应用概率统计　第 33 卷
第 1 期　2017 年 2 月

Chinese Journal of Applied Probability and Statistics
Feb., 2017, Vol. 33, No. 1, pp. 49-57
doi: 10.3969/j.issn.1001-4268.2017.01.004

一个关于布朗运动泛函极限定理的拟必然收敛速度 *

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摘　要: 本文利用大偏差小偏差研究得到布朗运动的一个局部泛函极限定理, 证明了在拟必然收敛意义下布朗运动增量关于(r,p)-容度局部泛函极限的收敛速度.

关键词: 布朗运动; 拟必然收敛; Hölder范数; 容度

中图分类号: O212.4

英文引用格式: Li Y H, Liu Y H. The rate of quasi sure convergence of a functional limitation theorem for a Brownian motion [J]. Chinese J. Appl. Probab. Statist., 2017, 33(1): 49–57. (in Chinese)

§1. 引　　言

拟必然分析是随机分析的一个重要分支. 自从Yoshida [1]得到关于(r,p)-容度的大偏差结论, 学者们开始关注布朗运动在Hölder范数下的拟必然泛函极限理论. Baldi和Roynette [2]利用大偏差工具得到了布朗运动在Hölder范数下的收敛速度结果, Chen和Balakrishnan [3]得到布朗运动关于(r,p)-容度的泛函重对数律结果, Liu和Ren [4]加强了布朗运动增量在Hölder范数下关于(r,p)-容度的大偏差结果, 得到在Hölder范数与(r,p)-容度下布朗运动的泛函连续模结论. 本文利用大偏差与小偏差, 得到布朗运动增量在拟必然收敛意义下关于(r,p)-容度的局部泛函极限的收敛速度.

考虑经典的Wiener空间(B,H,μ), 对$r\geqslant 0$, $p\geqslant 1$, 设$D^{r,p}$是Wiener泛函的Sobolev空间, 即

$$D^{r,p}=(1-\mathscr{L})^{-r/2}\boldsymbol{L}^p,\qquad \|F\|_{r,p}=\|(1-\mathscr{L})^{r/2}F\|_p,\qquad F\in\boldsymbol{L}^p,$$

其中$\boldsymbol{L}^p$记为(B,μ)上的实值函数的L^p空间, $\mathscr{L}$是(B,H,μ)上的Ornstein-Uhlenbeck算子. 对$r\geqslant 0$, $p>1$, (r,p)-容度定义如下: 对开集$O\subset B$,

$$C_{r,p}(O)=\inf\{\|F\|_{r,p}^p; F\in D_{r,p}, F\geqslant 1, \mu\text{ 几乎处处在 } O\text{ 上}\},$$

且对任意集合$A\subset B$,

$$C_{r,p}(A)=\inf\{C_{r,p}(O); A\subset O\subset B,\ O\text{ 是开集}\}.$$

* 国家自然科学基金项目(批准号: 71162017)和广西省自然科学基金项目(批准号: 2014GXNSFAA118010)资助.
本文2015年5月4日收到, 2015年12月31日收到修改稿.

设$\mathscr{C}^d$为从$[0,1]$到$\mathbb{R}^d$的连续函数空间, 赋予上确界范数$\|f\| = \sup\limits_{0\leqslant t\leqslant 1}|f(t)|$, 其中$|\cdot|$表示欧几里得范数. 记

$$\mathscr{C}_a^d = \{f \in \mathscr{C}^d; f(0) = a\},$$

$$\mathscr{H}^d = \Big\{f \in \mathscr{C}_0^d; f(t) = \int_0^t \dot{f}(s)\mathrm{d}s, \|f\|_{\mathscr{H}^d}^2 = \int_0^1 |\dot{f}(t)|^2\mathrm{d}t < \infty\Big\},$$

$\mathscr{H}^d$是一定义如下内积的Hilbert空间:

$$\langle \rho_1, \rho_2\rangle_{\mathscr{H}^d} = \int_0^1 (\dot{\rho}_1(s), \dot{\rho}_2(s))\mathrm{d}s.$$

设μ是$\mathscr{C}_0^d$上的Wiener测度, $(\mathscr{C}_0^d, \mathscr{H}^d, \mu)$是一经典Wiener空间. 对$0 < \alpha < 1/2$, 考虑两Banach空间

$$\mathscr{C}^\alpha = \Big\{f \in \mathscr{C}_0^d; \|f\| = \sup_{\substack{s,t\in[0,1]\\ s\neq t}} \frac{|f(t)-f(s)|}{|t-s|^\alpha} < \infty\Big\},$$

$$\mathscr{C}^{\alpha,0} = \Big\{f \in \mathscr{C}^\alpha; \lim_{\delta\to 0} \sup_{\substack{s,t\in[0,1]\\ 0<|t-s|<\delta}} \frac{|f(t)-f(s)|}{|t-s|^\alpha} = 0\Big\},$$

则$(\mathscr{C}^{\alpha,0}, \mathscr{H}^d, \mu)$也是一经典Wiener空间(参见文献[2]的定理2.4), $\omega \in \mathscr{C}^{\alpha,0}$是标准布朗运动. 定义$I: B \to [0,\infty]$为: 若$f \in \mathscr{H}^d$, $I(f) = \|f\|_{\mathscr{H}^d}^2/2$; 否则$I(f) = \infty$. 记$\mathscr{F} = \{f \in \mathscr{H}^d; I(f) \leqslant 1\}$.

Baldi和Roynette[2]证明了存在$k = k(\alpha) \geqslant 0$, 使得

$$\lim_{\varepsilon\to 0} \varepsilon^{2/(1-2\alpha)} \ln \mathsf{P}\{\|w\| \leqslant \varepsilon\} = -k, \tag{1}$$

进一步, 对任意$f \in \mathscr{F}$有

$$\lim_{\varepsilon\to 0} \varepsilon^{2/(1-2\alpha)} \ln \mathsf{P}\Big(\Big\|w - \frac{f}{\varepsilon^{1/(1-2\alpha)}}\Big\| \leqslant \tau\varepsilon\Big) = -I(f) - k\tau^{2/(2\alpha-1)}. \tag{2}$$

§2. 预备结论

引理 1 (见[3; 定理2.1]) 设$\{S_\varepsilon\}_{\varepsilon>0}$是$\mathscr{C}^{\alpha,0}$上一双射线性算子, 使得对任意$\varepsilon > 0$及$A \subset \mathscr{C}^{\alpha,0}$有$\mu(S_\varepsilon^{-1}A) = \mu(\varepsilon^{-1/2}A)$. 则对$(r,p) \in [0,\infty) \times (1,\infty)$, 有

$$-\inf_{f\in \mathring{A}} I(f) \leqslant \varliminf_{\varepsilon\to 0} \varepsilon \ln C_{r,p}(S_\varepsilon^{-1}A) \leqslant \varlimsup_{\varepsilon\to 0} \varepsilon \ln C_{r,p}(S_\varepsilon^{-1}A) \leqslant -\inf_{f\in\overline{A}} I(f).$$

引理 2 (见[6; 引理2.1]) 设$k \in \mathrm{N}$, $q_1, q_2 \in (1,\infty)$满足$1/p = 1/q_1 + 1/q_2$. 则存在常数$c = c(k,p,q_1,q_2) > 0$, 使得对任意$-\infty < a_i < b_i < \infty$, $\delta \in (0,1)$及$F_i \in D^{k,kq_1}$有

$$C_{k,p}\Big(\bigcap_{i=1}^n \{a_i < \widetilde{F}_i(z) < b_i\}\Big)^{1/p}$$

$$\leqslant cn^k\delta^{-k}\Big(1+\max_{1\leqslant i\leqslant n}\|F_i\|_{k,kq_1}\Big)^k\mu\Big(\bigcap_{i=1}^n\{a_i-\delta<F_i(z)<b_i+\delta\}\Big)^{1/q_2},$$

其中$\widetilde{F_i}$为F_i的拟连续修正.

引理 3　设k,p,q_1,q_2如引理2中定义. 对任意$\varepsilon>0$及$f\in\mathscr{F}$, 设

$$F_\varepsilon^{(i)}(w)=\Big\|\varepsilon\Big(\frac{w(t_i+h_i\cdot)-w(t_i)}{\sqrt{h_i}}\Big)-f\Big\|,\qquad h_i>0,\ t_i\geqslant 0,\ i=1,2,\ldots,n,$$

则存在一常数$c=c(k,p,q_1,f)>0$, 对任意$\delta\in(0,1]$, $\varepsilon\in(0,1]$, 有

$$C_{k,p}\Big(\bigcap_{i=1}^n\{z:a_i<F_\varepsilon^{(i)}(z)<b_i\}\Big)^{1/p}\leqslant cn^k\delta^{-2k^2-k}\mu\Big(\bigcap_{i=1}^n\{z:a_i-\delta<F_\varepsilon^{(i)}(z)<b_i+\delta\}\Big)^{1/q_2}.$$

证明:　利用引理2, 类似文献[6]中引理2.2易证.　□

引理 4　设$f\in\mathscr{F}$, α与$k>0$如(2)式中所定义, 则对$t\geqslant 0$, $\tau>0$有

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon^{2/(1-2\alpha)}\ln C_{r,p}\Big(\Big\|\frac{w(t+h\cdot)-w(t)}{\sqrt{h}}-\frac{f}{\varepsilon^{1/(1-2\alpha)}}\Big\|\leqslant\varepsilon\tau\Big)\\&=\lim_{\varepsilon\to 0}\varepsilon^{2/(1-2\alpha)}\ln\mu\Big(\Big\|\frac{w(t+h\cdot)-w(t)}{\sqrt{h}}-\frac{f}{\varepsilon^{1/(1-2\alpha)}}\Big\|\leqslant\varepsilon\tau\Big)\\&=-I(f)-k\tau^{2/(2\alpha-1)}.\end{aligned}$$

证明:　因容度具有性质$C_{r,p}(\cdot)\geqslant\mu(\cdot)$, 故只需证明下式即可:

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon^{2/(1-2\alpha)}\ln C_{r,p}\Big(\Big\|\frac{w(t+h\cdot)-w(t)}{\sqrt{h}}-\frac{f}{\varepsilon^{1/(1-2\alpha)}}\Big\|\leqslant\varepsilon\tau\Big)\\&\leqslant\lim_{\varepsilon\to 0}\varepsilon^{2/(1-2\alpha)}\ln\mu\Big(\Big\|\frac{w(t+h\cdot)-w(t)}{\sqrt{h}}-\frac{f}{\varepsilon^{1/(1-2\alpha)}}\Big\|\leqslant\varepsilon\tau\Big).\end{aligned}$$

设q_1,q_2,p,k如引理2所定义, 取$k=[r]+1$, 对任意$1>\delta>0$, $c_0>0$, 根据引理3有

$$\begin{aligned}&C_{r,p}\Big(\Big\|\frac{w(t+h\cdot)-w(t)}{\sqrt{h}}-\frac{f}{\varepsilon^{1/(1-2\alpha)}}\Big\|\leqslant\varepsilon\tau\Big)^{1/p}\\&\leqslant c_0(\varepsilon^{1/(1-2\alpha)+1}\delta)^{-2k^2-k}\mu\Big(\Big\|\varepsilon^{1/(1-2\alpha)}\frac{w(t+h\cdot)-w(t)}{\sqrt{h}}-f\Big\|\leqslant\varepsilon^{1/(1-2\alpha)+1}(\tau+\delta)\Big)^{1/q_2},\end{aligned}$$

再利用(2)式得

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon^{2/(1-2\alpha)}\ln C_{r,p}\Big(\Big\|\frac{w(t+h\cdot)-w(t)}{\sqrt{h}}-\frac{f}{\varepsilon^{1/(1-2\alpha)}}\Big\|\leqslant\varepsilon\tau\Big)\\&\leqslant\frac{p}{q_2}\lim_{\varepsilon\to 0}\varepsilon^{2/(1-2\alpha)}\ln\mu\Big(\Big\|w(\cdot)-\frac{f}{\varepsilon^{1/(1-2\alpha)}}\Big\|\leqslant\varepsilon(\tau+\delta)\Big)\\&=-\frac{p}{q_2}(I(f)+k(\tau+\delta)^{2/(2\alpha-1)}),\end{aligned}$$

最后令上式中$\delta\to 0$, $q_2\to p$, 引理4获证.　□

§3. 布朗运动增量泛函的局部拟必然收敛速度

设x_u是一从R^+到R^+的单调不减函数, 根据文献[8], 假设x_u满足$0 < x_u \leqslant u$且u/x_u单调不减. 引入记号$\rho_u = \ln(ux_u^{-1}\ln u)$, 且对$t \in [0, u-x_u]$记

$$M_{t,u}(s) = \frac{w(t+x_u s) - w(t)}{x_u^{1/2}(k\rho_u)^{\alpha-1/2}}, \qquad s \in [0,1],$$

其中k如(2)中所定义. 为表达方便起见, 引入记号$LL(x) = \ln\ln x$. 本文主要结果为定理8, 该结果的证明由引理5、引理6及引理7完成.

引理 5 $\varliminf\limits_{u\to\infty} \inf\limits_{t\in[0,u-x_u]} \|M_{t,u}(s)\| \geqslant 1$, $C_{r,p}$-q.s.

证明: 对$1 < \theta < (1-\varepsilon)^{-2}$, 设$u_n = \theta^n$. 选取适当$\eta > 0$使得$\delta_0 = 1/(\theta^{1/2}(1-\varepsilon))^{2/(1-2\alpha)} - \eta > 1$. 设$g(u) = x_u(\rho_u)^{-2/(1-2\alpha)}$, $h_n = [u_{n+1}/g(u_n)]$, $t_i = ig(u_n)$, $i = 0, 1, 2, \ldots, h_n$. 则有

$$\begin{aligned}
&\min_{0<i\leqslant h_n} \|M_{t_i,u_{n+1}}(s)\| \\
\leqslant& \sup_{0\leqslant i\leqslant h_n} \sup_{0\leqslant s\leqslant g(u_n)} \left\|x_{u_{n+1}}^{-1/2}(k\rho_{u_{n+1}})^{1/2-\alpha}(w(s+t_i+x_{u_{n+1}}\cdot) - w(t_i+x_u\cdot))\right\| \\
&+ \inf_{t\in[0,u_{n+1}-x_{u_{n+1}}]} \left\|x_{u_{n+1}}^{-1/2}(k\rho_{u_{n+1}})^{1/2-\alpha}(w(t+x_{u_{n+1}}\cdot) - w(t))\right\| \\
=& I_1 + I_2.
\end{aligned} \tag{3}$$

对任意$0 < \varepsilon < 1$, 由引理4, 当n充分大时有

$$\begin{aligned}
&C_{r,p}\Big(\min_{0\leqslant i\leqslant h_n} \|M_{t_i,u_{n+1}}(s)\| \leqslant 1-\varepsilon\Big) \\
\leqslant& \sum_{0\leqslant i\leqslant h_n} C_{r,p}\Big(\Big\|\frac{w(t_i+x_{u_{n+1}}\cdot) - w(t_i)}{(\rho_{u_{n+1}}x_u)^{1/2}}\Big\| \leqslant \frac{\theta^{1/2}(1-\varepsilon)}{(\rho_{u_{n+1}})^{1-\alpha}}\Big) \\
\leqslant& (1+h_n)\exp\{-\rho_{u_{n+1}}\delta_0\} \\
\leqslant& \frac{u_{n+1}+g(u_n)}{g(u_n)}\Big(\frac{x_{u_{n+1}}}{u_{n+1}\ln u_{n+1}}\Big)^{\delta_0},
\end{aligned}$$

因此根据Borel-Cantelli引理得到

$$\varliminf_{n\to\infty} \min_{0\leqslant i\leqslant h_n} \|M_{t_i,u_{n+1}}(s)\| \geqslant 1, \qquad C_{r,p}\text{-q.s.} \tag{4}$$

另一方面, 引入记号

$$V(j,n) = \sup_{0\leqslant T\leqslant 1} \|w(Tg(u_n)+t_i+jg(u_n)+g(u_n)\cdot) - w(t_i+jg(u_n)+g(u_n)\cdot)\|,$$

则对任意$\varepsilon > 0$, 有

$$
\begin{aligned}
& C_{r,p}\{I_1 > \varepsilon\} \\
&= C_{r,p}\Big\{x_{u_{n+1}}^{-1/2}(k\rho_{u_{n+1}})^{1/2-\alpha} \sup_{0\leqslant i\leqslant h_n} \sup_{0\leqslant T\leqslant 1} \|w(Tg(u_n)+t_i+x_{u_n}\cdot) - w(t_i+x_{u_n}\cdot)\| > \varepsilon\Big\} \\
&\leqslant \sum_{j=0}^{[x_{u_n}/g(u_n)]} \sum_{i=0}^{h_n} C_{r,p}\Big\{x_{u_{n+1}}^{-1/2}(k\rho_{u_{n+1}})^{1/2-\alpha} \frac{x_{u_n}^{\alpha}}{(g(u_n))^{\alpha}} V(j,n) > \varepsilon\Big\} \\
&\leqslant \sum_{j=0}^{[x_{u_n}/g(u_n)]} \sum_{i=0}^{h_n} C_{r,p}\Big\{\frac{k^{1/2-\alpha}\theta^{\alpha+3/2}}{(\rho_{u_{n+1}})^{1/2+\alpha}\sqrt{g(u_n)}} V(j,n) > \varepsilon\Big\},
\end{aligned}
$$

对于$\mathscr{F} = \big\{f \in \mathscr{C}^{\alpha,0}; \sup\limits_{0\leqslant t\leqslant 1} \|f((1/2)t + (1/2)\cdot) - f((1/2)\cdot)\| \geqslant \varepsilon\big\}$, 由于

$$
\mu\Big\{\frac{k^{1/2-\alpha}\theta^{\alpha+3/2}}{(\rho_{u_{n+1}})^{1/2+\alpha}\sqrt{g(u_n)}} V(j,n) > \varepsilon\Big\} = \mu\Big\{\frac{\sqrt{2}\, k^{1/2-\alpha}\theta^{\alpha+3/2}}{(\rho_{u_{n+1}})^{1/2+\alpha}} w \in \mathscr{F}\Big\},
$$

而$\inf_{f\in\mathscr{F}} I(f) \geqslant \varepsilon^2/32$, 故由引理1, 当$n$充分大时有

$$
C_{r,p}\Big\{\frac{k^{1/2-\alpha}\theta^{\alpha+3/2}}{(\rho_{u_{n+1}})^{1/2+\alpha}\sqrt{g(u_n)}} V(j,n) > \varepsilon\Big\} \leqslant \Big(\frac{x_{u_{n+1}}}{u_{n+1}\ln u_{n+1}}\Big)^{\varepsilon^2(\rho_{u_{n+1}})^{2\alpha}/(128\theta^{\alpha+3/2})}.
$$

考虑到当$n \to \infty$时, $\rho_{u_{n+1}} \to \infty$, 从而有

$$
\sum_n C_{r,p}\{I_1 > \varepsilon\} \leqslant \sum_n \frac{x_{u_{n+1}}}{g(u_n)} \frac{u_{n+1} + g(u_n)}{g(u_n)} \Big(\frac{x_{u_{n+1}}}{u_{n+1}\ln u_{n+1}}\Big)^{\varepsilon^2(\rho_{u_{n+1}})^{2\alpha}/(128\theta^{\alpha+3/2})} < \infty,
$$

再次利用Borel-Cantelli引理得

$$
\varlimsup_{n\to\infty} I_1 = 0, \qquad C_{r,p}\text{-q.s.} \tag{5}
$$

联合(3)–(5)式得

$$
\varliminf_{n\to\infty} I_2 \geqslant 1, \qquad C_{r,p}\text{-q.s.} \tag{6}
$$

进一步, 对于$u \in [u_n, u_{n+1})$有

$$
\begin{aligned}
& \inf_{t\in[0,u-x_u]} \|M_{t,u}(s)\| \\
&\geqslant k^{1/2-\alpha}\rho_u^{1-\alpha} \inf_{t\in[0,u_{n+1}-x_{u_{n+1}}]} (x_{u_{n+1}}\rho_{u_{n+1}})^{-1/2} \|w(t + x_{u_{n+1}}\cdot) - w(t)\| \\
&\geqslant \frac{1}{\theta^{2(1-\alpha)}} \cdot I_2,
\end{aligned} \tag{7}
$$

在上式中令$\theta \to 1$, 再结合(6)、(7)式证得引理5. □

引理 6 若 $\overline{\lim\limits_{u\to\infty}}\ln(u/x_u)(LL(u))^{-1}<\infty$, 则有

$$\underline{\lim_{u\to\infty}}\inf_{t\in[0,u-x_u]}\|M_{t,u}(s)\|\leqslant 1,\qquad C_{r,p}\text{ - q.s.}$$

证明: 记$A=\lim\limits_{u\to\infty}x_u/u$, 显然$A\leqslant 1$. 下面分两种情况讨论.

(i) 若$A<1$. 定义序列$\{u_n,n\geqslant 1\}$如下: $u_1=1$, $u_{n+1}=u_n+x_{u_{n+1}}$, 记

$$\Delta w=w(t_n+x_{u_n}\cdot)-w(t_n),\qquad t_n=u_n-x_{u_n},$$

从而对于$n\geqslant 1$, 有$\{\|(x_{u_n}\rho_{u_n})^{-1/2}\Delta w\|>\varepsilon\}$相互独立. 对任意$\varepsilon>0$, 设$\lambda=[x]+1$, 由引理3有

$$\begin{aligned}&C_{r,p}\Big(\bigcap_{n=m_0}^{z}(\|M_{t_n,u_n}(s)\|\geqslant 1+\varepsilon)\Big)^{1/p}\\
=&C_{r,p}\Big(\bigcap_{n=m_0}^{z}\Big(\Big\|\frac{\Delta w}{(\rho_{u_n}x_{u_n})^{1/2}}\Big\|\geqslant k^{(1-2\alpha)/2}(1+\varepsilon)\rho_{u_n}^{\alpha-1}\Big)\Big)^{1/p}\\
\leqslant& cz^{\lambda}\big(k^{(1-2\alpha)/2}\varepsilon\rho_{u_z}^{\alpha-1}\big)^{-2\lambda^2-\lambda}\prod_{n=m_0}^{z}\Big\{1-\mu\Big(\Big\|\frac{\Delta w}{(\rho_{u_n}x_{u_n})^{1/2}}\Big\|<k^{(1-2\alpha)/2}(1+\varepsilon)\rho_{u_n}^{\alpha-1}\Big)\Big\}^{1/q_2},\end{aligned}$$

进一步根据(2)式, 当n充分大时有

$$\mu\Big(\Big\|\frac{\Delta w}{(\rho_{u_n}x_{u_n})^{1/2}}\Big\|<(1+\varepsilon)k^{(1-2\alpha)/2}\rho_{u_n}^{\alpha-1}\Big)\geqslant\Big(\frac{x_{u_n}}{u_n\ln u_n}\Big)^{\delta_0},$$

其中对某个$\delta>0$, $\delta_0=(1+\varepsilon)^{2/(2\alpha-1)}+\delta<1$. 故有

$$\begin{aligned}&C_{r,p}\Big(\bigcap_{n=m_0}^{z}(\|M_{t_n,u_n}(s)\|\geqslant 1+\varepsilon)\Big)^{1/p}\\
\leqslant& cz^{\lambda}\Big(\frac{\rho_{u_z}^{1-\alpha}}{\varepsilon k^{(1-2\alpha)/2}}\Big)^{2\lambda^2+\lambda}\exp\Big(-\frac{1}{q_2}\sum_{n=m_0}^{z}\Big(\frac{x_{u_n}}{u_n\ln u_n}\Big)^{\delta_0}\Big),\end{aligned}$$

对于上式中的级数, 可以证明存在某个$\pi_0>0$, 使得

$$\sum_{n=m_0}^{z}\Big(\frac{x_{u_n}}{u_n\ln u_n}\Big)^{\delta_0}>\pi_0(\ln u_z)^{1-\delta_0},$$

又因$\lim\limits_{u\to\infty}\ln(u/x_u)(LL(u))^{-1}<\infty$, 故存在某个$N_0>0$使得$u/x_u\leqslant(\ln u)^{N_0}$. 取$\theta_0>2/(1-\delta_0)$, $u_0=\mathrm{e}^{(\ln z_0)^{\theta_0}}$, 则当$z_0$足够大, $z\geqslant z_0$时, 有$\ln u_z\geqslant(\ln z)^{\theta_0}$, 故而有$(\ln u_z)^{1-\delta_0}>(\ln z)^2$. 同理, 对$c_1>0$, 当$z$充分大时, 有$c_1z\geqslant\ln u_z$, 故而有

$$\begin{aligned}&C_{r,p}\Big(\bigcap_{n=m_0}^{z}(\|M_{t_n,u_n}(s)\|\geqslant 1+\varepsilon)\Big)^{1/p}\\
\leqslant& cz^{\lambda}(LL(u_z))^{(1-\alpha)(2\lambda^2+\lambda)}\exp\Big(-\frac{\pi_0}{q_2}(\ln u_z)^{1-\delta_0}\Big)\end{aligned}$$

$$\leqslant c_2 z^{\lambda}(\ln z)^{(1-\alpha)(2\lambda^2+\lambda)} \exp\Big(-\frac{\pi_0}{q_2}\ln^2 z\Big) \to 0, \qquad z \to \infty,$$

其中$c_2 = c_2(\lambda, p, q_1, k, \alpha)$. 进而有

$$C_{r,p}\Big\{\bigcup_{z=1}^{\infty}\bigcap_{n=z}^{\infty}(\|M_{t_n,u_n}(s)\| \geqslant 1+\varepsilon)\Big\} = 0,$$

等价的得到引理6结论.

(ii) 若$A=1$, 则$x_u = u$, 此时证明参见文献[3]中的定理3.2. 引理6获证. □

引理 7　若$\lim\limits_{u\to\infty} \ln(u/x_u)(LL(u))^{-1} = \infty$, 则有

$$\overline{\lim_{u\to\infty}} \inf_{t\in[0,u-x_u]} \|M_{t,u}(s)\| \leqslant 1, \qquad C_{r,p}\text{ - q.s.}$$

证明:　根据引理7条件可知存在一序列$\{u_n, n\geqslant 1\}$, 满足$u_n/x_{u_n} = n^{p_0}$, $p_0 > 1$. 显然$\{u_n\}$单调递增, 且当$n\to\infty$时$u_n\to\infty$. 设$t_i = ix_{u_{n+1}}$, $i = 0,1,2,\ldots$,

$$v_n = \Big[\frac{u_n}{x_{u_{n+1}}}\Big] - 1, \qquad h(n) = \frac{\ln(u_n/x_{u_n})}{LL(u_n)} = \frac{\ln n^{p_0}}{LL(u_n)},$$

则$u_n = \exp(n^{p_0/h(n)})$, 且当$n\to\infty$, $h(n)\to\infty$. 对任意$\theta > 0$, 当$n\to\infty$有$n^{\theta}(\ln u_n)^{-1} \to \infty$,

$$1 \leqslant \frac{u_{n+1}}{u_n} = \exp\{(n+1)^{p_0(h(n+1))^{-1}} - n^{p_0(h(n))^{-1}}\} \leqslant \exp\{n^{p_0(h(n))^{-1}-1}\} \to 1.$$

考虑选取适当的$\beta > 0$使得$\beta_0 = (1+\varepsilon)^{2/(2\alpha-1)} + \beta < 1$. 令$\lambda = [x]+1$, 记

$$\Omega_i = (x_{u_{n+1}}\rho_{u_{n+1}})^{-1/2}(w(t_i + x_{u_{n+1}}\cdot) - w(t_i)),$$

则根据引理3及(2)式, 当n充分大时有

$$\begin{aligned}
&\sum_{n=1}^{\infty} C_{r,p}\Big(\inf_{t\in[0,u_n-x_{u_{n+1}}]} \|M_{t,u_{n+1}}(s)\| \geqslant 1+2\varepsilon\Big)\\
\leqslant\ & \sum_{n=1}^{\infty} C_{r,p}\Big(\min_{0\leqslant i\leqslant v_n} \|\Omega_i\| \geqslant \frac{k^{(1-2\alpha)/2}(1+2\varepsilon)}{(\rho_{u_{n+1}})^{1-\alpha}}\Big)\\
\leqslant\ & \sum_{n=1}^{\infty} (1+v_n)^{p\lambda}\Big(\frac{\rho_{u_{n+1}}^{1-\alpha}}{\varepsilon k^{(1-2\alpha)/2}}\Big)^{p(2\lambda^2+\lambda)} \mu\Big(\min_{0\leqslant i\leqslant v_n} \|\Omega_i\| \geqslant \frac{k^{(1-2\alpha)/2}(1+\varepsilon)}{(\rho_{u_{n+1}})^{1-\alpha}}\Big)^{p/q_2}\\
\leqslant\ & \sum_{n=1}^{\infty} (1+v_n)^{p\lambda}\Big(\frac{\rho_{u_{n+1}}^{1-\alpha}}{\varepsilon k^{(1-2\alpha)/2}}\Big)^{p(2\lambda^2+\lambda)} \Big(1-\mu\Big(\|w(\cdot)\| < \frac{k^{(1-2\alpha)/2}(1+\varepsilon)}{(\rho_{u_{n+1}})^{1/2-\alpha}}\Big)\Big)^{(1+v_n)/q_2}\\
\leqslant\ & \sum_{n=1}^{\infty} (1+v_n)^{p\lambda}\Big(\frac{\rho_{u_{n+1}}^{1-\alpha}}{\varepsilon k^{(1-2\alpha)/2}}\Big)^{p(2\lambda^2+\lambda)} \Big(1-\Big(\frac{x_{u_{n+1}}}{u_{n+1}\ln u_{n+1}}\Big)^{\beta_0}\Big)^{(1+v_n)/q_2}\\
\leqslant\ & \sum_{n=1}^{\infty} cn^{p\lambda p_0}(\ln(n+1))^{p(1-\alpha)(2\lambda^2+\lambda)} \exp\Big\{-\frac{p}{q_2}\Big[\frac{u_n}{x_{u_{n+1}}}\Big]\Big(\frac{r_{u_{n+1}}}{u_{n+1}\ln u_{n+1}}\Big)^{\beta_0}\Big\},
\end{aligned}$$

其中常数$c>0$. 对上式通过选取适当的p_0可得到

$$\sum_{n=1}^{\infty} cn^{p\lambda p_0}(\ln(n+1))^{p(2\lambda^2+\lambda)(1-\alpha)}\exp\Big\{-\frac{p}{q_2}\Big[\frac{u_n}{x_{u_{n+1}}}\Big]\Big(\frac{x_{u_{n+1}}}{u_{n+1}\ln u_{n+1}}\Big)^{\beta_0}\Big\}<\infty,$$

再利用Borel-Cantelli引理从而证得存在一递增序列u_n, 当$n\to\infty$时$u_n\to\infty$, 使得

$$\varlimsup_{n\to\infty}\inf_{t\in[0,u_n-x_{u_{n+1}}]}\|M_{t,u_{n+1}}(s)\|\leqslant 1,\qquad C_{r,p}\text{ - q.s.}\tag{8}$$

而对$u\in[u_n,u_{n+1})$, 有

$$\inf_{t\in[0,u-x_u]}\|M_{t,u}(s)\|\leqslant\inf_{t\in[0,u_n-x_{u_{n+1}}]}\frac{(x_{u_{n+1}}\rho_{u_{n+1}})^{1/2}}{(x_{u_n}\rho_{u_n})^{1/2}}\|M_{t,u_{n+1}}(s)\|,$$

考虑到

$$\frac{x_{u_{n+1}}\rho_{u_{n+1}}}{x_{u_n}\rho_{u_n}}\to 1,\qquad n\to\infty,$$

因此根据(8)式进而得到

$$\varlimsup_{u\to\infty}\inf_{t\in[0,u-x_u]}\|M_{t,u}(s)\|\leqslant 1,\qquad C_{r,p}\text{ - q.s.}$$

从而等价地证得引理7. □

定理 8 若$\lim\limits_{u\to\infty}\ln(u/x_u)(LL(u))^{-1}<\infty$, 则有

$$\varliminf_{u\to\infty}\inf_{t\in[0,u-x_u]}\|M_{t,u}(s)\|=1,\qquad C_{r,p}\text{ - q.s.}\tag{9}$$

若$\lim\limits_{u\to\infty}\ln(u/x_u)(LL(u))^{-1}=\infty$, 则有

$$\lim_{u\to\infty}\inf_{t\in[0,u-x_u]}\|M_{t,u}(s)\|=1,\qquad C_{r,p}\text{ - q.s.}\tag{10}$$

证明: 根据引理5、引理6可得(9)式成立, 再根据引理5、引理7可得(10)式成立. □

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The Rate of Quasi Sure Convergence of a Functional Limitation Theorem for a Brownian Motion

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Abstract: In this paper the local functional limit theorem for increments of a Brownian motion is derived with large and small deviations, and the local functional convergence rate for increments of Brownian motion in Hölder norm with respect to (r, p)-capacity is estimated.

Keywords: Brownian motion; quasi sure convergence; Hölder norm; capacity

2010 Mathematics Subject Classification: 60F15; 60F17