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Optimal Dividend Strategy in a Jump-Diffusion Model with a Linear Barrier Constraint *

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Abstract: For a financial or insurance entity, the problem of finding the optimal dividend distribution strategy and optimal firm value function is a widely discussed topic. In the present paper, it is assumed that the firm faces two types of liquidity risks: a Brownian risk and a Poisson risk. The firm can control the time and amount of dividends paid out to shareholders. By sufficiently taking into account the safety of the company, bankruptcy is said to take place at time t if the cash reserve of the firm runs below the linear barrier $b + kt$ (not zero), see [1]. We deal with the problem of maximizing the expected total discounted dividends paid out until bankruptcy. The optimal dividend return (or, firm value) function is identified as the classical solution of the associated Hamilton-Jacobi-Bellman (HJB) equation where a second-order differential-integro equation is involved. By solving the corresponding HJB equation, the analytical solution of the optimal firm value function is obtained, the optimal dividend strategy is also characterized, which is of linear barrier type: at time t the firm keeps cash inside when the cash reserves level is less than a critical linear barrier $x_0 + kt$ and pays cash in excess of this linear barrier as dividends.

Keywords: optimal firm value function; dividend strategy; linear barrier dividend strategy

2010 Mathematics Subject Classification: 91B30; 97M30

§1. Introduction

This paper represents a model for financial valuation of a firm which has control of the dividend payment as well as potential profit by choosing different business activities among those available to it. We associate the value of the company with the expected present value of the net dividend payouts since, according to [2], the value of a company in a world of perfect capital markets is exactly the expected present value of future dividends. Our objective is to find a dividend payout scheme that maximizes the expected total discounted

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dividends payment until bankruptcy, i.e., until the capital runs below the linear barrier $b+kt$ for the first time.

During the recent decades, there has been an upsurge in dealing with market models that take into account dividend strategies, owing to the recent development of mathematical tools in finance. Højgaard and Taksar [3] allow the company to control its risk exposure by choosing its business activity and to invest in a risky asset. They show that the optimal value function is concave and find that the optimal policy is of a barrier type. Cadenillas et al. [4] provide the solution of an optimization problem where the cash reserves follow a mean reverting process. Li [5] analyzes the distribution of dividends payments of a compound risk model perturbed by diffusion for barrier strategy. Optimal dividend problems are also treated in [6–13] and the references therein.

However, most of these works are based on pure diffusion models. It has been observed that pure diffusion models are not robust enough to capture the appearance of jumps in the cash reserves. As a result, jump-diffusion processes have been gaining popularity for modelling finance. Belhaj [14] considered the problem of optimal dividend pay-outs for a firm of which the cash reserves follow a jump-diffusion process: the firm faces two types of risks, a Brownian risk that represents small movements in the cash flow and a Poisson risk that represents large movements in the cash flow and that corresponds to losses due to unexpected events. The optimal dividend payout policies (subject to maximization of the expected total discounted dividend payments until bankruptcy) with and with out insurance was obtained, which turned out to be barrier strategy.

In this paper, we will continue the study in this direction. Namely, we deal with the dividend optimization problem in the jump-diffusion financial model, while a linear barrier $b+kt$ is present, i.e., bankruptcy takes place at time t if the cash reserve of the firm runs below the linear barrier $b+kt$. The presence of the linear barrier $b+kt$ can be retrospected to [1] and can be explained as follows: the firm has to keep some cash reserves in order to protect itself against future losses since it's financially constrained. It turns out that the optimal firm value function is a classical solution to a Hamilton-Jacobi-Bellman (HJB) equation, and, the optimal dividend policy is a linear barrier strategy: at time t the firm keeps cash inside when the cash reserves level is less than a critical linear barrier x_0+kt and pays cash in excess of this linear barrier to shareholders as dividends.

We mention that, although a similar outline parallel to that of [14] is present in this paper, our new definition of bankruptcy requires a more delicate mathematical treatment, especially in the proof of the main result, i.e., Theorem 2. We also mention that, when our model degenerates to that of the existing literatures, our results coincides with the existing results, see Remarks 3–5.

The rest of this paper is organized as follows. In Section 2, we present the mathematical model. Section 3 is devoted to address the problem of finding the optimal firm value function and the optimal dividend distribution strategy.

§2. The Mathematical Model

We start with a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ endowed with a filtration $\{\mathscr{F}_t; t \geqslant 0\}$ satisfying the usual condition, a standard Brownian motion $\{w_t; t \geqslant 0\}$ and a Poisson process $\{N_t; t \geqslant 0\}$ which is independent of $\{w_t; t \geqslant 0\}$. It is assumed that both $\{w_t; t \geqslant 0\}$ and $\{N_t; t \geqslant 0\}$ are adapted to the filtration $\{\mathscr{F}_t; t \geqslant 0\}$. Our state variable is the reserve process which represents the liquid assets of the firm. In the absence of control, the reserves evolve according to

$$X_t = x + ct + \sigma w_t - \sum_{n=1}^{N_t} Y_n,$$

where $x > 0$ is the initial capital, c is the expected cash flow, σ is the volatility of the cash flow, $\{Y_n; n \geqslant 1\}$ are a sequence of i.i.d. positive random variables with cumulative distribution function $F(y) \,\widehat{=}\, \int_0^y \beta \mathrm{e}^{-\beta z} \mathrm{d}z$. We denote the intensity and the n-th jump time of the Poisson process $\{N_t; t \geqslant 0\}$ by λ and T_n, respectively. According to [14], the Brownian motion represents small movements of the cash reserves and corresponds to the continuous part in the cash flow, while the Poisson process represents large movements in the cash reserves and corresponds to big losses.

The manager can control the timing and the amount of dividends paid out to the shareholders. Under the manager's control the reserves evolve according to

$$X_t^L = x + cdt + \sigma w_t - \sum_{n=1}^{N_t} Y_n - L_t,$$

where L_t represents the cumulated dividend paid up to time t. L_t is said to be admissible if it is an adapted right-continuous nondecreasing process, and, $0 \leqslant \Delta L_t \leqslant X_{t-}^L - (b+kt)$, i.e., the manager cannot pay an amount of dividends which bring the reserve level below the linear barrier $b+kt$. Here, we get the linear dividend barrier $b+kt$ (b is a constant dividend barrier and k is a constant linear slope) involved in our model, in order to sufficiently take into account the safety of the company. Actually, the presence of the constant dividend barrier b is natural, since the constant dividend barrier strategy has already been proved the optimal dividend strategy in a large number of risk models. While, if $k = 0$, then it would probably result to an ultimate ruin probability (probability of the event that the value of the objective risk process drops below 0 in finite time) of 1. Hence, we would rather have a positive slope k in our present risk model, which do good to protecting the insurer from going bankruptcy.

Let the time of bankruptcy be defined by $\tau^L := \inf\{t \geqslant 0;\, X_t^L \leqslant b + kt\}$. With this definition of bankruptcy, it's natural to assume that $c > k$, in which case we have a positive net cash flow per unit of time.

With each admissible dividend strategy, the associated firm value function is defined as follows

$$V_L(x) := \mathsf{E}_x\Big[\int_{0-}^{\tau^L} \mathrm{e}^{-\delta t}\mathrm{d}L_t\Big],$$

i.e., the expected discounted net dividends paid out to shareholders until the time of bankruptcy. We mention that $\delta > 0$ is a discount factor, which can also be interpreted as a measure of the preference of shareholders to receive dividend payments earlier rather than later.

Denote the class of all admissible dividend policies by Π. Then the optimal firm value function of the maximization problem can be defined through $V(x) := \sup\limits_{L\in\Pi} V_L(x)$. Hence, $V(x) = 0$, for all $x \leqslant b$. Our objective is to characterize the optimal firm value function $V(x)$ and find an admissible optimal dividend policy $\{L_t^*; t \geqslant 0\}$ such that $V(x) = V_{L^*}(x)$. It needs to be mentioned that, throughout the paper, the positive safety loading condition (the expected cash flow per unit of time is larger than the expected loss: $c > \lambda/\beta$) is not needed.

§3. Constructing Optimal Firm Value Function and Optimal Dividend Distribution Strategy

The computation of the optimal return function $V(x)$ and the optimal dividend policy is based on some Hamilton-Jacobi-Bellman (HJB) equation. Before motivating the HJB equation, let us define an operator which would be important in overlapping the main results of the present paper.

Suppose that $v \in \mathbb{C}^2(\mathbb{R}) : \mathbb{R} \mapsto \mathbb{R}$ is a candidate optimal return function. Let $\mathscr{A}$ be defined as follows.

$$\mathscr{A}v(x) := \frac{\sigma^2}{2}v''(x) + (c-k)v'(x) - \lambda v(x) + \lambda\int_0^x v(x-y)\mathrm{d}F(y).$$

Using similar arguments as used in [3], we can verify that V satisfies the following dynamic programming principle.

$$V(x)=\sup_{L\in\Pi}\mathsf{E}_x\Big[\int_{0-}^{\tau\wedge\tau^L}\mathrm{e}^{-\delta t}\mathrm{d}L_t+I_{(\tau<\tau^L)}\mathrm{e}^{-\delta\tau}V(X_\tau^L-k\tau)\Big],\qquad x\geqslant b,\tag{1}$$

for any stopping time τ.

For any $h>0$ and admissible strategy $\{L_t;t\geqslant 0\}$ associated with initial capital $x>b$, define $v_h^L=h\wedge\inf\{t\geqslant 0;\ X_t^L-kt\notin(x-h,x+h)\}$, then $v_h^L\to 0$ as $h\to 0$. Choose $h<x-b$, we get from (1) that

$$V(x)\geqslant\mathsf{E}_x\big[\mathrm{e}^{-\delta v_h^L}V\big(X_{v_h^L}^L-kv_h^L\big)\big],\qquad x>b,$$

since $v_h^L<\tau^L$. Applying Itô's formula and noting that $X_s^L\neq X_{s-}^L$ only at the arrival of a claim, we get

$$\begin{aligned}V(x)\geqslant{}&V(x)-\int_{0-}^{v_h^L}\delta\mathrm{e}^{-\delta s}V(X_s^L-ks)\mathrm{d}s+\int_{0-}^{v_h^L}(c-k)\mathrm{e}^{-\delta s}V'(X_s^L-ks)\mathrm{d}s\\&+\int_{0-}^{v_h^L}\sigma\mathrm{e}^{-\delta s}V'(X_s^L-ks)\mathrm{d}w_s+\int_{0-}^{v_h^L}\frac{\sigma^2}{2}\mathrm{e}^{-\delta s}V''(X_s^L-ks)\mathrm{d}s\\&+\sum_{s\leqslant v_h^L}^{\Delta X_s\neq 0}\mathrm{e}^{-\delta s}[V(X_{s-}^L-ks+\Delta X_s^L)-V(X_{s-}^L-ks)]\\&-\lambda\int_{0-}^{v_h^L}\int_0^\infty\mathrm{e}^{-\delta s}[V(X_{s-}^L-ks-y)-V(X_{s-}^L-ks)]\mathrm{d}F(y)\mathrm{d}s\\&+\lambda\int_{0-}^{v_h^L}\int_0^\infty\mathrm{e}^{-\delta s}[V(X_{s-}^L-ks-y)-V(X_{s-}^L-ks)]\mathrm{d}F(y)\mathrm{d}s\\={}&V(x)+\int_{0-}^{v_h^L}\mathrm{e}^{-\delta s}(\mathscr{A}-\delta)V(X_s^L-ks)\mathrm{d}s+\int_{0-}^{v_h^L}\sigma\mathrm{e}^{-\delta s}V'(X_s^L-ks)\mathrm{d}w_s\\&+\int_{0-}^{v_h^L}\int_0^\infty\mathrm{e}^{-\delta s}[V(X_{s-}^L-ks-y)-V(X_{s-}^L-ks)](N(\mathrm{d}s,\mathrm{d}y)-\lambda\mathrm{d}F(y)\mathrm{d}s),\end{aligned}\tag{2}$$

where $\Delta X_s=X_s-X_{s-}$ and $N(t,A):=\sum\limits_{s\leqslant t}I_{\{N_s-N_{s-}=1,Y_{N_s}\in A\}}$, $A\in\mathscr{B}(R_+)$, $t\geqslant 0$, is the Poisson random measure associated with the compound Poisson process $\Big\{\sum\limits_{k=1}^{N_t}Y_k,t\geqslant 0\Big\}$. By [15; page 62, line 19],

$$\Big\{\int_0^t\int_0^\infty\mathrm{e}^{-\delta s}[V(X_{s-}^L-ks-y)-V(X_{s-}^L-ks)](N(\mathrm{d}s,\mathrm{d}y)-\lambda\mathrm{d}F(y)\mathrm{d}s);\ t\geqslant 0\Big\}$$

is a martingale with mean zero. Also,

$$\Big\{\int_{0-}^t\sigma\mathrm{e}^{-\delta s}V'(X_s^L-ks)\mathrm{d}w_s;\ t\geqslant 0\Big\}$$

is a martingale with zero mean. Taking expectations in (2) leads to

$$V(x) \geqslant V(x) + \mathsf{E}_x\Big[\int_{0-}^{v_h^L} \mathrm{e}^{-\delta s}(\mathscr{A}-\delta)V(X_s^L - ks)\mathrm{d}s\Big]. \tag{3}$$

Subtracting $V(x)$ from both sides of (3), dividing them by $\mathsf{E}[v_h^L]$ and letting h tends to zero, we have

$$(\mathscr{A}-\delta)V(x) \leqslant 0, \qquad x > b.$$

In addition, for any $0 < h < x - b$ we can find strategy $\{L_t^h; t \geqslant 0\}$ corresponding to initial reserve $x - h$ such that $V_{L^h}(x-h) \geqslant V(x-h) - h^2$, define a strategy corresponding to initial reserve x as $\{L_t \,\widehat{=}\, h + L_t^h;\, t \geqslant 0\}$. This corresponds to an initial payout of size h, and then following an h^2-optimal strategy. Then,

$$V(x) \geqslant V_L(x) = h + V_{L^h}(x-h) \geqslant h + V(x-h) - h^2, \qquad x > b.$$

Subtracting $V(x-h)$ from both sides, dividing by h, and letting $h \downarrow 0$ yields

$$V'(x) \geqslant 1, \qquad x > b.$$

Definition 1 A continuous and increasing function $v \in \mathbb{C}^2((b,\infty)) : (b,\infty) \mapsto (b,\infty)$ satisfies the HJB equation of the optimal dividend problem if

$$(\mathscr{A}-\delta)v(x) \leqslant 0, \qquad x > b, \tag{4}$$

$$v'(x) \geqslant 1, \qquad x > b, \tag{5}$$

$$\max\{(\mathscr{A}-\delta)v(x), 1 - v'(x)\} = 0, \qquad x > b. \tag{6}$$

The following Theorem 2 devotes to motivating a classical solution to the HJB equation, and, verifying that the candidate dividend strategy characterized in (22) bellow is the optimal dividend strategy.

Theorem 2 Let $v(x)$ be given by (17) with x_0 given by (18) or (19), then it characterizes a classical $\mathbb{C}^2((b,\infty))$ solution to the HJB equations (4)–(6). In addition, if we define a dividend strategy as in (22), then this dividend strategy is the optimal dividend strategy, and the corresponding dividend return function is exactly $v(x)$.

Proof of Theorem 2 For any solution of the HJB equation $v(x)$, define $x_0 \,\widehat{=}\, \sup\{z \geqslant b \,|\, (\mathscr{A}-\delta)v(x) \geqslant 1 - v'(x) \text{ for all } x \in [b,z]\}$, then we claim that $x_0 > b$. If it is not the case, then for any $\epsilon > 0$, there exist $x_\epsilon \in [b, b+\epsilon)$ such that $1 - v'(x_\epsilon) > (\mathscr{A}-\delta)v(x_\epsilon)$. By the continuity of the functions $1 - v'(x)$ and $(\mathscr{A}-\delta)v(x)$ (since $v(x) \in \mathbb{C}^2((b,\infty))$), there exists $\varrho > 0$ such that $1 - v'(x) > (\mathscr{A}-\delta)v(x)$ for all $x \in (x_\epsilon - \varrho, x_\epsilon + \varrho) \subseteq [b, b+\epsilon)$, hence

$(\mathscr{A}-\delta)v(x) < 1-v'(x) = \max\{(\mathscr{A}-\delta)v(x), 1-v'(x)\} = 0$, for all $x \in (x_\epsilon-\varrho, x_\epsilon+\varrho) \subseteq [b, b+\epsilon)$, which implies $v(x) = v(x_\epsilon-\varrho)+x-(x_\epsilon-\varrho)$, for all $x \in (x_\epsilon-\varrho, x_\epsilon+\varrho) \subseteq [b, b+\epsilon)$. Now we choose the above $\epsilon > 0$ small enough such that $v(b+\epsilon) < (c-k)/(\lambda+\delta)$, so, by the increasing property of $v(x)$,

$$v(x) \leqslant v(b+\epsilon) < \frac{c-k}{\lambda+\delta}, \qquad \text{for all } \ x \in (x_\epsilon-\varrho, x_\epsilon+\varrho) \subseteq [b, b+\epsilon),$$

therefore, for all $x \in (x_\epsilon-\varrho, x_\epsilon+\varrho) \subseteq [b, b+\epsilon)$ we have

$$\begin{aligned} 0 = 1-v'(x) > (\mathscr{A}-\delta)v(x) &\geqslant \frac{1}{2}\sigma^2 v''(x) + (c-k)v'(x) - (\lambda+\delta)v(x) \\ &\geqslant (c-k) - (\lambda+\delta)\frac{c-k}{\lambda+\delta} = 0, \end{aligned}$$

which is a contradiction. Thus $x_0 > b$.

In what follows immediately, we motivate solution of the HJB equation by considering the cases $b < x_0 < +\infty$ and $b < x_0 = +\infty$, respectively.

Case 1: $b < x_0 < +\infty$. In this case, we consider the following system of integro-differential equation:

$$\begin{cases} (\mathscr{A}-\delta)v(x) = 0, & x \in [b, x_0]; \\ 1-v'(x) = 0, & x \in [x_0, +\infty]; \\ \max\{(\mathscr{A}-\delta)v(x), 1-v'(x)\} = 0, & x > b. \end{cases} \tag{7}$$

Simple algebraic manipulations turn the first equation of (7) into

$$\frac{1}{2}\sigma^2 v''(x) + (c-k)v'(x) - (\lambda+\delta)v(x) + \lambda\beta e^{-\beta x}\int_b^x v(y)e^{\beta y}dy = 0, \qquad x \in [b, x_0]. \tag{8}$$

Multiplying both sides of (8) by $e^{\beta x}$ gives rise to

$$\Big[\frac{1}{2}\sigma^2 v''(x) + (c-k)v'(x) - (\lambda+\delta)v(x)\Big]e^{\beta x} = -\lambda\beta\int_b^x v(y)e^{\beta y}dy, \qquad x \in [b, x_0]. \tag{9}$$

Taking derivatives on both sides of (9) and then rearranging yields

$$\frac{1}{2}\sigma^2 v'''(x) + \Big[\frac{1}{2}\sigma^2\beta + (c-k)\Big]v''(x) + [(c-k)\beta - (\lambda+\delta)]v'(x) - \beta\delta v(x) = 0, \qquad x \in [b, x_0]. \tag{10}$$

Let $P(\theta) = \sigma^2\theta^3/2 + [\sigma^2\beta/2 + (c-k)]\theta^2 + [(c-k)\beta - (\lambda+\delta)]\theta - \beta\delta$. Then it can be verified that

$$\lim_{\theta\to-\infty} P(\theta) = -\infty; \qquad P(-\beta) = \beta\lambda > 0; \qquad P(0) = -\beta\delta < 0, \qquad \lim_{\theta\to+\infty} P(\theta) = +\infty.$$

Hence, $P(\theta)$ has three zeroes θ_1, θ_2, θ_3 with $-\infty < \theta_1 < -\beta < \theta_2 < 0 < \theta_3 < +\infty$. Additionally,

$$\theta_1 + \theta_2 + \theta_3 = -\beta - \frac{2(c-k)}{\sigma^2}.$$

Thus, the solution of the HJB equation can be expressed as

$$v(x) = C_1 \mathrm{e}^{\theta_1 x} + C_2 \mathrm{e}^{\theta_2 x} + C_3 \mathrm{e}^{\theta_3 x},$$

with boundary condition

$$v(b) = 0, \tag{11}$$

and

$$\frac{1}{2}\sigma^2 v''(b) + (c-k)v'(b) = 0, \tag{12}$$

the latter of which is derived from letting $x = b$ in (8). Combining (11) and (12) we get

$$C_1 \mathrm{e}^{\theta_1 b} + C_2 \mathrm{e}^{\theta_2 b} + C_3 \mathrm{e}^{\theta_3 b} = 0, \tag{13}$$

$$C_1 \mathrm{e}^{\theta_1 b}\Big[\frac{1}{2}\sigma^2\theta_1^2 + (c-k)\theta_1\Big] + C_2 \mathrm{e}^{\theta_2 b}\Big[\frac{1}{2}\sigma^2\theta_2^2 + (c-k)\theta_2\Big] + C_3 \mathrm{e}^{\theta_3 b}\Big[\frac{1}{2}\sigma^2\theta_3^2 + (c-k)\theta_3\Big] = 0. \tag{14}$$

Solving (13)–(14), we arrive at

$$\begin{aligned} C_1 \mathrm{e}^{\theta_1 b} &= C_3 \mathrm{e}^{\theta_3 b} \frac{[\sigma^2\theta_3^2/2 + (c-k)\theta_3] - [\sigma^2\theta_2^2/2 + (c-k)\theta_2]}{[\sigma^2\theta_2^2/2 + (c-k)\theta_2] - [\sigma^2\theta_1^2/2 + (c-k)\theta_1]}, \\ C_2 \mathrm{e}^{\theta_2 b} &= C_3 \mathrm{e}^{\theta_3 b} \frac{[\sigma^2\theta_1^2/2 + (c-k)\theta_1] - [\sigma^2\theta_3^2/2 + (c-k)\theta_3]}{[\sigma^2\theta_2^2/2 + (c-k)\theta_2] - [\sigma^2\theta_1^2/2 + (c-k)\theta_1]}. \end{aligned}$$

Let

$$r_i = \frac{1}{2}\sigma^2\theta_i^2 + (c-k)\theta_i, \qquad i = 1, 2, 3,$$

then the solution of the first equation of the HJB equation (7) can be expressed as

$$v(x) = C_3 \mathrm{e}^{\theta_3 b}\frac{1}{r_2 - r_1}[(r_3 - r_2)\mathrm{e}^{\theta_1(x-b)} + (r_1 - r_3)\mathrm{e}^{\theta_2(x-b)} + (r_2 - r_1)\mathrm{e}^{\theta_3(x-b)}], \quad x \in [b, x_0].$$

Solving (7) we arrive at

$$v(x) = \begin{cases} C_3 \mathrm{e}^{\theta_3 b}\dfrac{1}{r_2 - r_1}[(r_3 - r_2)\mathrm{e}^{\theta_1(x-b)} + (r_1 - r_3)\mathrm{e}^{\theta_2(x-b)} + (r_2 - r_1)\mathrm{e}^{\theta_3(x-b)}], & x \in [b, x_0]; \\ v(x_0) + x - x_0, & x \in [x_0, +\infty). \end{cases} \tag{15}$$

Due to the continuity of $v'(x)$ at x_0, we have $v'(x_0-)=1$, which implies,

$$C_3\mathrm{e}^{\theta_3 b}\frac{1}{r_2-r_1}[(r_3-r_2)\theta_1\mathrm{e}^{\theta_1(x_0-b)}+(r_1-r_3)\theta_2\mathrm{e}^{\theta_2(x_0-b)}+(r_2-r_1)\theta_3\mathrm{e}^{\theta_3(x_0-b)}]=1. \tag{16}$$

With (16), $v(x)$ in (15) can be re-expressed as

$$v(x)=\begin{cases}\dfrac{(r_3-r_2)\mathrm{e}^{\theta_1(x-b)}+(r_1-r_3)\mathrm{e}^{\theta_2(x-b)}+(r_2-r_1)\mathrm{e}^{\theta_3(x-b)}}{(r_3-r_2)\theta_1\mathrm{e}^{\theta_1(x_0-b)}+(r_1-r_3)\theta_2\mathrm{e}^{\theta_2(x_0-b)}+(r_2-r_1)\theta_3\mathrm{e}^{\theta_3(x_0-b)}}, & \\ & x\in[b,x_0]; \\ v(x_0)+x-x_0, & x\in[x_0,+\infty).\end{cases} \tag{17}$$

Since $v\in\mathbb{C}^2((b,\infty))$, the solution $v(x)$ in (23) must satisfy also $v''(x_0-)=0$, which implies

$$\frac{(r_3-r_2)\theta_1^2\mathrm{e}^{\theta_1(x_0-b)}+(r_1-r_3)\theta_2^2\mathrm{e}^{\theta_2(x_0-b)}+(r_2-r_1)\theta_3^2\mathrm{e}^{\theta_3(x_0-b)}}{(r_3-r_2)\theta_1\mathrm{e}^{\theta_1(x_0-b)}+(r_1-r_3)\theta_2\mathrm{e}^{\theta_2(x_0-b)}+(r_2-r_1)\theta_3\mathrm{e}^{\theta_3(x_0-b)}}=0, \tag{18}$$

that is,

$$(r_3-r_2)\theta_1^2\mathrm{e}^{\theta_1(x_0-b)}+(r_1-r_3)\theta_2^2\mathrm{e}^{\theta_2(x_0-b)}+(r_2-r_1)\theta_3^2\mathrm{e}^{\theta_3(x_0-b)}=0. \tag{19}$$

In fact, the existence and uniqueness of solution x_0 of the equation (19) can be verified as follows. Facts that

$$\begin{aligned}r_2-r_1&=\frac{1}{2}\sigma^2(\theta_2^2-\theta_1^2)+(c-k)(\theta_2-\theta_1)=\Big[\frac{1}{2}\sigma^2(\theta_2+\theta_1)+(c-k)\Big](\theta_2-\theta_1)\\&=-\frac{1}{2}\sigma^2(\beta+\theta_3)(\theta_2-\theta_1)<0;\\r_3-r_2&=-\frac{1}{2}\sigma^2(\beta+\theta_1)(\theta_3-\theta_2)>0;\\r_1-r_3&=-\frac{1}{2}\sigma^2(\beta+\theta_2)(\theta_1-\theta_3)>0,\end{aligned}$$

lead to

$$\mathrm{e}^{\theta_3 b}\frac{1}{r_2-r_1}[(r_3-r_2)\theta_1\mathrm{e}^{\theta_1(x-b)}+(r_1-r_3)\theta_2\mathrm{e}^{\theta_2(x-b)}+(r_2-r_1)\theta_3\mathrm{e}^{\theta_3(x-b)}]>0, \tag{20}$$

and

$$\mathrm{e}^{\theta_3 b}\frac{1}{r_2-r_1}[(r_3-r_2)\theta_1^3\mathrm{e}^{\theta_1(x-b)}+(r_1-r_3)\theta_2^3\mathrm{e}^{\theta_2(x-b)}+(r_2-r_1)\theta_3^3\mathrm{e}^{\theta_3(x-b)}]>0, \tag{21}$$

In addition, from (12) and (20), we get

$$\frac{1}{2}\sigma^2\Big[\mathrm{e}^{\theta_3 b}\frac{1}{r_2-r_1}[(r_3-r_2)\theta_1^2\mathrm{e}^{\theta_1(x-b)}+(r_1-r_3)\theta_2^2\mathrm{e}^{\theta_2(x-b)}+(r_2-r_1)\theta_3^2\mathrm{e}^{\theta_3(x-b)}]\Big]$$

$$= -(c-k)\Big[\mathrm{e}^{\theta_3 b}\frac{1}{r_2-r_1}[(r_3-r_2)\theta_1\mathrm{e}^{\theta_1(x-b)}+(r_1-r_3)\theta_2\mathrm{e}^{\theta_2(x-b)}+(r_2-r_1)\theta_3\mathrm{e}^{\theta_3(x-b)}]\Big]$$
$$< 0.$$

While (21) implies that the function

$$l(x) \mathrel{\widehat{=}} \mathrm{e}^{\theta_3 b}\frac{1}{r_2-r_1}[(r_3-r_2)\theta_1^2\mathrm{e}^{\theta_1(x-b)}+(r_1-r_3)\theta_2^2\mathrm{e}^{\theta_2(x-b)}+(r_2-r_1)\theta_3^2\mathrm{e}^{\theta_3(x-b)}]$$

is strictly increasing with $\lim\limits_{x\to+\infty} l(x)=+\infty$. Hence there exist a unique x_0 such that (18) holds.

Fortunately, with the solution (17) with x_0 given by (18) or (19) (if it is indeed a solution to the HJB equation), we can propose a candidate dividend strategy as follows:

$$\Delta L_t^* = \begin{cases} X_{t-}^{L^*}-(x_0+kt), & \text{if } X_{t-}^{L^*} > kt+x_0; \\ 0, & \text{otherwise.}\end{cases} \tag{22}$$

Hence, we can for the moment put aside the arguments of the case $b<x_0=\infty$ and forward with the verification of the optimality of strategy (22).

We now first check that the function given by (17) with x_0 given by (18) or (19) is exactly a solution to the HJB equations (4)–(6). By construction it is easy to see that $v(x)$ satisfy $(\mathscr{A}-\delta)v(x)=0$ on $(b,x_0]$ and

$$v'(x)=\frac{(r_3-r_2)\theta_1\mathrm{e}^{\theta_1(x-b)}+(r_1-r_3)\theta_2\mathrm{e}^{\theta_2(x-b)}+(r_2-r_1)\theta_3\mathrm{e}^{\theta_3(x-b)}}{(r_3-r_2)\theta_1\mathrm{e}^{\theta_1(x_0-b)}+(r_1-r_3)\theta_2\mathrm{e}^{\theta_2(x_0-b)}+(r_2-r_1)\theta_3\mathrm{e}^{\theta_3(x_0-b)}} \geqslant 1,$$
$$x\in(b,x_0], \tag{23}$$

since we have already checked that the function

$$g(x)=-[(r_3-r_2)\theta_1\mathrm{e}^{\theta_1(x-b)}+(r_1-r_3)\theta_2\mathrm{e}^{\theta_2(x-b)}+(r_2-r_1)\theta_3\mathrm{e}^{\theta_3(x-b)}],$$

attains it's minimum on $(b,x_0]$ at x_0, during the construction of solution (17) of case 1. We need still to check that $(\mathscr{A}-\delta)v(x)\leqslant 0$ on $[x_0,+\infty)$. Actually, for $x\in[x_0,+\infty)$,

$$\begin{aligned}(\mathscr{A}-\delta)v(x) &= \frac{1}{2}\sigma^2 v''(x)+(c-k)v'(x)-(\lambda+\delta)v(x)+\lambda\beta\mathrm{e}^{-\beta x}\int_b^x v(y)\mathrm{e}^{\beta y}\mathrm{d}y \\ &= (c-k)-(\lambda+\delta)v(x)+\lambda\beta\mathrm{e}^{-\beta x}\int_b^x v(y)\mathrm{e}^{\beta y}\mathrm{d}y \\ &= (c-k)-(\lambda+\delta)[v(x_0)+x-x_0]+\lambda\beta\mathrm{e}^{-\beta x}\int_b^x v(y)\mathrm{e}^{\beta y}\mathrm{d}y,\end{aligned}$$

which implies

$$\lambda\beta\mathrm{e}^{-\beta x}\int_b^x v(y)\mathrm{e}^{\beta y}\mathrm{d}y = (\mathscr{A}-\delta)v(x)-(c-k)+(\lambda+\delta)v(x), \tag{24}$$

and

$$\begin{aligned}[(\mathscr{A}-\delta)v(x)]' &= -(\lambda+\delta)-\beta\Big[\lambda\beta\mathrm{e}^{-\beta x}\int_b^x v(y)\mathrm{e}^{\beta y}\mathrm{d}y\Big]+\lambda\beta v(x)\\ &= -(\lambda+\delta)-\beta(\mathscr{A}-\delta)v(x)+\beta(c-k)-\delta\beta[v(x_0)+x-x_0],\end{aligned} \tag{25}$$

where we have used (24) in the latter equation. Rearranging (25) yields

$$\begin{aligned}[(\mathscr{A}-\delta)v(x)]'+\beta(\mathscr{A}-\delta)v(x) &= -(\lambda+\delta)+\beta(c-k)-\delta\beta v(x_0)-\delta\beta(x-x_0)\\ &= -\frac{1}{2}\sigma^2v'''(x_0-)-\delta\beta(x-x_0)<0,\end{aligned} \tag{26}$$

where we have used (10) (let $x\uparrow x_0$ in (10), $v''(x_0)=0$) and the fact that $v'''(x_0-)>0$. With (26) we see that $[(\mathscr{A}-\delta)v(x)]'<0$ whenever $(\mathscr{A}-\delta)v(x)=0$. Together with $(\mathscr{A}-\delta)v(x_0)=0$, it is obvious that $(\mathscr{A}-\delta)v(x)\leqslant 0$, for any $x\in[x_0,+\infty)$. Therefore, the function defined by equation (17) is indeed a solution of the HJB equations (4)–(6).

In the sequel, we will check the optimality of the dividend strategy given by (22). For a given dividend strategy $L\in L$, define the following set.

$$\mathscr{D}_t^L=\{s\leqslant t;\ \Delta L_s\neq 0\}/\{T_n;n\geqslant 1\},$$

which represents the jumps up to time t in the dividend process, that do not occur at the same time as a jump in the compound Poisson process. Let the continuous part of the processes $\{X_t^L-kt;t\geqslant 0\}$ and $\{L_t;t\geqslant 0\}$ be denoted by $\{X_t^{L,c};t\geqslant 0\}$ and $\{L_t^c;t\geqslant 0\}$, respectively. From Theorem 4.57 (Itô's formula) in page 57 of [16], we have

$$\begin{aligned}&\mathrm{e}^{-\delta(t\wedge\tau^L)}v(X^L_{t\wedge\tau^L}-k(t\wedge\tau^L))\\ &= v(x)-\int_{0-}^{t\wedge\tau^L}\delta\mathrm{e}^{-\delta s}v(X_s^L-ks)\mathrm{d}s+\int_{0-}^{t\wedge\tau^L}\mathrm{e}^{-\delta s}v'(X_s^L-ks)\mathrm{d}(X_s^L-ks)\\ &\quad+\frac{1}{2}\int_{0-}^{t\wedge\tau^L}\mathrm{e}^{-\delta s}v''(X_s^L-ks)\mathrm{d}\langle X_\cdot^{L,c},X_\cdot^{L,c}\rangle_s\\ &\quad+\sum_{s\leqslant t\wedge\tau^L}\mathrm{e}^{-\delta s}[v(X_{s-}^L-ks+\Delta X_s^L)-v(X_{s-}^L-ks)-v'(X_{s-}^L-ks)\Delta X_s^L]\\ &= v(x)-\int_{0-}^{t\wedge\tau^L}\delta\mathrm{e}^{-\delta s}v(X_s^L-ks)\mathrm{d}s+\int_{0-}^{t\wedge\tau^L}(c-k)\mathrm{e}^{-\delta s}v'(X_s^L-ks)\mathrm{d}s\\ &\quad+\int_{0-}^{t\wedge\tau^L}\sigma\mathrm{e}^{-\delta s}v'(X_s^L-ks)\mathrm{d}w_s+\int_{0-}^{t\wedge\tau^L}\frac{\sigma^2}{2}\mathrm{e}^{-\delta s}v''(X_s^L-ks)\mathrm{d}s\\ &\quad-\int_{0-}^{t\wedge\tau^L}\mathrm{e}^{-\delta s}v'(X_s^L-ks)\mathrm{d}L_s^c+\sum_{s\in\mathscr{D}_{t\wedge\tau^L}}\mathrm{e}^{-\delta s}[v(X_{s-}^L-ks+\Delta X_s^L)-v(X_{s-}^L-ks)]\end{aligned}$$

$$
\begin{aligned}
&+\int_{0-}^{t\wedge\tau^L}\int_0^\infty \mathrm{e}^{-\delta s}[v(X_{s-}^L-ks-y)-v(X_{s-}^L-ks)]N(\mathrm{d}s,\mathrm{d}y)\\
&+\int_{0-}^{t\wedge\tau^L}\int_0^\infty \mathrm{e}^{-\delta s}[v(X_{s}^L-ks)-v(X_{s-}^L-ks-y)]N(\mathrm{d}s,\mathrm{d}y)\\
=&\ v(x)+\int_{0-}^{t\wedge\tau^L}\mathrm{e}^{-\delta s}(\mathscr{A}-\delta)v(X_s^L-ks)\mathrm{d}s+\int_{0-}^{v_h^L}\sigma\mathrm{e}^{-\delta s}v'(X_s^L-ks)\mathrm{d}w_s\\
&+\int_{0-}^{t\wedge\tau^L}\int_0^\infty \mathrm{e}^{-\delta s}[v(X_{s}^L-ks)-v(X_{s-}^L-ks-y)]N(\mathrm{d}s,\mathrm{d}y)\\
&+\int_{0-}^{t\wedge\tau^L}\int_0^\infty \mathrm{e}^{-\delta s}[v(X_{s-}^L-ks-y)-v(X_{s-}^L-ks)](N(\mathrm{d}s,\mathrm{d}y)-\lambda\beta\mathrm{e}^{-\beta y}\mathrm{d}y\mathrm{d}s)\\
&-\int_{0-}^{t\wedge\tau^L}\mathrm{e}^{-\delta s}v'(X_s^L-ks)\mathrm{d}L_s^c-\sum_{s\leqslant t\wedge\tau^L}\mathrm{e}^{-\delta s}\Delta L_s\\
&+\sum_{s\in\mathscr{D}_{t\wedge\tau^L}}\mathrm{e}^{-\delta s}[v(X_{s-}^L-ks+\Delta X_s^L)-v(X_{s-}^L-ks)+\Delta L_s]\\
&+\int_{0-}^{t\wedge\tau^L}\int_0^\infty \mathrm{e}^{-\delta s}[-(X_s^L-(X_{s-}^L-y))]I_{\{\Delta L_s\neq 0\}}N(\mathrm{d}s,\mathrm{d}y).
\end{aligned}
$$

Since $v'(x)\geqslant 1$ we have $v(X_s^L-ks)+(X_{s-}^L-X_s^L)\leqslant v(X_{s-}^L-ks)$ and $v(X_s^L-ks)+(X_{s-}^L-y-X_s^L)\leqslant v(X_{s-}^L-ks-y)$, we conclude that

$$
\begin{aligned}
&\sum_{s\in\mathscr{D}_{t\wedge\tau^L}}\mathrm{e}^{-\delta s}[v(X_{s-}^L-ks+\Delta X_s^L)-v(X_{s-}^L-ks)+\Delta L_s]\\
=&\sum_{s\in\mathscr{D}_{t\wedge\tau^L}}\mathrm{e}^{-\delta s}[v(X_{s-}^L-ks+\Delta X_s^L)-v(X_{s-}^L-ks)+(X_{s-}^L-X_s^L)]\leqslant 0,
\end{aligned}
$$

and

$$
\int_{0-}^{t\wedge\tau^L}\int_0^\infty \mathrm{e}^{-\delta s}\{v(X_s^L-ks)-v(X_{s-}^L-ks-y)+[-(X_s^L-(X_{s-}^L-y))]I_{\{\Delta L_s\neq 0\}}\}N(\mathrm{d}s,\mathrm{d}y)\leqslant 0.
$$

Therefore, we have

$$
\begin{aligned}
&\mathrm{e}^{-\delta(t\wedge\tau^L)}v(X_{t\wedge\tau^L}^L-k(t\wedge\tau^L))\\
\leqslant&\ v(x)+\int_0^{t\wedge\tau^L}\mathrm{e}^{-\delta s}(\mathscr{A}-\delta)v(X_s^L-ks)\mathrm{d}s\\
&+\int_0^{t\wedge\tau^L}\int_0^\infty \mathrm{e}^{-\delta s}[v(X_{s-}^L-ks-y)-v(X_{s-}^L-ks)](N(\mathrm{d}s,\mathrm{d}y)-\lambda\beta\mathrm{e}^{-\beta y}\mathrm{d}y\mathrm{d}s)\\
&+\int_{0-}^{v_h^L}\sigma\mathrm{e}^{-\delta s}v'(X_s^L-ks)\mathrm{d}w_s-\int_{0-}^{t\wedge\tau^L}\mathrm{e}^{-\delta s}v'(X_s^L-ks)\mathrm{d}L_s^c-\sum_{s\leqslant t\wedge\tau^L}\mathrm{e}^{-\delta s}\Delta L_s\\
\leqslant&\ v(x)+\int_0^{t\wedge\tau^L}\mathrm{e}^{-\delta s}(\mathscr{A}-\delta)v(X_s^L-ks)\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{t\wedge\tau^L}\int_0^{\infty}\mathrm{e}^{-\delta s}[v(X_{s-}^L-ks-y)-v(X_{s-}^L-ks)](N(\mathrm{d}s,\mathrm{d}y)-\lambda\beta\mathrm{e}^{-\beta y}\mathrm{d}y\mathrm{d}s)\\
&+\int_{0-}^{v_h^L}\sigma\mathrm{e}^{-\delta s}v'(X_s^L-ks)\mathrm{d}w_s-\int_{0-}^{t\wedge\tau^L}\mathrm{e}^{-\delta s}\mathrm{d}L_s^c-\sum_{s\leqslant t\wedge\tau^L}\mathrm{e}^{-\delta s}\Delta L_s. \qquad (27)
\end{aligned}
$$

Additionally, it follows from [15; page 62, line 19] that the compensated sum:

$$
t\longmapsto\int_0^t\int_0^{\infty}\mathrm{e}^{-\delta s}[v(X_{s-}^L-ks-y)-v(X_{s-}^L-ks)](N(\mathrm{d}s,\mathrm{d}y)-\lambda\beta\mathrm{e}^{-\beta y}\mathrm{d}y\mathrm{d}s)
$$

is an $(\mathscr{F}_t)$-martingale. Taking expectations on both sides of (27) and recalling $(\mathscr{A}-\delta)\cdot v(X_s^L-ks)\leqslant 0$ results in

$$
\begin{aligned}
v(x)&\geqslant\mathsf{E}_x[\mathrm{e}^{-\delta(t\wedge\tau^L)}v(X_{t\wedge\tau^L}^L-k(t\wedge\tau^L))]+\mathsf{E}_x\Big[\int_{0-}^{t\wedge\tau^L}\mathrm{e}^{-\delta s}\mathrm{d}L_s^c+\sum_{s\leqslant t\wedge\tau^L}\mathrm{e}^{-\delta s}\Delta L_s\Big]\\
&\geqslant\mathsf{E}_x\Big[\int_{0-}^{t\wedge\tau^L}\mathrm{e}^{-\delta s}\mathrm{d}L_s^c+\sum_{s\leqslant t\wedge\tau^L}\mathrm{e}^{-\delta s}\Delta L_s\Big]\\
&=\mathsf{E}_x\Big[\int_{0-}^{t\wedge\tau^L}\mathrm{e}^{-\delta s}\mathrm{d}L_s\Big]. \qquad (28)
\end{aligned}
$$

Finally, taking limits as $t\to\infty$ in (28), we have

$$
v(x)\geqslant\mathsf{E}_x\Big[\int_{0-}^{\tau^L}\mathrm{e}^{-\delta s}\mathrm{d}L_s\Big]=V_L(x). \qquad (29)
$$

Since the strategy L is arbitrary, it follows that

$$
v(x)\geqslant V(x). \qquad (30)
$$

In addition, if we replace L with L^* and noting the condition

$$
\begin{aligned}
&\lim_{t\to+\infty}\mathsf{E}_x[\mathrm{e}^{-\delta(t\wedge\tau^{L^*})}v(X_{t\wedge\tau^{L^*}}^{L^*}-k(t\wedge\tau^{L^*}))]\\
=&\lim_{t\to+\infty}\mathsf{E}_x[\mathrm{e}^{-\delta t}v(X_t^{L^*}-kt)I_{\{t<\tau^{L^*}\}}+\mathrm{e}^{-\delta\tau^{L^*}}v(X_{\tau^{L^*}}^{L^*}-k\tau^{L^*})I_{\{t\geqslant\tau^{L^*}\}}]\\
=&\lim_{t\to+\infty}\mathsf{E}_x[\mathrm{e}^{-\delta t}v(X_t^{L^*}-kt)I_{\{t<\tau^{L^*}\}}]\leqslant\lim_{t\to+\infty}\mathsf{E}_x[\mathrm{e}^{-\delta t}v(X_t^{L^*}-kt)]
\end{aligned}
$$

(since $v(X_{\tau^{L^*}}^{L^*}-k\tau^{L^*})=0$ and $v(X_t^{L^*}-kt)$ is bounded due to $X_t^{L^*}-kt\leqslant x_0$),

then the inequalities in (27)–(30) are all equalities, hence

$$
v(x)=V_{L^*}(x)\leqslant V(x),
$$

which together with (30) leads to $v(x)=V(x)=V_{L^*}(x)$. Theorem 2 is proved. □

Remark 3 If $b = k = 0$, then the corresponding optimal return function is,

$$v(x) = \begin{cases} \dfrac{(r_3 - r_2)e^{\theta_1 x} + (r_1 - r_3)e^{\theta_2 x} + (r_2 - r_1)e^{\theta_3 x}}{(r_3 - r_2)\theta_1 e^{\theta_1 x_0} + (r_1 - r_3)\theta_2 e^{\theta_2 x_0} + (r_2 - r_1)\theta_3 e^{\theta_3 x_0}}, & x \in [0, x_0]; \\ v(x_0) + x - x_0, & x \in [x_0, +\infty), \end{cases} \tag{31}$$

with x_0 being the unique solution of the following equation,

$$(r_3 - r_2)\theta_1^2 e^{\theta_1 x_0} + (r_1 - r_3)\theta_2^2 e^{\theta_2 x_0} + (r_2 - r_1)\theta_3^2 e^{\theta_3 x_0} = 0,$$

$\{\theta_i, i = 1, 2, 3\}$ satisfying $-\infty < \theta_1 < -\beta < \theta_2 < 0 < \theta_3 < +\infty$ being the set of zeroes of the following equation,

$$\frac{1}{2}\sigma^2\theta^3 + \Big[\frac{1}{2}\sigma^2\beta + c\Big]\theta^2 + [c\beta - (\lambda + \delta)]\theta - \beta\delta = 0,$$

and $r_i = \sigma^2\theta_i^2/2 + c\theta_i$, $i = 1, 2, 3$. And, the corresponding optimal dividend strategy is

$$\Delta L_t^* = \begin{cases} X_{t-}^{L^*} - x_0, & \text{if } X_{t-}^{L^*} > x_0; \\ 0, & \text{otherwise}. \end{cases} \tag{32}$$

Equations (31)–(32) coincides with the corresponding results of Theorem 3.3 in [14].

Remark 4 If $b = k = \sigma = 0$, then by the fact that $v'(x_0) = 1$ the corresponding optimal return function is,

$$v(x) = \begin{cases} \dfrac{[c\theta_2 - (\lambda + \delta)]e^{\theta_1 x} - [c\theta_1 - (\lambda + \delta)]e^{\theta_2 x}}{[c\theta_2 - (\lambda + \delta)]\theta_1 e^{\theta_1 x_0} - [c\theta_1 - (\lambda + \delta)]\theta_2 e^{\theta_2 x_0}}, & x \in [0, x_0]; \\ v(x_0) + x - x_0, & x \in [x_0, +\infty), \end{cases}$$

with $\{\theta_i, i = 1, 2\}$ satisfying $-\beta < \theta_1 < 0 < \theta_2 < (\lambda + \delta)/c < +\infty$ being the set of zeroes of the following equation,

$$c\theta^2 + [c\beta - (\lambda + \delta)]\theta - \beta\delta = 0.$$

In addition, x_0 is the unique minimizer of the function $[c\theta_2 - (\lambda + \delta)]\theta_1 e^{\theta_1 x} - [c\theta_1 - (\lambda + \delta)] \cdot \theta_2 e^{\theta_2 x}$, i.e.,

$$\begin{aligned} &[c\theta_2 - (\lambda + \delta)]\theta_1^2 e^{\theta_1 x_0} - [c\theta_1 - (\lambda + \delta)]\theta_2^2 e^{\theta_2 x_0} = 0 \\ \text{or} \quad & x_0 = \frac{1}{\theta_2 - \theta_1} \log\Big\{\frac{[c\theta_2 - (\lambda + \delta)]\theta_1^2}{[c\theta_1 - (\lambda + \delta)]\theta_2^2}\Big\}, \end{aligned} \tag{33}$$

or $x_0 = 0$ if the solution of (33) does not exist. Note that there should be a unique solution of (33) if it's solution does exist, since for all $x \geqslant 0$ we have

$$\{[c\theta_2 - (\lambda + \delta)]\theta_1^2 e^{\theta_1 x} - [c\theta_1 - (\lambda + \delta)]\theta_2^2 e^{\theta_2 x}\}'$$

$$= [c\theta_2 - (\lambda+\delta)]\theta_1^3 \mathrm{e}^{\theta_1 x} - [c\theta_1 - (\lambda+\delta)]\theta_2^3 \mathrm{e}^{\theta_2 x} \geqslant 0,$$

by the facts that $[c\theta_2 - (\lambda+\delta)] < 0$, $[c\theta_1 - (\lambda+\delta)] < 0$, $\theta_1 < 0$ and $\theta_2 > 0$. Further, the corresponding optimal dividend strategy is

$$\Delta L_t^* = \begin{cases} X_{t-}^{L^*} - x_0, & \text{if } X_{t-}^{L^*} > x_0; \\ 0, & \text{otherwise.} \end{cases} \tag{34}$$

Equations (33)–(34) coincides well with the corresponding results of Theorem 2.39, Theorem 2.41 and (2.34) in [17], or coincides with the corresponding results of [18] if we additionally let the interest rate $i = 0$ in that paper.

Remark 5 If $b = k = 0$ and $\beta \uparrow \infty$ which means the compound poisson term vanishes in our model, in this case (8) degenerates to

$$\frac{1}{2}\sigma^2 v''(x) + cv'(x) - (\lambda+\delta)v(x) = 0, \qquad x \in [0, x_0],$$

with boundary conditions $v(0) = 0$, $v'(x_0) = 1$ and $v''(x_0) = 0$. Then the corresponding optimal return function is,

$$v(x) = \begin{cases} \dfrac{\mathrm{e}^{\theta_1 x} - \mathrm{e}^{\theta_2 x}}{\theta_1 \mathrm{e}^{\theta_1 x_0} - \theta_2 \mathrm{e}^{\theta_2 x_0}}, & x \in [0, x_0]; \\ v(x_0) + x - x_0, & x \in [x_0, +\infty), \end{cases} \tag{35}$$

with x_0 being the unique solution of the following equation,

$$\theta_1^2 \mathrm{e}^{\theta_1 x_0} - \theta_2^2 \mathrm{e}^{\theta_2 x_0} = 0 \qquad \text{or} \qquad x_0 = \frac{2}{\theta_2 - \theta_1} \log\left\{\left|\frac{\theta_1}{\theta_2}\right|\right\},$$

$\{\theta_i, i = 1, 2\}$ satisfying $-\infty < \theta_1 < 0 < \theta_2 < +\infty$ being the set of zeroes of the following equation,

$$\frac{1}{2}\sigma^2\theta^2 + c\theta - (\lambda+\delta) = 0.$$

And, the corresponding optimal dividend strategy is

$$\Delta L_t^* = \begin{cases} X_{t-}^{L^*} - x_0, & \text{if } X_{t-}^{L^*} > x_0; \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

Equations (35)–(36) coincides with the corresponding results of Theorem 3.2 in [19].

§4. Conclusion

For a financially constrained firm with both small and large movements of cash flow, we use a jump-diffusion model with exponentially distributed jumps to characterize it's asset process. Meanwhile, a linear barrier is present in the model in order to characterize risk aversion of the firm, bankruptcy takes place whenever the reserve of the firm runs below the linear barrier. It's found that the optimal dividend policy is of a linear barrier type: whenever the firm's cash accumulates beyond a target linear barrier, the amount in excess of this linear ascending line is distributed to shareholders as dividends. As to our knowledge, this is different from the results of existing literature, where the optimal dividend strategy is mostly of barrier type. We also characterize the optimal firm value function, where the firm value function or the value of a firm is exactly the expected present value of future dividends according to [2].

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带线性约束条件的跳扩散风险模型中的最优分红策略

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摘　要: 对于一个金融或保险公司而言, 寻求最优分红策略和最优分红值函数是一个受到广泛讨论的热点问题. 在本文中, 我们假设公司面临两类风险: Brownian风险和Poisson风险. 公司可以控制其对股东的分红数额和分红时间. 为了充分考虑公司经营的安全性, 文中定义破产时间为公司盈余水平首次低于线性门槛$b+kt$的时刻, 而非首次低于0的时刻, 参见文献[1]. 本文解决了最大化公司从开始运营直至破产期间总分红折现值的期望的问题. 通过求解一个含有二阶微分–积分算子的HJB方程, 本文刻画出来了最优的分红值函数和最优的分红策略. 结果表明, 最优分红策略为线性门槛分红策略. 即, 当公司的盈余水平低于某线性门槛x_0+kt时, 公司不分红; 而当公司的盈余水平超过该线性门槛时, 超过部分将全部作为红利分出.

关键词: 最优分红值函数; 分红策略; 线性门槛分红策略

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