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L^2 Rate of Algebraic Convergence for Diffusion Processes on Non-Convex Manifold *

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Abstract

Algebraic convergence in L^2-sense is studied for the reflecting diffusion processes on noncompact manifold with non-convex boundary. A series of sufficient and necessary conditions for the algebraic convergence are presented.

Keywords: Non-convex manifold, algebraic convergence, diffusion process.

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§1. Introduction

Let (M, g) be a d-dimensional noncompact connected Riemannian manifold with non-convex boundary ∂M. Consider the operator $L = \triangle + \nabla h$ for some $h \in C^1(M)$ with $Z := \int_M \mathrm{e}^{h(x)}\mathrm{d}x < \infty$, where $\triangle$ and ∇ are, respectively, the Laplace and the gradient operators associated to the Riemannian metric g. Let $\pi(\mathrm{d}x) = Z^{-1}\mathrm{e}^{h(x)}\mathrm{d}x$, where $\mathrm{d}x$ is the Riemannian volume measure. It follows that L is symmetric with respect to π for all functions in $C_0^\infty(M)$.

Let X_t be the reflecting L-diffusion process on M, which is assumed to be non-explosive (see the conditions, for example, in Ikeda and Watanabe, 1981; Chapter V and Wang, 2013; Theorem 3.1.1), and P_t be the corresponding Markov semigroup. It is known that the process has algebraic convergence in L^2-sense if there exist a functional $V : L^2(\pi) \to [0, \infty]$ and constants $C > 0$, $q > 1$ such that

$$\|P_t f - \pi(f)\|^2 \le CV(f)/t^{q-1}, \qquad t > 0,\ f \in L^2(\pi), \tag{1.1}$$

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where $\|\cdot\|$ denotes the L^2-norm, and $\pi(f) = \int_M f\mathrm{d}\pi$. From now on, we use $\int$ to represent $\int_M$ for convenience.

The main purpose of the paper is to work out some explicit criteria for the algebraic convergence of the reflecting diffusion process on noncompact connected manifold M with non-convex boundary. Actually, what we are doing is to find out some $q > 1$ such that $\sup\limits_{t>0} t^{q-1}\|P_t f - \pi(f)\|$ is bounded within a class of functions. So the algebraic convergence depends heavily on the functional V. When ∂M is convex or empty, we can use the Kendal-Cranston's coupling to construct the functional V and obtain some criteria for the algebraic convergence, see Wang (2006) for detail. But when ∂M is non-convex, the coupling method can not be adopted directly as the minimal geodesic between two points may not be entirely contained in M, see Kendall (1986) for the original reference. This paper is devoted to dealing with this difficulty. The main idea of the paper is to make a proper conformal change of the metric to reduce the non-convex case to the convex, see the following Lemma 2.1. And we find a relatively good distance to construct the function V, which is also one of innovations of this paper. The second innovation is that using the polar coordinates transformation, we obtain a series of sufficient and necessary conditions of algebraic convergence for the reflecting diffusion processes on the noncompact manifold M with non-convex boundary.

§2. Main Results

Consider the reflecting L-diffusion process. Let N be the outward unit normal vector field of ∂M and $\mathscr{H} = \{f|f \in L^2(\pi), Nf|_{\partial M} = 0\}$. According to (1.1), the definition of algebraic convergence, the diffusion process will have algebraic convergence rate if $V(f) := \sup\limits_{t\geq 0} t^{q-1}\|P_t f - \pi f\|^2$ is bounded within the class of functions $\mathscr{H}$ for some constant $q > 1$. Obviously this V satisfies the following conditions:

$$V(cf + d) = c^2 V(f), \qquad \text{for all } c, d \in \mathbb{R}. \tag{2.1}$$

$$\text{For some constant } c > 0, \quad V(P_t f) \leq cV(f), \quad \text{for all } t \geq 0,\ f \in L^2(\pi). \tag{2.2}$$

The inequality (2.2) is the contraction condition. Conditions (2.1) and (2.2) are the prerequisites for us to construct V. Since it is difficult for us to use coupling method to construct the functional V as before (see for instance Wang, 2006; Chen and Wang, 2003; Wang, 2004), we are trying to construct it in another way without using coupling method. Prior to this, we need make the conformal change of Riemannian metric, so that the manifold M under the new metric is much easier to handle. We have the following lemma from Wang (2007):

Lemma 2.1 Let Sect be the sectional curvature of M. Suppose M satisfies the following conditions:

Sect $\le k$, where $k \ge 0$;

For some constants $K \in \mathbb{R}$, there is $\mathrm{Ric}(Y,Y) \ge (d-1)K|Y|^2$ for all $Y \in TM$;

For some constants $\sigma, r \ge 0$, there is $-\sigma|X|^2 \le \langle \nabla_X N, X\rangle \le r|X|^2$ for all $X \in T\partial M$.

Then there is a function $\varphi \in C^\infty(M)$ being strictly positive, such that ∂M is convex under the new metric $\widetilde{g} = \varphi^{-2} g$.

For the detailed construction of φ, the reader can refer to Wang (2007). Now we can discuss the convergence problem of the diffusion processes under the new metric $\widetilde{g}$. For $(M, \widetilde{g})$, let $\widetilde{\nabla}$, $\widetilde{\triangle}$ be the corresponding gradient operator and Laplace-Beltrami operator, $\widetilde{\mathrm{Ric}}$ be the new Ricci curvature and $\rho(x,y)$ be the new Riemannian distance from x to y. It's easy to get the following equations:

$$\triangle = \varphi^{-2}\big(\widetilde{\triangle} + (d-2)\widetilde{\nabla}\log\varphi\big), \qquad \nabla h = \varphi^{-2}\widetilde{\nabla} h.$$

Then the operator L can be rewritten as

$$L = \varphi^{-2}\big[\widetilde{\triangle} + (d-2)\widetilde{\nabla}\log\varphi + \widetilde{\nabla} h\big] = \varphi^{-2}(\widetilde{\triangle} + W),$$

where $W = (d-2)\widetilde{\nabla}\log\varphi + \widetilde{\nabla} h$.

Let J be a positive function satisfying

$$J(r) = \sup\big\{\widetilde{\triangle}\rho(\cdot,x)(y) + \langle W(y), \widetilde{\nabla}\rho(\cdot,x)(y)\rangle_{\widetilde{g}},\ \rho(x,y) = r,\ (x,y) \notin \mathrm{Cut}\big\} < \infty,$$

where $\mathrm{Cut} := \{(x,y) : x \text{ and } y \text{ are conjugate points}\}$. How to get the quantitative expression of $J(r)$? Let's give an example. Recall that $\widetilde{\mathrm{Ric}} < 0$ for the noncompact manifold. In particular, if $\widetilde{\mathrm{Ric}} \ge (d-1)k_0$ for some $k_0 < 0$ and $\|W\|_\infty < \infty$, using Laplacian comparison theorem, we have $\widetilde{\triangle}\rho(\beta(r)) \le (d-1)\eta'(r)/\eta(r)$ and $\langle W, \widetilde{\nabla}\rho\rangle_{\widetilde{g}} < \|W\|_\infty < \infty$. So we can just take $J(r) = (d-1)\eta'(r)/\eta(r) + \|W\|_\infty$, where

$$\eta(r) = \sinh(\sqrt{-k_0}\, r)/\sqrt{-k_0}, \qquad k_0 < 0. \tag{2.3}$$

For any continuous function $\psi(x) > 0$, let

$$C(r) := \int_0^r J(s)\mathrm{d}s; \quad F(r) := \int_0^r \mathrm{e}^{-C(s)}\mathrm{d}s \int_s^\infty \mathrm{e}^{C(u)}\psi(u)\mathrm{d}u, \qquad r \ge 0.$$

Especially, if we take $\psi(x) \equiv 1$, $F(r)$ can be just taken as $\int_0^r \mathrm{e}^{-C(s)}\mathrm{d}s$.

Now, we can construct the function V as follows.

Lemma 2.2 Let $V(f) = \sup\limits_{x\neq y}\big|[f(x)-f(y)]/[F\circ\rho(x,y)]\big|^2$. Then $V(f)$ satisfies conditions (2.1), (2.2).

Let us consider the geodesic polar coordinates of metric on M: (r,θ), which has the expression: $\mathrm{d}s^2 = \mathrm{d}r^2 + \sum\limits_{i,j} h_{ij}(r,\theta)\mathrm{d}\theta_i\mathrm{d}\theta_j$, $1\le i,j\le d-1$, where θ is $(d-1)$-dimensional unit spherical coordinates and (h_{ij}) is a positive matrix. The volume element of M under geodesic polar coordinates is denoted by $A(r,\theta)\mathrm{d}r\wedge\mathrm{d}\theta$, where $\mathrm{d}\theta$ is the volume element on the $(d-1)$-dimensional unit spherical $S^{d-1}(1)$. Assume that $o\in M$, and denote $\rho(o,x) = \rho(x)$. As θ is the spherical coordinate of $\exp_o^{-1}(x)/\rho(x)$, without confusion, we can denote $\theta = \exp_o^{-1}(x)/\rho(x)$. Let $\beta : [0,\rho(o,x)] \to M$ be the minimal geodesic from o to x under $(M,\widetilde{g})$. So $\beta(t) = \exp_o(t\theta)$, $t\in[0,R(\theta))$, where $R(\theta) = \sup\{t : \beta(t)\in M\}$. By the geodesic completeness of M, we have that $\beta(R(\theta))\in\partial M\cup\mathrm{cut}(o)$, $\mathrm{cut}(o)$ is the set of conjugate points of o.

Let $\xi(x)>0$ be a test function on the manifold M. Denote

$$\sigma_1(x) := \frac{\varphi^2(x)}{\pi(x)A(\rho(x),\theta)}\int_{\rho(x)}^{R(\theta)}\frac{A(r,\theta)}{\xi^2(\beta(r))}\rho(\beta(r))\pi(\beta(r))\mathrm{d}r.$$

The following result is the main theorem of algebraic convergence for the reflecting L-diffusion process on M with non-convex boundary.

Theorem 2.1 Suppose that M satisfies the conditions of Lemma 2.1. Let V be as in Lemma 2.2.

(i) If there exists a positive function $\xi(x)$ on M and a constant $q>1$, such that

$$\sup_{x\in M}\sigma_1(x)<\infty, \qquad \int F(\rho(x))^2\xi(x)^{2q-2}\pi(\mathrm{d}x)<\infty.$$

Then the process has algebraic convergence, i.e. (1.1) holds.

(ii) On the contrary, assume that the process has algebraic convergence with respect to V. If there is a monotone increasing function $\zeta : M\to[0,\infty)$ satisfying

(a) there exists sufficiently large positive integer m so that $\int\zeta^{2m}\mathrm{d}\pi = \infty$;

(b) $\|\nabla\zeta\|^2$ has upper bound;

(c) there exists $\varepsilon<2q$ such that $V(\zeta\wedge N)\le N^{\varepsilon}$ for all $N\in\mathbb{N}$.

Then for all $0\le k\le 2q-\varepsilon$, we have $\int\zeta^k(x)\pi(\mathrm{d}x)<\infty$.

Using a different method from the proof of Theorem 2.1, we can obtain another criterion of algebraic convergence. Let $\xi>0$ as before. Define

$$\sigma_2(x) := \frac{\xi(x)\varphi(x)}{\pi(x)A(\rho(x),\theta)}\int_{\rho(x)}^{R(\theta)}\frac{A(r,\theta)}{\xi^2(\beta(r))}\pi(\beta(r))\mathrm{d}r.$$

Theorem 2.2 Suppose that M satisfies the conditions of Lemma 2.1. Let V be as in Lemma 2.2. If there exists a positive function $\xi(x)$ and a constant $q>1$, satisfying

$$\sup_{x\in M}\sigma_2(x)<\infty,\qquad \int F(\rho(x))^2\xi(x)^{2q-2}\pi(\mathrm{d}x)<\infty,$$

then the process has algebraic convergence with respect to V.

According to Theorem 2.2, we select a particular test function to get a more convenient criterion.

Corollary 2.1 Suppose that M satisfies the conditions of Lemma 2.1 and define V as in Lemma 2.2. If there exists $q>1$, such that

$$\int_0^\infty \mathrm{e}^{C(u)}\psi(u)\mathrm{d}u\cdot\int\varphi^{-2}(x)\sigma_2(x)^{2q}\pi(\mathrm{d}x)<\infty,$$

then the process has algebraic convergence with respect to V.

In particular, if $(M,\widetilde{g})$ has constant curvature K_0, then we have $A(r,\theta)=\eta(r)^{d-1}$, where $\eta(r)$ defined as (2.3) with K_0 instead of k_0. So we can get more explicit expressions of $\sigma_1(x)$ and $\sigma_2(x)$ which are not very difficult to computed.

For more general case, we can use the comparison theorem to simplify σ_1 and σ_2. For example, if $\widetilde{\mathrm{Ric}}\geq(d-1)k_0$ for some $k_0<0$, denote η by (2.3) and define

$$\delta_1(x):=\frac{\varphi^2(x)}{\pi(x)\eta(\rho(x))^{d-1}}\int_{\rho(x)}^{R(\theta)}\frac{\eta(r)^{d-1}}{\xi^2(\beta(r))}\rho(\beta(r))\pi(\beta(r))\mathrm{d}r;$$
$$\delta_2(x):=\frac{\xi(x)\varphi(x)}{\pi(x)\eta(\rho(x))^{d-1}}\int_{\rho(x)}^{R(\theta)}\frac{\eta(r)^{d-1}}{\xi^2(\beta(r))}\pi(\beta(r))\mathrm{d}r.$$

In fact, $\delta_1(x)$, $\delta_2(x)$ are the new expressions of $\sigma_1(x)$, $\sigma_2(x)$ by replacing $A(r,\theta)$ with $\eta(r)^{d-1}$ respectively. Then we have the corollary as follows.

Corollary 2.2 Under the additional condition of $\widetilde{\mathrm{Ric}}\geq(d-1)k_0$ for some $k_0<0$, Theorem 2.1 and Theorem 2.2 are still valid by using $\delta_i(x)$ instead of $\sigma_i(x)$, $i=1,2$, respectively.

In fact, our results are also suitable for the convex manifolds because $\varphi\equiv1$ in this case. Especially, when $M=\mathbb{R}^d$, we have $\mathrm{Ric}\equiv0$, $\beta(r)=ry/\rho(y)$, Cut is null, $R(\theta)\equiv+\infty$ and $\varphi\equiv1$. Thus,

$$\begin{aligned}\sigma_2(y)&=\frac{\xi(y)}{\pi(y)\rho(y)^{d-1}}\int_{\rho(y)}^{+\infty}r^{d-1}\xi^{-2}(ry/\rho(y))\pi(ry/\rho(y))\mathrm{d}r\\&=\frac{\xi(y)\rho(y)}{\pi(y)}\int_1^{+\infty}t^{d-1}\xi^{-2}(ty)\pi(ty)\mathrm{d}t,\end{aligned}$$

which is consistent with Theorem 2.1 of Wang (2004). Therefore, the above conclusions are more general.

Furthermore, our conclusion can be used for the hyperplane. We have known that many manifolds with boundary are diffeomorphic to the hyperplane. Consider the special case that M is d-dimensional upper half-plane $\mathbb{H}^d$. Let o as the origin of the coordinates. To judge the algebraic convergence, we only need to take supremum of σ_i $(i = 1, 2)$ for $y \in \mathbb{H}^d$.

§3. Proofs of the Results

Our starting point is the following theorem which gives a sufficient (necessary) condition of algebraic convergence for general Markov semigroup. See Liggett (1991) for details.

Theorem 3.1 (Liggett-Stroock) Let $1 < p, q < \infty$, such that $1/q + 1/p = 1$. Suppose that $V : L^2(\pi) \to [0, \infty]$ satisfies $V(cf + d) = c^2 V(f)$ for all $c, d \in \mathbb{R}$. Consider the following two statements.

(i) There exists a constant $C' > 0$, such that

$$\|f - \pi f\|^2 \le C' D(f)^{1/p} V(f)^{1/q}, \qquad f \in \mathscr{D}(D), \tag{3.1}$$

where $D(f) = \int |\nabla f|^2 \pi(x) \mathrm{d}x$ is the Dirichlet form of the operator L with domain $\mathscr{D}(D) = \{f \in L^2(\pi) : D(f) < \infty\}$.

(ii) There exists a constant $C > 0$, such that $\|P_t f - \pi f\|^2 \le CV(f)/t^{q-1}$, for all $t > 0$, $f \in L^2(\pi)$.

Then we have the following conclusions:

(a) If (i) holds and V satisfies the contraction condition $V(P_t(f)) \le cV(f)$, $c > 0$, then (ii) holds;

(b) If the process is reversible with respect to π and (ii) holds, then (i) holds.

Remark 1 In the original Liggett-Stroock theorem in Liggett (1991), the contraction property of V is represented as $V(P_t f) \le V(f)$. Here we use $V(P_t(f)) \le cV(f)$ instead of $V(P_t f) \le V(f)$. It is easy to check that the Liggett-Stroock theorem still holds.

In fact, the criteria for algebraic convergence of this paper are some more explicit conditions for (3.1) of the Liggett-Stroock theorem in the context of diffusion processes on M. Now, we will start from the contraction property of V.

Proof of Lemma 2.2 Apparently, $V(f)$ satisfies (2.1). When $x \neq y$, we have

$$\Big|\frac{P_tf(x)-P_tf(y)}{F\circ\rho(x,y)}\Big|^2 \le 3\Big[\Big|\frac{P_tf(x)-f(y)}{F\circ\rho(x,y)}\Big|^2+\Big|\frac{f(x)-f(y)}{F\circ\rho(x,y)}\Big|^2+\Big|\frac{P_tf(y)-f(x)}{F\circ\rho(x,y)}\Big|^2\Big]. \tag{3.2}$$

By symmetricity, we only need to prove

$$\sup_{x\neq y}\Big|\frac{P_tf(x)-f(y)}{F\circ\rho(x,y)}\Big|^2 \le V(f). \tag{3.3}$$

For some fixed y, we temporarily abbreviate $\rho(x,y)$ to $\rho(x)$ for convenience. Firstly, we have

$$\begin{aligned} LF(\rho(x)) &= \varphi^{-2}[\widetilde{\triangle}F(\rho(x))+WF(\rho(x)] = \varphi^{-2}[F''(\rho(x))+F'(\rho(x))(\widetilde{\triangle}\rho(x)+W\rho(x))] \\ &\le \varphi^{-2}[F''(\rho(x))+J(\rho(x))F'(\rho(x))] = -\varphi^{-2}\psi(\rho(x)) \le 0. \end{aligned}$$

Next, let X_t be the L-diffusion process with $X_0=x$. We have

$$\mathsf{E}^xF(\rho(X_t,y)) = F(\rho(x,y)) + \int_0^t \mathsf{P}_sLF(\rho(x))\mathrm{d}s \le F(\rho(x,y)).$$

Thus, we have

$$\begin{aligned} &\Big|\frac{P_tf(x)-f(y)}{F\circ\rho(x,y)}\Big|^2 = \Big|\mathsf{E}^x\Big(\frac{f(X_t)-f(y)}{F(\rho(X_t,y))}\cdot\frac{F(\rho(X_t,y))}{F(\rho(x,y))}\Big)\Big|^2 \\ \le &\Big(\sup_{x\neq y}\Big|\frac{f(x)-f(y)}{F(\rho(x,y))}\Big|^2\Big)\cdot\Big|\frac{\mathsf{E}^xF(\rho(X_t,y))}{F(\rho(x,y))}\Big|^2 = V(f)\cdot\Big|\frac{\mathsf{E}^xF(\rho(X_t,y))}{F(\rho(x,y))}\Big|^2 \le V(f), \end{aligned}$$

which yields (3.3) by making the supremum over all $x\neq y$ on the left-hand side.

Therefore, taking the supremum over all $x \neq y$ on the left-hand side of (3.2) yields $V(P_tf)\le 9V(f)$ for all $t\ge 0$. Hence the proof of Lemma 2.2 is completed. □

Proof of Theorem 2.1 We only prove the part (i) of the theorem since the proof of part (ii) is quite similar as the diffusion processes on $\mathbb{R}^n$. See Wang (2007; Theorem 2.1) for details.

By the Liggett-Stroock theorem, we only to check condition (3.1). So we will start from the variance $\|f-\pi f\|^2$, $f\in L^2(\pi)$. Because the diffusion process which we studied is reflecting on ∂M, f must satisfy Neumann boundary, i.e. $Nf|_{\partial M}=0$. Since $C_0^\infty(M)$ is dense in the function class $\mathscr{H}=\{f|f\in L^2(\pi), Nf|_{\partial M}=0\}$, it is easy to see that $C_0^\infty(M)$ is large enough for us to discuss the problem by an approximation argument.

Suppose that $o\in M$ is the original point, and recall that $\rho(o,x)=\rho(x)$. Let $f\in$

$C_0^\infty(M)$, $\xi > 0$. We have

$$\begin{aligned}\|f-\pi f\|^2 &= \inf_c \int \pi(\mathrm{d}x)(f(x)-c)^2 \le \int \pi(\mathrm{d}x)(f(x)-f(o))^2 \\ &\le \Big\{\int \Big(\frac{f(x)-f(o)}{\xi(x)}\Big)^2 \pi(\mathrm{d}x)\Big\}^{1/p} \Big\{\int \Big(\frac{f(x)-f(o)}{\xi(x)}\Big)^2 \xi(x)^{2q}\pi(\mathrm{d}x)\Big\}^{1/q} \\ &=: \mathrm{I}^{1/p}\cdot \mathrm{II}^{1/q}. \end{aligned} \tag{3.4}$$

Let $\beta:[0,\rho(x)]\to M$ $(\beta(0)=o,\ \beta(\rho(x))=x)$ be the minimal geodesic between o and x under the new metric $\widetilde{g}$. Then $|\mathrm{d}\beta(t)/\mathrm{d}t|_{\widetilde{g}}=1$, and

$$\frac{\mathrm{d}f(\beta(t))}{\mathrm{d}t} = \Big\langle \widetilde{\nabla} f, \frac{\mathrm{d}\beta(t)}{\mathrm{d}t}\Big\rangle_{\widetilde{g}} \le |\widetilde{\nabla} f|_{\widetilde{g}}\cdot \Big|\frac{\mathrm{d}\beta(t)}{\mathrm{d}t}\Big|_{\widetilde{g}} = |\widetilde{\nabla} f|_{\widetilde{g}}.$$

We get

$$\begin{aligned}\mathrm{I} &= \int \xi^{-2}(x)(f(x)-f(o))^2\pi(\mathrm{d}x) \le \int \xi^{-2}(x)\Big(\int_0^{\rho(x)} \frac{\mathrm{d}f(\beta(t))}{\mathrm{d}t}\mathrm{d}t\Big)^2\pi(\mathrm{d}x) \\ &\le \int \xi^{-2}(x)\Big(\int_0^{\rho(x)} |\widetilde{\nabla} f|_{\widetilde{g}}(\beta(t))\mathrm{d}t\Big)^2\pi(\mathrm{d}x) \le \int \xi^{-2}(x)\rho(x)\pi(\mathrm{d}x)\int_0^{\rho(x)} |\widetilde{\nabla} f|^2_{\widetilde{g}}(\beta(t))\mathrm{d}t.\end{aligned}$$

Then, we establish geodesic polar coordinates (r,θ) with o as the original point. We have

$$\begin{aligned}\mathrm{I} &\le \int_{S^{d-1}(1)}\mathrm{d}\theta\int_0^{R(\theta)} A(r,\theta)\xi^{-2}(\beta(r))\rho(\beta(r))\pi(\beta(r))\mathrm{d}r\int_0^r |\widetilde{\nabla} f|^2_{\widetilde{g}}(\beta(s))\mathrm{d}s \\ &= \int_{S^{d-1}(1)}\mathrm{d}\theta\int_0^{R(\theta)} |\widetilde{\nabla} f|^2_{\widetilde{g}}(\beta(s))\mathrm{d}s\int_s^{R(\theta)} A(r,\theta)\xi^{-2}(\beta(r))\rho(\beta(r))\pi(\beta(r))\mathrm{d}r \\ &= \int A^{-1}(\rho(y),\theta)|\widetilde{\nabla} f|^2_{\widetilde{g}}(y)\mathrm{d}y\int_{\rho(y)}^{R(\theta)} A(r,\theta)\xi^{-2}(\beta(r))\rho(\beta(r))\pi(\beta(r))\mathrm{d}r \\ &= \int \sigma_1(y)\varphi^{-2}(y)\pi(y)|\widetilde{\nabla} f|^2_{\widetilde{g}}(y)\mathrm{d}y.\end{aligned}$$

Here in the third step, θ is the spherical coordinates of $\exp_o^{-1}(y)/\rho(y)$. By the conformal change, we have

$$|\widetilde{\nabla} f|^2_{\widetilde{g}} = \varphi^2\langle \nabla f, \nabla f\rangle.$$

Combining the above results with $\sup\limits_{y\in M}\sigma_1(y)<\infty$ gives that there exists constant C_1 such that

$$\mathrm{I} \le \sup_{y\in M}\sigma_1(y)\int |\nabla f|^2\pi(y)\mathrm{d}y = C_1\int |\nabla f|^2\pi(y)\mathrm{d}y = C_1 D(f). \tag{3.5}$$

On the other hand,

$$\mathrm{II} = \int \Big(\frac{f(x)-f(o)}{\xi(x)}\Big)^2\xi(x)^{2q}\pi(\mathrm{d}x) \le V(f)\int F(\rho(x))^2\xi(x)^{2q-2}\pi(\mathrm{d}x). \tag{3.6}$$

Hence we get II $\leq C_2V(f)$ for some constant C_2 when $\int F(\rho(x))^2\xi(x)^{2q-2}\pi(\mathrm{d}x) < \infty$.

Combining (3.5), (3.6) with (3.4), we obtain that there exists constant C such that $\|f-\pi(f)\|^2 \leq CD(f)^{1/p}V(f)^{1/q}$. By Theorem 3.1 (Liggett-Stroock theorem), the reflecting L-diffusion process has algebraic convergence. $\square$

Proof of Theorem 2.2 Let the symbols I and II be the same as in the proof of Theorem 2.1. We have

$$\begin{aligned}
\mathrm{I} &\leq 2\int_{S^{d-1}(1)}\mathrm{d}\theta\int_0^{R(\theta)} A(r,\theta)\xi^{-2}(\beta(r))\pi(\beta(r))\mathrm{d}r\cdot\int_0^r|\widetilde{\nabla} f|_{\widetilde{g}}(\beta(t))\mathrm{d}t\int_0^t|\widetilde{\nabla} f|_{\widetilde{g}}(\beta(s))\mathrm{d}s\\
&= 2\int_{S^{d-1}(1)}\mathrm{d}\theta\int_0^{R(\theta)}|\widetilde{\nabla} f|_{\widetilde{g}}(\beta(t))\mathrm{d}t\cdot\int_t^{R(\theta)} A(r,\theta)\xi^{-2}(\beta(r))\pi(\beta(r))\mathrm{d}r\int_0^t|\widetilde{\nabla} f|_{\widetilde{g}}(\beta(s))\mathrm{d}s\\
&= 2\int|\widetilde{\nabla} f|_{\widetilde{g}}(x)A^{-1}(\rho(x),\theta)\Big\{\int_{\rho(x)}^{R(\theta)}A(r,\theta)\xi^{-2}(\beta(r))\pi(\beta(r))\mathrm{d}r\cdot\int_0^{\rho(x)}|\widetilde{\nabla}(f)|_{\widetilde{g}}(\beta(s))\mathrm{d}s\Big\}\mathrm{d}x.
\end{aligned}$$

By Schwarz's inequality, we obtain

$$\begin{aligned}
\mathrm{I} &\leq 2\Big[\int\xi^{-2}(x)\Big(\int_0^{\rho(x)}|\widetilde{\nabla} f|_{\widetilde{g}}(\beta(s))\mathrm{d}s\Big)^2\pi(x)\mathrm{d}x\Big]^{1/2}\\
&\quad\cdot\Big[\int\Big(|\widetilde{\nabla} f|_{\widetilde{g}}(x)\frac{1}{\pi(x)}\xi(x)\int_{\rho(x)}^{R(\theta)}A^{-1}(\rho(x),\theta)A(r,\theta)\xi^{-2}(\beta(r))\pi(\beta(r))\mathrm{d}r\Big)^2\pi(x)\mathrm{d}x\Big]^{1/2}\\
&\leq 4\int|\nabla f|^2(x)\frac{\varphi^2(x)\xi^2(x)}{\pi^2(x)A^2(\rho(x),\theta)}\Big(\int_{\rho(x)}^{R(\theta)}A(r,\theta)\xi^{-2}(\beta(r))\pi(\beta(r))\mathrm{d}r\Big)^2\pi(x)\mathrm{d}x\\
&= 4\int|\nabla f|^2(x)\sigma_2(x)\pi(x)\mathrm{d}x.
\end{aligned}$$

This means when $\sup\limits_{x\in M}\sigma_2(x)<\infty$, $\mathrm{I}\leq CD(f)$ for some constant C. The rest of the proof is the same as the corresponding part of the proof of Theorem 2.1. $\square$

Proof of Corollary 2.1 We select a simple test function $\xi\equiv 1$. By a proof similar to that of Theorem 2.2 and Schwarz's inequality, we obtain that

$$\begin{aligned}
\|f-\pi(f)\|^2 &\leq \int\pi(\mathrm{d}x)(f(x)-f(o))^2 \leq 4\int|\nabla f|^2(x)\sigma_2(x)^2\pi(\mathrm{d}x)\\
&\leq 4\Big[\int|\nabla f|^2(x)\pi(\mathrm{d}x)\Big]^{1/p}\Big[\int|\nabla f|^2(x)\sigma_2(x)^{2q}\pi(\mathrm{d}x)\Big]^{1/q}.
\end{aligned}$$

Let γ be the minimal geodesic satisfying $\gamma(t)=\exp_x(tv)$ and $v=\widetilde{\nabla} f(x)$. So we have $\gamma(0)=x$, $\gamma'(0)=\widetilde{\nabla} f(x)$. Thus,

$$|\nabla f|^2=\varphi^{-2}|\widetilde{\nabla} f|^2_{\widetilde{g}}=\varphi^{-2}\Big|\lim_{t\to 0}\frac{f(\gamma(t))-f(x)}{\rho(\gamma(t),x)}\Big|^2\leq V(f)\varphi^{-2}(x)|F'(0)|^2.$$

Then we obtain

$$\|f-\pi(f)\|^2\leq 4D(f)^{1/p}V(f)^{1/q}\Big\{\int\varphi^{-2}(x)|F'(0)|^2\sigma_2(x)^{2q}\pi(\mathrm{d}x)\Big\}^{1/q}.$$

According to Theorem 3.1, Corollary 2.1 is proved. □

Proof of Corollary 2.2 From the conditions of Lemma 2.1, using comparison theorem (see Cheeger and Ebin, 1992), we deduce that $A(r,\theta)/\eta(r)^{d-1}$ is the non-increasing function of r when $\widetilde{\mathrm{Ric}} \ge (d-1)k_0$ for some $k_0 \le 0$, which means $A(r,\theta)/A(\rho(y),\theta) \le (\eta(r)/\eta(\rho(y)))^{d-1}$ for $r \ge \rho(y)$. Thus we have $\delta_1(x) \ge \sigma_1(x)$ and $\delta_2(x) \ge \sigma_2(x)$. Using this inequality, we can replace $\sigma_1(x)$ by $\delta_1(x)$ in Theorem 2.1. Similarly, for $\sigma_2(x)$ in Theorem 2.2, we can replace it with $\delta_2(x)$ to determine whether the processes have algebraic convergence. □

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非凸流形上扩散过程的代数式收敛性

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本文研究带非凸边界的非紧流形上的反射扩散过程在L^2范数下的代数式收敛性, 给出了若干过程代数式收敛的充分的和必要的判定条件.

关键词: 非凸流形, 代数式收敛, 扩散过程.

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