# Efficient Estimation for the Partially Linear Models with Random Effects＊ 

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#### Abstract

In this paper an efficient estimation methodology for the partially linear models with random effects is proposed．For this，we use the generalized least square estimate（GLSE）and the B－splines methods to estimate the unknowns，and employ the penalized least square method to obtain the estimators of the random effects item．Further，we also consider the estimation for the variance components．Compared with the existing methods，our proposed methodology performs well．The asymptotic properties of the estimators are obtained．A simulation study is carried out to assess the performance of our proposed methodology．


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## §1．Introduction

We consider the partially linear model with random effects：

$$
\begin{equation*}
Y_{i}=X_{i}^{\top} \beta+g\left(U_{i}\right)+Z_{i}^{\top} b+\varepsilon_{i}, \quad i=1,2, \ldots, n \tag{1}
\end{equation*}
$$

where we assume that the function $g(\cdot)$ is a smooth function on a generic domain $\chi \cdot \beta$ is a $p \times 1$ parameter vector，$b$ is a random－effect vector with $\mathrm{E}(b)=0$ and $\operatorname{Var}(b)=D$ ．Here $D$ is a positive definite matrix depending on a parameter vector $\phi, \varepsilon_{i}$ is a independent random variables and has $\mathrm{E}\left(\varepsilon_{i}\right)=0$ and $\operatorname{Var}\left(\varepsilon_{i}\right)=\sigma_{\varepsilon}^{2}>0$ ．We usually assume that

[^0]$U_{i} \in[0,1]$. There $X_{i}$ and $Y_{i}$ are the random variables, which can be observed. $Z_{i}$ is $n$-dimensional vector. We describe the general problem of estimating $\beta, g(\cdot), b, \sigma_{b}^{2}$ and $\sigma_{\varepsilon}^{2}$.

Mixed effects models are widely applied in the analysis of relevant data, such as longitudinal data and repeated measures data. Some scholars studied the linear and nonparametric mixed effects model in literatures. In model (1), when $g(\cdot)=0$, it is linear mixed-effect models and it is also known as the variance components model. Zhong et al. ${ }^{[1]}$ proposed a unified approach to estimate the linear mixed models with errors-in-variables. When $\beta=0$, model (1) becomes a nonparametric mixed effects model. Wang ${ }^{[2]}$ studied the smoothing spline analysis of variance in nonparametric mixed effect model, they used the smoothing splines to model the fixed effects and constructed the penalized likelihood function. Gu and $\mathrm{Ma}{ }^{[3]}$ used penalized least squares method to study the estimation problem of nonparametric mixed effects model. When $b=0$, model (1) become a partially linear model, see [4-7]. These proposed estimation method was proved to be effective. There are many literatures about partially linear mixed effect models. Tang and Duan ${ }^{[8]}$ studied a semiparametric Bayesian approach to generalize partially linear mixed models for longitudinal data, they presented a semiparametric Bayesian approach by simultaneously utilizing an approximation truncation Dirichlet process prior of the random effects and a Pspline approximation. Li and Xue ${ }^{[9]}$ studied the partially linear varying coefficient model with random effect for longitudinal data, they proposed the estimators for the variance component and profile weighted semiparametric least squares techniques to estimate the parametric component efficiently, they also use B-spline to estimate the function. Li and Xue ${ }^{[10]}$ studied the statistical inference for the generalized partially linear mixed effects models, they proposed a class of semiparametric estimators for the parametric and variance components and they used the local linear smoother method to present the nonparametric estimator. Other literatures about the random effects, see [11-13].

The purpose of this article is to study the estimation problem of model (1). We express the function as the linear form by using B-spline, then we use the generalized least squares to estimate the parameters and function, and use the penalized (unweighted) least squares method to estimate the random effect. We construct the variance estimation for $\sigma_{\varepsilon}^{2}$ and $\sigma_{b}^{2}$, and they are root- $n$ consistent estimators. Our algorithm is fast and stable in numerically. Simulation will be used to show the application of the methodology.

Compared with existing methods, our method has some advantages as follows: In most of the literatures, the relevant scholars have not considered the estimation of random effect in the case of B-spline, while we use the penalized (unweighted) least squares method to
estimate the random effect $b$. Comparing with [9], our method is more stable and faster. The algorithm is more simple in our paper.

We organize the rest of the paper as follows. In Section 2, we introduce the estimation method and main results. In Section 3, we present the simulation study. The proofs of main results are shown in the Appendix.

## §2. Estimation Methodology

In this section, we focus on the estimation methodology. We use B-spline to express the function $g(\cdot)$ as the following approximate form,

$$
g(u)=\sum_{j=1}^{q} c_{j} B_{j}(u)=B(u) c
$$

Therefore, the estimation can be expressed as $\widehat{g}(u)=\sum_{j=1}^{q} \widehat{c}_{j} B_{j}(u)$, where $\left\{B_{j}(u)\right\}_{j=1}^{q}$ is a B-spline basis, $B(u)=\left(B_{1}(u), B_{2}(u), \ldots, B_{q}(u)\right), c=\left(c_{1}, c_{2}, \ldots, c_{q}\right)^{\top}$. The estimation process of $\beta, g(\cdot), b, \sigma_{\varepsilon}^{2}$ and $\sigma_{b}^{2}$ is as follows.
Step 1: Estimation of $\beta$ and $g(\cdot)$
We let $Y=\left(Y_{1}, Y_{2}, \ldots, Y_{n}\right)^{\top}, X=\left(X_{1}, X_{2}, \ldots, X_{n}\right)^{\top}, X_{i}=\left(X_{i 1}, X_{i 2}, \ldots, X_{i p}\right)^{\top}, \beta=$ $\left(\beta_{1}, \beta_{2}, \ldots, \beta_{p}\right)^{\top}, B=\left(B\left(U_{1}\right), B\left(U_{2}\right), \ldots, B\left(U_{n}\right)\right)^{\top}, Z=\left(Z_{1}, Z_{2}, \ldots, Z_{n}\right), Z_{i}=\left(Z_{i 1}, Z_{i 2}\right.$, $\left.\ldots, Z_{\text {in }}\right)^{\top}, b=\left(b_{1}, b_{2}, \ldots, b_{n}\right)^{\top}$ and $\varepsilon=\left(\varepsilon_{1}, \varepsilon_{2}, \ldots, \varepsilon_{n}\right)^{\top}$, then model (1) can be expressed as a vector form: $Y=X \beta+B c+Z^{\top} b+\varepsilon$. We can obtain $\widehat{\beta}$ and $\widehat{c}$ by minimizing the following problem: $[Y-(X \beta+B c)]^{\top} V^{-1}[Y-(X \beta+B c)]$, where $V=\operatorname{Var}\left(Z^{\top} b+\varepsilon\right)=Z D Z^{\top}+\sigma_{\varepsilon}^{2} I_{n}$, they are also the solution of (2):

$$
\left\{\begin{array}{l}
X^{\top} V^{-1}[Y-(X \beta+B c)]=0,  \tag{2}\\
B^{\top} V^{-1}[Y-(X \beta+B c)]=0 .
\end{array}\right.
$$

We let $S=B\left(B^{\top} V^{-1} B\right)^{-1} B^{\top} V^{-1}$ and through formula (2), we will get the estimations of $\beta$ and $g: \widehat{\beta}=\left[X^{\top} V^{-1}(I-S) X\right]^{-1} X^{\top} V^{-1}(I-S) Y, \widehat{g}=B\left(B^{\top} V^{-1} B\right)^{-1} B^{\top} V^{-1}(Y-X \widehat{\beta})=$ $S(Y-X \widehat{\beta})$. We will prove that $\sigma_{b}^{2}$ and $\sigma_{\varepsilon}^{2}$ can be estimated at root- $n$ rates in Step 3, so we take $V$ as known quantity. Then through iterative until convergence, we obtain the final estimates of $\beta$ and $g(\cdot)$.

## Step 2: Estimation of $b$

We use the penalized (unweighted) least squares method to estimate the random effect. Through minimize the following problem:

$$
\frac{1}{n} \sum_{i=1}^{n}\left(Y_{i}-X_{i}^{\top} \widehat{\beta}-\widehat{g}\left(U_{i}\right)-Z_{i}^{\top} b\right)^{2}+\frac{1}{n} b^{\top} \Sigma b,
$$

we get the estimation of $b$. We plug $g(u)=B(u) c$ in the above minimization problem and get $\left(Y-X \widehat{\beta}-B \widehat{c}-Z^{\top} b\right)^{\top}\left(Y-X \widehat{\beta}-B \widehat{c}-Z^{\top} b\right)+b^{\top} \Sigma b$, where $\Sigma>0$ is $p \times p$ general positive definite matrix, we usually use $\sigma_{\varepsilon}^{2} D^{-1}$ to replace $\Sigma$, we will prove that the $D$ and $\sigma_{\varepsilon}^{2}$ can be estimated at root- $n$ rates, then $\Sigma$ is known, so the estimation of $b$ is $\widehat{b}=$ $\left(Z Z^{\top}+\Sigma\right)^{-1}(Z Y-Z X \widehat{\beta}-Z B \widehat{c})$.
Step 3: Estimation of $\sigma_{\varepsilon}^{2}$ and $\sigma_{b}^{2}$
Finally, we estimate the variance components. The estimation method of the variance components is similar to [13]. Assumption that the covariance matrix of model (1) is $V=\sigma_{b}^{2} J_{n} J_{n}^{\top}+\sigma_{\varepsilon}^{2} I_{n}$, where $J_{n}=(1,1, \ldots, 1)^{\top}$ is a $n$-dimensional vector about ones. If the random effects $b$ and the error term $\varepsilon_{i}$ has a normal distribution, then the distribution of observation $Y$ submit to $\mathrm{N}(X \beta+g(U), V)$. Replacing $\beta$ and $g(\cdot)$ with their estimators $\widehat{\beta}$ and $\widehat{g}(\cdot)$, respectively. We can write the normal likelihood function about $\sigma_{b}^{2}$ and $\sigma_{\varepsilon}^{2}$ :

$$
\begin{aligned}
& -n(n-1) \ln \left(\sigma_{\varepsilon}^{2}\right)-n \ln \left(\sigma_{\varepsilon}^{2}+n \sigma_{b}^{2}\right)-\frac{n}{\sigma_{\varepsilon}^{2}+n \sigma_{b}^{2}}(\bar{Y}-\bar{X} \overline{\widehat{\beta}}-\overline{\widehat{g}})^{2} \\
& -\frac{1}{\sigma_{\varepsilon}^{2}} \sum_{i=1}^{n}\left\{Y_{i}-X_{i}^{\top} \widehat{\beta}-\widehat{g}\left(U_{i}\right)-(\bar{Y}-\bar{X} \overline{\widehat{\beta}}-\overline{\widehat{g}})\right\}^{2}
\end{aligned}
$$

where

$$
\bar{Y}=n^{-1} \sum_{i=1}^{n} Y_{i}, \quad \overline{\widehat{g}}=n^{-1} \sum_{i=1}^{n} \widehat{g}\left(U_{i}\right), \quad \bar{X}=n^{-1} \sum_{i=1}^{n} \bar{X}_{i}, \quad \bar{X}_{i}=\frac{1}{p} \sum_{j=1}^{p} X_{i j}, \quad \overline{\widehat{\beta}}=\frac{1}{p} \sum_{j=1}^{p} \widehat{\beta}_{j} .
$$

When $\widehat{\sigma}_{\varepsilon}^{2}>0$, the maximum of the likelihood function can be obtained in the following points:
$\widehat{\sigma}_{\varepsilon}^{2}=\frac{1}{n(n-1)} \sum_{i=1}^{n}\left\{Y_{i}-X_{i}^{\top} \widehat{\beta}-\widehat{g}\left(U_{i}\right)-(\bar{Y}-\bar{X} \overline{\widehat{\beta}}-\overline{\widehat{g}})\right\}^{2}, \quad \widehat{\sigma}_{b}^{2}=\frac{1}{n}(\bar{Y}-\bar{X} \overline{\widehat{\beta}}-\overline{\widehat{g}})^{2}-\frac{1}{n} \widehat{\sigma}_{\varepsilon}^{2}$.
When $\widehat{\sigma}_{b}^{2}=0, \widehat{\sigma}_{\varepsilon}^{2}=n^{-2} \sum_{i=1}^{n}\left\{Y_{i}-X_{i}^{\top} \widehat{\beta}-\widehat{g}\left(U_{i}\right)\right\}^{2}$, where the large sample properties of $\sigma_{\varepsilon}^{2}$ and $\sigma_{b}^{2}$ is as follows:

$$
\begin{align*}
& \widehat{\sigma}_{\varepsilon}^{2}-\sigma_{\varepsilon}^{2}=O_{P}\left(n^{-1 / 2}\right)  \tag{3}\\
& \widehat{\sigma}_{b}^{2}-\sigma_{b}^{2}=O_{P}\left(n^{-1 / 2}\right) \tag{4}
\end{align*}
$$

## §3. Numerical Simulation Studies

In this section, our purpose is to illustrate the performance of estimation methods for model (1). In our simulation, we use the root mean squared errors (RMSE) to assess the precision of $\widehat{\beta}$ and $\widehat{g}(\cdot)$. It is defined as

$$
\mathrm{RMSE}=\left\{n_{\text {grid }}^{-1} \sum_{k=1}^{n_{\text {grid }}}\left[\widehat{g}\left(u_{k}\right)-g\left(u_{k}\right)\right]^{2}\right\}^{1 / 2}
$$

where $n_{\text {grid }}$ is the number of grid point, and $\left\{u_{k}, k=1,2, \ldots, n_{\text {grid }}\right\}$ are equidistant grid points.

Example 1 We consider the model of the form

$$
Y_{i}=X_{i}^{\top} \beta+6 \sin \left(\pi U_{i}\right)+b+\varepsilon_{i}, \quad i=1,2, \ldots, n
$$

where $\beta=(1.5,0.5)^{\top}, X_{i}$ is a two-dimensional random vector with standard normal components, $U_{i}$ is a variable with uniform $[0,1], b$ is a random-effect variable with standard normal components. $\varepsilon_{i}$ is a normal variable with mean 0 and variance 0.36 . Here $g(u)=6 \sin (\pi u)$. The number of subjects, $n$, is taken as 30,50 and 100 . In our simulation, we repeat the simulation 500 times, and compute the bias, standard deviation (SD) and root mean square error (RMSE). The simulation results are obtained from Table 1.

Table 1 Simulation result for Example 1. The biases, standard deviations (SD) and root mean squared error (RMSE) for the estimates of $\beta_{1}$ and $\beta_{2}$.

| Parameter | $n$ | Bias | SD | RMSE |
| :---: | :---: | :---: | :---: | :---: |
| $\beta_{1}$ | 30 | 0.0215 | 0.5147 | 0.5147 |
|  | 50 | 0.0122 | 0.3542 | 0.3541 |
|  | 100 | 0.0085 | 0.2249 | 0.2248 |
| $\beta_{2}$ | 30 | 0.0205 | 0.5329 | 0.5328 |
|  | 50 | 0.0088 | 0.3737 | 0.3734 |
|  | 100 | 0.0031 | 0.2242 | 0.2240 |

From Table 1, we see that the small sample size will lead to larger SD and RMSE. We also find that the Bias, SD and RMSE decrease as $n$ increases, so the improvement is significant. When $n=100$, the means of the estimates of $\beta_{1}$ and $\beta_{2}$ are 1.5086 and 0.4969 , respectively. The estimates of $\sigma_{b}^{2}$ and $\sigma_{\varepsilon}^{2}$ are 1.0346 and 0.3726 , respectively. Figure 1 shows the asymptotic normality of these estimators. When $n=100$, Figure 1 (a) and (b) shows the Q-Q plots of the 500 estimates of $\beta_{1}$ and $\beta_{2}$, respectively.

We also considered the estimation of $g(u)$. When $n=100$, Figure 2(a) shows the real link function curve and the estimated link function curve, the solid curve shows the real link function curve, the dashed curve shows the estimated curve of link function. Figure 2(b) gives the boxplot of the 500 RMSEs of the estimates for link function when $n=100$. From Figure 2(a), we find that the estimated curve is close to the real link function curve, so the estimation methods for data fitting is ideal. Figure $2(\mathrm{~b})$ tell us that the RMSEs of the estimates for link function are small.


Figure 1 When $n=100$, the Q-Q plots of parameter for Example 1.


Figure 2 When $n=100$, the fitted curve and boxplot of function for Example 1.

Finally, we compare our method with [9]. When $n=30,50,100,150$, we calculate the RMSEs of the estimates of $g(\cdot)$ and $\beta$ under the two methods, respectively. The results are shown in Table 2. When $n=100$, from Table 2, our method about the RMSEs of the estimates of $g(\cdot), \beta_{1}$ and $\beta_{2}$ are $0.2394,0.2248$ and 0.2240 , respectively. [9] about the RMSEs of the estimates of $g(\cdot), \beta_{1}$ and $\beta_{2}$ are $0.2559,0.2382$ and 0.2439 , respectively. Under our method, the RMSE is smaller, it tell us that our method is superior to [9] about the estimate of $g(\cdot)$ and $\beta$. So you can see that our method is better than [9]. And in the small sample, our method performs better. From Table 2, we can find the RMSE of the two methods decrease as $n$ increases. Finally, we also find that our calculation speed is faster than [9].

Table 2 The results of the comparison of the estimation method for our method with [9] about $g(\cdot)$ and $\beta$

|  | $n$ | RMSE $_{\text {Que }}$ | RMSE $_{\mathrm{LX}}$ |
| :---: | :---: | :---: | :---: |
| $g(\cdot)$ | 30 | 0.4443 | 0.4529 |
|  | 50 | 0.3501 | 0.3618 |
|  | 100 | 0.2394 | 0.2559 |
|  | 150 | 0.1970 | 0.2048 |
| $\beta_{1}$ | 30 | 0.5147 | 0.5367 |
|  | 50 | 0.3541 | 0.3702 |
|  | 100 | 0.2248 | 0.2382 |
|  | 150 | 0.1879 | 0.1919 |
| $\beta_{2}$ | 30 | 0.5328 | 0.5686 |
|  | 50 | 0.3734 | 0.3758 |
|  | 100 | 0.2240 | 0.2439 |
|  | 150 | 0.1860 | 0.1889 |

## Appendix

We will use the following regular conditions.
$\mathrm{C}_{1} \tau_{1}, \tau_{2}, \ldots, \tau_{J}$ are the internal nodes of spline function, namely, we assume $z_{0}=0$, $z_{k+1}=1, h_{i}=z_{i}-z_{i-1}, h=\max _{1 \leqslant i \leqslant k+1} h_{i}$ and exist a constant $M_{0}$, making $h / \min _{i} \leqslant M_{0}$ and $\max _{i}\left|h_{i+1}-h_{i}\right|=o(1 / k)$, so $h=o(1 / k)$.
$\mathrm{C}_{2}$ The design point sequence $\left\{U_{i}, i=1,2, \ldots, n\right\}$ have bounded support set $\Re$ and the density function $f(u)$ satisfy $0<\inf _{\Re} f(\cdot) \leqslant \sup _{\Re} f(\cdot)<\infty$.
$\mathrm{C}_{3}$ If $u \in \Re$, the eigenvalue of $\Sigma=\mathrm{E}\left(X X^{\boldsymbol{\top}}\right)$ is $\lambda_{0} \leqslant \lambda_{0} \leqslant \cdots \leqslant \lambda_{k}$. The eigenvalue is not equal to zero and bounded, the distribution of $X_{i}$ is compactly supported set.
$\mathrm{C}_{4}$ There is a constant $c_{0}$, making $\mathrm{E}\left(e^{2}\right) \leqslant c_{0}<\infty$, where $e=Z^{\top} b+\varepsilon$.
$\mathrm{C}_{5}$ As for any $i, X_{i}$ and $U_{i}$ are the random variables with independent identically distributed, respectively. And there is a constant $\delta>0$ and $M_{1}, M_{2}$, making $\mathrm{E}\left\|X_{i}\right\|^{2+\delta} \leqslant M_{1}<\infty$ and $\mathrm{E}\left\|U_{i}\right\|^{2+\delta} \leqslant M_{2}<\infty$.

Under the condition of $\left(\mathrm{C}_{1}\right)-\left(\mathrm{C}_{5}\right)$, the Proof of (3) and (4) as follows. First, we prove (3). For convenience, we let

$$
\bar{\varepsilon}=n^{-1} \sum_{i=1}^{n} \varepsilon_{i}, \quad \widetilde{g}_{i}=g\left(U_{i}\right)-\widehat{g}\left(U_{i}\right), \quad \overline{\widetilde{g}}=n^{-1} \sum_{i=1}^{n} \widetilde{g}_{i}
$$

$$
X_{i}^{\top} \widetilde{\beta}=X_{i}^{\top} \beta-X_{i}^{\top} \widehat{\beta}, \quad \overline{X^{\top} \widetilde{\beta}}=n^{-1} \sum_{i=1}^{n}\left(X_{i}^{\top} \widetilde{\beta}\right)
$$

By

$$
\widehat{\sigma}_{\varepsilon}^{2}=\frac{1}{n(n-1)} \sum_{i=1}^{n}\left\{Y_{i}-X_{i}^{\top} \widehat{\beta}-\widehat{g}\left(U_{i}\right)-(\bar{Y}-\bar{X} \overline{\widehat{\beta}}-\overline{\widehat{g}})\right\}^{2}
$$

we get

$$
\begin{aligned}
& \widehat{\sigma}_{\varepsilon}^{2}=\frac{1}{n(n-1)} \sum_{i=1}^{n}\left[\left(\varepsilon_{i}-\bar{\varepsilon}\right)+\left(\widetilde{g}_{i}-\overline{\widetilde{g}}\right)+\left(X_{i}^{\top} \widetilde{\beta}-\overline{X^{\top} \widetilde{ }}\right)\right]^{2} \\
& =\frac{1}{n(n-1)} \sum_{i=1}^{n}\left(\varepsilon_{i}-\bar{\varepsilon}\right)^{2}+\frac{1}{n(n-1)} \sum_{i=1}^{n}\left(\widetilde{g}_{i}-\overline{\widetilde{g}}\right)^{2}+\frac{1}{n(n-1)} \sum_{i=1}^{n}\left(X_{i}^{\top} \widetilde{\beta}-\overline{X^{\top} \widetilde{\widetilde{\beta}}}\right)^{2} \\
& +\frac{2}{n(n-1)} \sum_{i=1}^{n}\left(\varepsilon_{i}-\bar{\varepsilon}\right)\left(\widetilde{g}_{i}-\overline{\widetilde{g}}\right)+\frac{2}{n(n-1)} \sum_{i=1}^{n}\left(\varepsilon_{i}-\bar{\varepsilon}\right)\left(X_{i}^{\top} \widetilde{\beta}-\overline{X^{\top} \widetilde{\beta}}\right) \\
& +\frac{2}{n(n-1)} \sum_{i=1}^{n}\left(\widetilde{g}_{i}-\overline{\widetilde{g}}\right)\left(X_{i}^{\top} \widetilde{\beta}-\overline{X^{\top} \widetilde{\widetilde{\beta}}}\right) \\
& \equiv M_{1}+M_{2}+M_{3}+M_{4}+M_{5}+M_{6} \text {. }
\end{aligned}
$$

By the law of large numbers, we can prove that $M_{1}=\sigma_{\varepsilon}^{2}+O_{P}\left(n^{-1 / 2}\right), M_{2}=O_{P}\left(n^{-1 / 2}\right)$, $M_{3}=O_{P}\left(n^{-1 / 2}\right), M_{4}=O_{P}\left(n^{-1 / 2}\right), M_{5}=O_{P}\left(n^{-1 / 2}\right)$ and $M_{6}=O_{P}\left(n^{-1 / 2}\right)$, so the formula (3) is established. Now, we can prove the formula (4). Through calculation, we can get

$$
\begin{aligned}
\widehat{\sigma}_{b}^{2} & =\frac{1}{n}\left(b+\bar{\varepsilon}+\overline{\widetilde{g}}+\overline{X^{\top} \widetilde{\widetilde{\beta}}}\right)^{2}-\widehat{\sigma}_{\varepsilon}^{2} / n \\
& =\frac{1}{n}(b+\bar{\varepsilon})^{2}+\frac{1}{n} \overline{\widetilde{g}}^{2}+\frac{1}{n}\left(\overline{X^{\top} \widetilde{\beta}}\right)^{2}+\frac{2}{n}(b+\bar{\varepsilon}) \overline{\widetilde{g}}+\frac{2}{n}(b+\bar{\varepsilon})\left(\overline{X^{\top} \widetilde{\widetilde{\beta}}}\right)+\frac{2}{n} \overline{\widetilde{g}}\left(\overline{\left.X^{\top} \widetilde{\widetilde{\beta}}\right)-\widehat{\sigma}_{\varepsilon}^{2} / n}\right. \\
& \equiv N_{1}+N_{2}+N_{3}+N_{4}+N_{5}+N_{6}-\widehat{\sigma}_{\varepsilon}^{2} / n
\end{aligned}
$$

It is easy to prove that $N_{1}=\sigma_{b}^{2}+\sigma_{\varepsilon}^{2} / n+O_{P}\left(n^{-1 / 2}\right), N_{2}=O_{P}\left(n^{-1 / 2}\right), N_{3}=O_{P}\left(n^{-1 / 2}\right)$, $N_{4}=O_{P}\left(n^{-1 / 2}\right), N_{5}=O_{P}\left(n^{-1 / 2}\right)$ and $N_{6}=O_{P}\left(n^{-1 / 2}\right)$. We can prove (4) by using (3).

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# 部分线性混合效应模型的有效估计 

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[^1]
[^0]:    ${ }^{*}$ The project was supported by the National Natural Science Foundation of China（Grant Nos．11471160； 1110 1114），the National Statistical Science Research Key Program of China（Grant No．2013LZ45），the Fundamental Research Funds for the Central Universities（Grant No．30920130111015），the Jiangsu Provincial Basic Research Program（Natural Science Foundation）（Grant No．BK20131345）and sponsored by Qing Lan Project．
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[^1]:    摘 要：本文我们给出部分线性混合效应模型的有效估计方法。首先，我们使用广义最小二乘估计和 B 样条方法去估计未知量，然后利用惩罚最小二乘方法得到随机效应项的估计。接着我们还考虑了方差分量的估计。和现有的方法相比，我们的方法表现更好。此外，我们还给出了估计量的渐近性质。最后，模拟研究被用来评价我们的估计方法的表现。
    关键词：渐近性质；B 样条方法；混合效应模型；部分线性模型；惩罚最小二乘方法
    中图分类号：O212．7

