\theta};y,y)\geq P^{(n)}(\overrightarrow{\theta};y,x)P^{(s)}(T^n\overrightarrow{\theta};x,x)P^{(m)}(T^{n+s}\overrightarrow{\theta};x,y)>0,$$

$$P^{(n+2s+m)}(\overrightarrow{\theta};y,y)\geq P^{(n)}(\overrightarrow{\theta};y,x)P^{(2s)}(T^n\overrightarrow{\theta};x,x)P^{(m)}(T^{n+2s}\overrightarrow{\theta};x,y)>0.$$

It follows that d_y divides $n+2s+m-(n+s+m)=s$, so d_y divides d_x. By the symmetry of d_x and d_y we have $d_x=d_y$. □

Definition 2.3 Define $C(x)=\{y:y$ and x strongly communicate$\}$.

Corollary 2.1 If $y\in C(x)$ and $z\in C(x)$, then $d_y=d_z$.

Theorem 2.2 If the period of x exists and $d_x=d$, then there exists a positive integer N such that $n\geq N$ implies that

$$\pi(\{\overrightarrow{\theta}:P^{(nd)}(\overrightarrow{\theta};x,x)>0\})=1.$$

Proof By the definition of a period there exists finitely many positive integers $s_1, s_2, \ldots, s_m$, with $\pi(\{\overrightarrow{\theta} : P^{(s_i)}(\overrightarrow{\theta}; x, x) > 0\}) = 1$, $1 \le i \le m$ and such that d is their greatest common divisor. Let

$$\overrightarrow{\Theta}_i = \bigcap_{k=0}^{\infty} \{\overrightarrow{\theta} : P^{(s_i)}(T^k \overrightarrow{\theta}; x, x) > 0\}, \quad 1 \le i \le m, \qquad \overrightarrow{\Theta}_0 = \bigcap_{i=1}^{m} \overrightarrow{\Theta}_i.$$

Then $\pi(\overrightarrow{\Theta}_0) = 1$. By an elementary result from number theory there exists a positive integer N such that $n \ge N$ implies the existence of positive integers $c_1, c_2, \ldots, c_m$ and such that $nd = \sum\limits_{i=1}^{m} c_i s_i$. To every $\overrightarrow{\theta} \in \overrightarrow{\Theta}_0$, we have

$$P^{(nd)}(\overrightarrow{\theta}; x, x) \ge P^{(c_1 s_1)}(\overrightarrow{\theta}; x, x) P^{(c_2 s_2)}(T^{c_1 s_1} \overrightarrow{\theta}; x, x) \cdots P^{(c_m s_m)}\Big(T^{\sum\limits_{i=1}^{m-1} c_i s_i} \overrightarrow{\theta}; x, x\Big).$$

It follows that $P^{(nd)}(\overrightarrow{\theta}; x, x) > 0$. $\square$

Theorem 2.3 To every $y \in C(x)$ there corresponds a unique residue class r_y modulo d_x such that

$$\pi(\{\overrightarrow{\theta} : P^{(n)}(\overrightarrow{\theta}; x, y) > 0\}) = 1$$

implies that $n \equiv r_y \pmod{d_x}$. Furthermore there exists an $N(y)$ such that $n \ge N$ implies that

$$\pi(\{\overrightarrow{\theta} : P^{(nd_x + r_y)}(\overrightarrow{\theta}; x, y) > 0\}) = 1.$$

Proof Let $\pi(\{\overrightarrow{\theta} : P^{(m)}(\overrightarrow{\theta}; x, y) > 0\}) = 1$ and $\pi(\{\overrightarrow{\theta} : P^{(m')}(\overrightarrow{\theta}; x, y) > 0\}) = 1$. There exists n such that $\pi(\{\overrightarrow{\theta} : P^{(n)}(\overrightarrow{\theta}; y, x) > 0\}) = 1$. Hence, by $\pi \circ T^{-1} = \pi$,

$$\pi(\{\overrightarrow{\theta} : P^{(n)}(T^m \overrightarrow{\theta}; y, x) > 0\}) = 1.$$

Let

$$\overrightarrow{\Theta} = (\{\overrightarrow{\theta} : P^{(m)}(\overrightarrow{\theta}; x, y) > 0, P^{(n)}(T^m \overrightarrow{\theta}; y, x) > 0\}).$$

Then $\pi(\overrightarrow{\Theta}) = 1$. To every $\overrightarrow{\theta} \in \overrightarrow{\Theta}$,

$$P^{(m+n)}(\overrightarrow{\theta}; x, x) \ge P^{(m)}(\overrightarrow{\theta}; x, y) P^{(n)}(T^m \overrightarrow{\theta}; y, x) > 0.$$

So $\pi(\{\overrightarrow{\theta} : P^{(m+n)}(\overrightarrow{\theta}; x, x) > 0\}) = 1$. Simiarly $\pi(\{\overrightarrow{\theta} : P^{(m'+n)}(\overrightarrow{\theta}; x, x) > 0\}) = 1$ holds. It follows that d_x divides $m - m'$. This proves the first assertion.

By Theorem 2.2 there exists a positive integer N such that $n \ge N$ implies

$$\pi(\{\overrightarrow{\theta} : P^{(nd_x)}(\overrightarrow{\theta}; x, x) > 0\}) = 1.$$

Now if $y \in C(x)$ corresponds a residue class r_y, then there exists a positive m such that

$$\pi(\{\overrightarrow{\theta} : P^{(md_x+r_y)}(\overrightarrow{\theta}; x, y) > 0\}) = 1.$$

Let $N(y) = N + m$. If $n \geq N(y)$, then $nd_x + r_y = n'd_x + md_x + r_y$ where $n' \geq N$. Hence

$$\pi(\{\overrightarrow{\theta} : P^{(nd_x+r_y)}(\overrightarrow{\theta}; x, y) > 0\}) = 1.$$

This proves the second assertion. $\square$

Definition 2.4 (Orey, 1991) Define $C_x = \{\overrightarrow{\theta} : \sum\limits_{k=1}^{\infty} P(\theta_{-k} \cdots \theta_{-1}; x, x) = \infty\}$.

An open problem (Orey, 1991; Problem 1.3.3) When does the zero-one property hold on C_x?

Lemma 2.1 $\pi(C_x) = \pi(\overrightarrow{\theta} : G(\overrightarrow{\theta}; x, x) = \infty)$.

Proof It is obvious by $\pi = \pi \circ T^{-1}$. $\square$

Theorem 2.4 If the transformation T about π be ergodic and $d_x = 1$, then $\pi(C_x) = 0$ or 1.

Proof By Theorem 2.2 and $d_x = 1$, there exist a positive integers N such that $n \geq N$ implies

$$\pi(\{\overrightarrow{\theta} : P^{(n)}(\overrightarrow{\theta}; x, x) > 0\}) = 1.$$

Let

$$B = \{\overrightarrow{\theta} : G(\overrightarrow{\theta}; x, x) = \infty\}.$$

If $\pi(B) > 0$, then by the Pointcare recurrence theorem there exists $F \subset B$, $\pi(F) = \pi(B) > 0$ such that $\overrightarrow{\theta} \in F$ implies that there exists a sequence $n_1 < n_2 < \cdots$ such that $T^{n_i}\overrightarrow{\theta} \in F$, $i = 1, 2, \ldots$. Since π is ergodic we have $\pi(\bigcup\limits_{n=0}^{\infty} T^{-n}F) = 1$ by the ergodic theorems. It follows that to almost everywhere $\overrightarrow{\theta}$ there exists an $m \geq 0$ such that $T^m\overrightarrow{\theta} \in F$. So there exists a sequence $n'_1 < n'_2 < \cdots$ such that $T^{m+n'_i}\overrightarrow{\theta} \in F$. Let n'_k be large enough such that $m + n'_k \geq N$, $T^{m+n'_k}\overrightarrow{\theta} \in F$ then we have

$$\begin{aligned} G(\overrightarrow{\theta}; x, x) &= \sum_{n=1}^{\infty} P_{(x,\overrightarrow{\theta})}(X_n = x) \geq \sum_{n=m+n'_k+1}^{\infty} P_{(x,\overrightarrow{\theta})}(X_{m+n'_k} = x, X_n = x) \\ &\geq P^{(m+n'_k)}(\overrightarrow{\theta}; x, x) \sum_{n=m+n'_k+1}^{\infty} P_{(x,T^{m+n'_k}\overrightarrow{\theta})}(X_n = x) = \infty. \end{aligned}$$

So $\pi(\overrightarrow{\theta} : G(\overrightarrow{\theta}; x, x) = \infty) = 1$, it follows that $\pi(C_x) = 1$ by Lemma 2.1. $\square$

Definition 2.5 (Durrett, 2005) A measure-preserving transformation T on $(\Omega, \mathscr{F}, \pi)$ is called mixing about π if for all $A, B \in \mathscr{F}$

$$\lim_{n\to\infty} \pi(AT^{-n}B) = \pi(A)\pi(B).$$

Lemma 2.2 Let transformation T about π be mixing. If $\pi(B) > 0$, then to every positive integers d and N we have $\pi\big(\bigcup\limits_{n=N}^{\infty} T^{-nd}B\big) = 1$.

Proof $\pi(B) > 0$ implies that $\pi(B^c) < 1$. The transformation T about π is mixing implies that for all $A, B \in \mathscr{F}$

$$\lim_{n\to\infty} \pi(AT^{-n}B) = \pi(A)\pi(B).$$

Hence given any $\varepsilon > 0$ there exists an N such that $n \geq N$ implies

$$\pi(AT^{-n}B) \leq \pi(A)\pi(B) + \varepsilon.$$

Set $n_1 = N$ and let $n_2 - n_1$ be large enough such that

$$\pi(T^{-n_1 d}B^c \cap T^{-n_2 d}B^c) \leq \pi(T^{-n_1 d}B^c)\pi(T^{-n_2 d}B^c) + \frac{\varepsilon}{2} = (\pi(B^c))^2 + \frac{\varepsilon}{2}.$$

Let $n_3 - n_2$ be large enough such that

$$\begin{aligned}\pi(T^{-n_1 d}B^c \cap T^{-n_2 d}B^c \cap T^{-n_3 d}B^c) &\leq \pi(T^{-n_1 d}B^c \cap T^{-n_2 d}B^c)\pi(T^{-n_3 d}B^c) + \frac{\varepsilon}{4} \\ &\leq (\pi(B^c))^3 + \frac{\varepsilon}{2} + \frac{\varepsilon}{4}.\end{aligned}$$

Simiarly,

$$\pi(T^{-n_1 d}B^c \cap T^{-n_2 d}B^c \cap \cdots \cap T^{-n_k d}B^c) \leq (\pi(B^c))^k + \varepsilon$$

holds for every positive integers k. It follows that

$$\pi\Big(\bigcap_{n=N}^{\infty} T^{-nd}B^c\Big) \leq \pi(T^{-n_1 d}B^c \cap T^{-n_2 d}B^c \cap \cdots \cap T^{-n_k d}B^c) \leq (\pi(B^c))^k + \varepsilon.$$

Letting $n \to \infty$ and then $\varepsilon \downarrow 0$, we conclude that $\pi\big(\bigcap\limits_{n=N}^{\infty} T^{-nd}B^c\big) = 0$. It follows that $\pi\big(\bigcup\limits_{n=N}^{\infty} T^{-nd}B\big) = 1$. $\square$

Theorem 2.5 If the transformation T about π be mixing and the period of x exists, then $\pi(C_x) = 0$ or 1.

Proof Let $d_x = d$. By Theorem 2.2 there exists a positive integer N such that $n \geq N$ implies

$$\pi(\{\overrightarrow{\theta} : P^{(nd)}(\overrightarrow{\theta}; x, x) > 0\}) = 1.$$

Let

$$B = \{\overrightarrow{\theta} : G(\overrightarrow{\theta}; x, x) = \infty\}.$$

If $\pi(B) > 0$, then by the Lemma 2.2 we have

$$\pi\Big(\bigcup_{n=N}^{\infty} T^{-nd}B\Big) = 1.$$

To almost everywhere $\overrightarrow{\theta}$ there exists an $n' \geq N$ such that $T^{n'd}\overrightarrow{\theta} \in B$, it follows that

$$G(\overrightarrow{\theta}; x, x) = \sum_{n=1}^{\infty} P_{(x,\overrightarrow{\theta})}(X_n = x) \geq P^{(n'd)}(\overrightarrow{\theta}; x, x) \sum_{n=n'd+1}^{\infty} P_{(x, T^{n'd}\overrightarrow{\theta})}(X_n = x) = \infty.$$

So $\pi(\overrightarrow{\theta} : G(\overrightarrow{\theta}; x, x) = \infty) = 1$, it follows that $\pi(C_x) = 1$ by Lemma 2.1. □

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随机环境中马氏链的状态周期

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引入了随机环境中马氏链的状态周期, 研究了周期的一些性质, 在假定状态周期存在的条件下, 研究了一个未解决问题(Orey, 1991; 问题1.3.3).

关键词: 随机环境中马尔可夫链, 周期, 分类.

学科分类号: O211.4.