应用概率统计　第 34 卷
第 1 期　2018 年 2 月

Chinese Journal of Applied Probability and Statistics
Feb. 2018, Vol. 34, No. 1, pp. 1-7
doi: 10.3969/j.issn.1001-4268.2018.01.001

A Counterexample on Local Limits of Galton–Watson Trees *

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Abstract: Take any subcritical offspring distribution with bounded support and consider the corresponding Galton–Watson tree. In this short note we condition this Galton–Watson tree on large width and show that the conditioned tree does not converge locally to any random tree with at most one infinite spine.

Keywords: random tree; conditioning; width; immortal tree

2010 Mathematics Subject Classification: 60J80; 60B10

Citation: He X. A counterexample on local limits of Galton–Watson trees [J]. Chinese J. Appl. Probab. Statist., 2018, 34(1): 1–7.

§1. Introduction

For an introduction of Galton–Watson processes and Galton–Watson trees (GW trees), refer to the standard reference [1]. Regarding the local limits of conditioned G-W trees, it was Kesten [2] who first studied the local limits of critical or subcritical GW trees conditioned to have large height. These local limits are certain size-biased trees with a unique infinite spine, which we call *immortal trees* throughout the present note. Recently, Jonsson and Stefánsson [3] discovered a different type of local limits by conditioning some subcritical GW trees to have large total progeny. These different local limits are certain size-biased trees with a unique node with countably infinite offsprings, which we call *condensation trees*. Also we call a node with countably infinite offsprings an *infinite node*. Shortly after, Janson [4] completed this result by proving that any subcritical GW tree conditioned to have large total progeny converges locally to a condensation tree or an immortal tree, depending on the offspring distribution of the GW tree. Very recently, Abraham and Delmas [5,6] and He [7,8] provided more results on local limits of GW trees,

*The project was supported by the National Natural Science Foundation of China (Grant Nos. 11401012; 11671145) and the Natural Science Foundation of Shanghai (Grant No. 16ZR1409700).

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Received August 29, 2015. Revised October 23, 2015.

by considering other different conditionings. Now, the general picture of all known results is the following: For any critical GW tree, essentially the conditioned tree always converges locally to an immortal tree, for details see [8]; For a subcritical GW tree, the conditioned tree converges locally to an immortal tree or a condensation tree, depending on the conditioning and the offspring distribution.

Naturally one would be interested in the following general question: Is it true that the conditioned subcritical GW trees always converge locally to immortal trees or condensation trees, under any "reasonable" conditioning? In this note we give a negative answer to this question. Specifically, we take any subcritical offspring distribution with bounded support, then condition the corresponding GW tree on large width. See (1) for the definition of the width. We prove in Theorem 2 that the conditioned tree does not converge locally to any random tree with at most one infinite spine. Note that the offspring distribution under consideration has bounded support, so trivially the corresponding conditioned GW tree does not converge locally to any random tree with infinite nodes, for details on this assertion one may refer to Section 2 in [6] for the local convergence of trees with infinite nodes. Thus we have arrived to a negative answer to our motivating question. We may also use an argument in the proof of Theorem 2 to prove a positive result. Specifically, we take any critical offspring distribution with bounded support, then condition the corresponding GW tree to have width equal to a large value. We prove in Proposition 3 that the conditioned tree converges locally to an immortal tree. Note that this result was first proved in [8] by a different method.

This note is organized as follows. In Section 2, we review several topics of GW trees. Section 3 is devoted to the proofs of our main results, Theorem 2 for the subcritical case and Proposition 3 for the critical case.

§2. Preliminaries

This section is extracted from [5]. For more details and proofs, refer to Section 2 in [5]. We denote by $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ the set of non-negative integers and by $\mathbb{N} = \{1, 2, \ldots\}$ the set of positive integers.

2.1 Local Convergence of Random Trees

We first review some notations of discrete trees. Use $\mathscr{U} = \bigcup_{n\geqslant 0} \mathbb{N}^n$ to denote the set of finite sequences of positive integers with the convention $\mathbb{N}^0 = \{\emptyset\}$. For $u \in \mathbb{N}^n$, we call $|u| = n$ the generation or the height of u. If u and v are two sequences of $\mathscr{U}$, denote by

uv the concatenation of the two sequences, with the convention that $uv = u$ if $v = \emptyset$ and $uv = v$ if $u = \emptyset$. The set of ancestors of u is the set

$$A_u = \{v \in \mathscr{U} : \text{ there exists } w \in \mathscr{U}, w \neq \emptyset, \text{ such that } u = vw\}.$$

A tree $\boldsymbol{t}$ is a subset of $\mathscr{U}$ that satisfies:

- $\emptyset \in \boldsymbol{t}$.
- If $u \in \boldsymbol{t}$, then $A_u \subset \boldsymbol{t}$.
- For every $u \in \boldsymbol{t}$, there exists $k_u(\boldsymbol{t}) \in \mathbb{Z}_+$ such that, for every positive integer $i, ui \in \boldsymbol{t}$ if and only if $1 \leqslant i \leqslant k_u(\boldsymbol{t})$.

The node $\emptyset$ is called the root of $\boldsymbol{t}$. The integer $k_u(\boldsymbol{t})$ represents the number of offsprings of the node u in $\boldsymbol{t}$. The set of leaves of the tree $\boldsymbol{t}$ is $\mathcal{L}(\boldsymbol{t}) = \{u \in \boldsymbol{t} : k_u(\boldsymbol{t}) = 0\}$. Write $X(\boldsymbol{t}) = (X_n(\boldsymbol{t}), n \in \mathbb{Z}_+)$ for the Galton–Watson process corresponding to the tree $\boldsymbol{t}$, that is, $X_n(\boldsymbol{t})$ is the total number of nodes of the tree $\boldsymbol{t}$ at height n. Then define the width of the tree $\boldsymbol{t}$ by

$$W(\boldsymbol{t}) = \sup_{n \in \mathbb{Z}_+} X_n(\boldsymbol{t}). \tag{1}$$

For $u \in \boldsymbol{t}$, by $\mathscr{S}_u(\boldsymbol{t})$ we mean the subtree of $\boldsymbol{t}$ "above" u. Denote by $\mathbb{T}$ the set of trees, by $\mathbb{T}_0$ the subset of finite trees, and by $\mathbb{T}_1$ the subset of trees with a unique infinite spine. An infinite spine is a genealogical line which never ends. For precise definitions of all these notations in the present paragraph, refer to Section 2 in [5].

For the general framework of the local convergence of discrete trees, also refer to Section 2 in [5]. Here we only recall some essential notations and a crucial lemma. Let $(T_n, n \in \mathbb{N})$ and T be some random discrete trees. We denote by $\operatorname{dist}(T)$ the distribution of the random discrete tree T, and we denote

$$\operatorname{dist}(T_n) \to \operatorname{dist}(T) \qquad \text{as } n \to \infty$$

for the convergence in distribution of the sequence $(T_n, n \in \mathbb{N})$ to T, with respect to the local distance on the space of discrete trees. If $\boldsymbol{t}, \boldsymbol{s} \in \mathbb{T}$ and $x \in \mathscr{L}(\boldsymbol{t})$, we denote by

$$\boldsymbol{t} \circledast (\boldsymbol{s}, x) = \{u \in \boldsymbol{t}\} \cup \{xv, v \in \boldsymbol{s}\} \tag{2}$$

the tree obtained by grafting the tree $\boldsymbol{s}$ on the leaf x of the tree $\boldsymbol{t}$. For every $\boldsymbol{t} \in \mathbb{T}$ and every $x \in \mathscr{L}(\boldsymbol{t})$, we shall consider the set of trees obtained by grafting a tree on the leaf x of $\boldsymbol{t}$,

$$\mathbb{T}(\boldsymbol{t}, x) = \{\boldsymbol{t} \circledast (\boldsymbol{s}, x), \boldsymbol{s} \in \mathbb{T}\}. \tag{3}$$

We recall Lemma 2.1 in [5], which is a very convenient characterization of convergence in distribution in $\mathbb{T}_0 \cup \mathbb{T}_1$.

Lemma 1 Let $(T_n, n \in \mathbb{N})$ and T be $\mathbb{T}$-valued random variables. The sequence $(T_n, n \in \mathbb{N})$ converges in distribution to T implies that for every $\boldsymbol{t} \in \mathbb{T}_0$ and every $x \in \mathscr{L}(\boldsymbol{t})$,

$$\lim_{n\to\infty} \mathsf{P}(T_n \in \mathbb{T}(\boldsymbol{t}, x)) = \mathsf{P}(T \in \mathbb{T}(\boldsymbol{t}, x)) \quad \text{and} \quad \lim_{n\to\infty} \mathsf{P}(T_n = \boldsymbol{t}) = \mathsf{P}(T = \boldsymbol{t}). \tag{4}$$

If $(T_n, n \in \mathbb{N})$ and T belong a.s. to $\mathbb{T}_0 \cup \mathbb{T}_1$, then (4) also implies that $(T_n, n \in \mathbb{N})$ converges in distribution to T.

2.2 GW Trees and Immortal Trees

Let $p = (p_0, p_1, p_2, \ldots)$ be an offspring distribution, which is just a probability distribution on $\mathbb{Z}_+$. A $\mathbb{T}$-valued random variable τ is a Galton–Watson tree (GW tree) with offspring distribution p if the distribution of $k_\emptyset(\tau)$ is p and for any $n \in \mathbb{N}$, conditionally on $\{k_\emptyset(\tau) = n\}$, the subtrees $(\mathscr{S}_1(\tau), \mathscr{S}_2(\tau), \ldots, \mathscr{S}_n(\tau))$ are independent and distributed as the original tree τ. By the definition of GW trees, we have for every $\boldsymbol{t} \in \mathbb{T}_0$, $x \in \mathscr{L}(\boldsymbol{t})$, and $\widetilde{\boldsymbol{t}} \in \mathbb{T}_0$,

$$\mathsf{P}(\tau = \boldsymbol{t} \circledast (\widetilde{\boldsymbol{t}}, x)) = \frac{1}{p_0} \mathsf{P}(\tau = \boldsymbol{t}) \mathsf{P}(\tau = \widetilde{\boldsymbol{t}}). \tag{5}$$

Denote by μ the expectation of p. The GW tree is called critical (resp. subcritical, supercritical) if $\mu = 1$ (resp. $\mu < 1$, $\mu > 1$). In the critical or subcritical case, it is well-known that a.s. $\tau \in \mathbb{T}_0$.

We recall the following definition of immortal trees from Section 1 in [6], which first appeared in Section 5 of [4]. Let p be a critical or subcritical offspring distribution. Let $\tau^*(p)$ denote the random tree which is defined by:

(i) There are two types of nodes: *normal* and *special*.

(ii) The root is special.

(iii) Normal nodes have offspring distribution p.

(iv) Special nodes have biased offspring distribution $\widehat{p}$ on $\mathbb{Z}_+$ defined by $\widehat{p}_n = np_n/\mu$ for any $n \in \mathbb{Z}_+$.

(v) The offsprings of all the nodes are independent of each others.

(vi) All the offsprings of a normal node are normal.

(vii) When a special node gets several offsprings (note that $\widehat{p}_0 = 0$), one of them is selected uniformly at random and is special while the others are normal.

Note that a.s. $\tau^*(p)$ has exactly one infinite spine and no infinite nodes. We call it an *immortal tree*. By the definitions of GW trees and immortal trees, we have for every $\boldsymbol{t} \in \mathbb{T}_0$ and $x \in \mathscr{L}(\boldsymbol{t})$,

$$\mathsf{P}(\tau^*(p) \in \mathbb{T}(\boldsymbol{t}, x)) = \frac{1}{\mu^{|x|} p_0} \mathsf{P}(\tau = \boldsymbol{t}). \tag{6}$$

§3. The Results

By (1), the definition of the width, it is not hard to see that $\mathsf{P}(W(\tau) > n) > 0$ for all n if and only if $p_0 + p_1 < 1$. So we need the assumption $p_0 + p_1 < 1$ in order for the conditional distribution $\operatorname{dist}(\tau \mid W(\tau) > n)$ to make sense.

Theorem 2 Consider a subcritical offspring distribution $p = (p_0, p_1, p_2, \ldots)$ with bounded support. Assume that $p_0 + p_1 < 1$. Then for the GW tree τ with the offspring distribution p, and for any random tree $T \in \mathbb{T}_1$ a.s., the local convergence $\operatorname{dist}(\tau \mid W(\tau) > n) \to \operatorname{dist}(T)$ does not hold.

Proof Assume that for some random tree T, the local convergence $\operatorname{dist}(\tau \mid W(\tau) > n) \to \operatorname{dist}(T)$ holds. Let N be the supremum of the support of p, that is, $N = \sup\{n : p_n > 0\} < \infty$.

Recall (2) and (3), the definitions of $\boldsymbol{t} \circledast (\widetilde{\boldsymbol{t}}, x)$ and $\mathbb{T}(\boldsymbol{t}, x)$. By (5) we get

$$\begin{aligned}\mathsf{P}(\tau \in \mathbb{T}(\boldsymbol{t}, x), W(\tau) > n) &= \sum_{\widetilde{\boldsymbol{t}} \in \mathbb{T}_0} \mathsf{P}(\tau = \boldsymbol{t} \circledast (\widetilde{\boldsymbol{t}}, x)) \mathbf{1}_{\{W(\boldsymbol{t} \circledast (\widetilde{\boldsymbol{t}}, x)) > n\}} \\ &= \sum_{\widetilde{\boldsymbol{t}} \in \mathbb{T}_0} \frac{1}{p_0} \mathsf{P}(\tau = \boldsymbol{t}) \mathsf{P}(\tau = \widetilde{\boldsymbol{t}}) \mathbf{1}_{\{W(\boldsymbol{t} \circledast (\widetilde{\boldsymbol{t}}, x)) > n\}}.\end{aligned}$$

Write $W_n(\boldsymbol{t})$ for the width of the tree which consists of the first n generations of $\boldsymbol{t}$. Since p has bounded support with N being the supremum of the support of p, we see that if $\mathsf{P}(\tau = \widetilde{\boldsymbol{t}}) > 0$ then $W_{H(\boldsymbol{t})}(\widetilde{\boldsymbol{t}}) \leqslant N^{H(\boldsymbol{t})}$. So for any $n > W(\boldsymbol{t}) + N^{H(\boldsymbol{t})}$, we have

$$n > W(\boldsymbol{t}) + N^{H(\boldsymbol{t})} \geqslant W(\boldsymbol{t}) + W_{H(\boldsymbol{t})}(\widetilde{\boldsymbol{t}}).$$

This inequality implies that $W(\boldsymbol{t} \circledast (\widetilde{\boldsymbol{t}}, x)) > n$ can only be realized after generation $H(\boldsymbol{t})$ of the tree $\boldsymbol{t} \circledast (\widetilde{\boldsymbol{t}}, x)$ (or above height $H(\boldsymbol{t})$ of the tree $\boldsymbol{t} \circledast (\widetilde{\boldsymbol{t}}, x)$, in different words), which then implies that

$$W(\boldsymbol{t} \circledast (\widetilde{\boldsymbol{t}}, x)) = W(\widetilde{\boldsymbol{t}}) > n.$$

By this observation we get for $n > W(\boldsymbol{t}) + N^{H(\boldsymbol{t})}$,

$$\begin{aligned}\sum_{\widetilde{\boldsymbol{t}} \in \mathbb{T}_0} \frac{1}{p_0} \mathsf{P}(\tau = \boldsymbol{t}) \mathsf{P}(\tau = \widetilde{\boldsymbol{t}}) \mathbf{1}_{\{W(\boldsymbol{t} \circledast (\widetilde{\boldsymbol{t}}, x)) > n\}} &= \frac{1}{p_0} \mathsf{P}(\tau = \boldsymbol{t}) \sum_{\widetilde{\boldsymbol{t}} \in \mathbb{T}_0} \mathsf{P}(\tau = \widetilde{\boldsymbol{t}}) \mathbf{1}_{\{W(\widetilde{\boldsymbol{t}}) > n\}} \\ &= \frac{1}{p_0} \mathsf{P}(\tau = \boldsymbol{t}) \mathsf{P}(W(\tau) > n).\end{aligned}$$

Therefore we have

$$\lim_{n \to \infty} \mathsf{P}(\tau \in \mathbb{T}(\boldsymbol{t}, x) \mid W(\tau) > n) = \lim_{n \to \infty} \frac{\mathsf{P}(\tau \in \mathbb{T}(\boldsymbol{t}, x), W(\tau) > n)}{\mathsf{P}(W(\tau) > n)}$$

$$= \lim_{n\to\infty} \frac{1}{p_0} \frac{\mathsf{P}(\tau = \boldsymbol{t})\mathsf{P}(W(\tau) > n)}{\mathsf{P}(W(\tau) > n)} = \frac{1}{p_0}\mathsf{P}(\tau = \boldsymbol{t}).$$

Then Lemma 1 asserts that

$$\mathsf{P}(T \in \mathbb{T}(\boldsymbol{t}, x)) = \frac{1}{p_0}\mathsf{P}(\tau = \boldsymbol{t}). \tag{7}$$

For every $\boldsymbol{t} \in \mathbb{T}_0$ and every $n > W(\boldsymbol{t})$, we have $\mathsf{P}(\tau = \boldsymbol{t} \,|\, W(\tau) > n) = 0$. So Lemma 1 asserts that for every $\boldsymbol{t} \in \mathbb{T}_0$, $\mathsf{P}(T = \boldsymbol{t}) = 0$, which implies that a.s. T has at least one infinite spine. If the root of T has only one offspring, that is the node 1, then $T \in \mathbb{T}(\{\emptyset, 1\}, 1)$ and (7) asserts that

$$\mathsf{P}(T \in \mathbb{T}(\{\emptyset, 1\}, 1)) = \frac{1}{p_0}\mathsf{P}(\tau = \{\emptyset, 1\}) = p_1,$$

so $\mathsf{P}(T \in \mathbb{T}_1, k_\emptyset(T) = 1) \leqslant p_1$. Recall that $k_\emptyset(T)$ is the number of offsprings of the root $\emptyset$ in the tree T. If the root of T has two offsprings, the node 1 and the node 2, and all the infinite spines of T go through the node 1, then $T \in \bigcup\limits_{\boldsymbol{t}\in\mathbb{T}_0} \mathbb{T}(\{\emptyset, 1, 2\} \circledast (\boldsymbol{t}, 2), 1)$, and (7) and (5) assert that

$$\begin{aligned} \mathsf{P}\Big(T \in \bigcup_{\boldsymbol{t}\in\mathbb{T}_0} \mathbb{T}(\{\emptyset, 1, 2\} \circledast (\boldsymbol{t}, 2), 1)\Big) &= \sum_{\boldsymbol{t}\in\mathbb{T}_0} \frac{1}{p_0}\mathsf{P}(\tau = \{\emptyset, 1, 2\} \circledast (\boldsymbol{t}, 2)) \\ &= \frac{1}{(p_0)^2}\mathsf{P}(\tau = \{\emptyset, 1, 2\}) \sum_{\boldsymbol{t}\in\mathbb{T}_0} \mathsf{P}(\tau = \boldsymbol{t}) = p_2. \end{aligned}$$

The same is true for the probability that the root of T has two offsprings and all the infinite spines go through the node 2, that is,

$$\mathsf{P}\Big(T \in \bigcup_{\boldsymbol{t}\in\mathbb{T}_0} \mathbb{T}(\{\emptyset, 1, 2\} \circledast (\boldsymbol{t}, 1), 2)\Big) = p_2,$$

so $\mathsf{P}(T \in \mathbb{T}_1, k_\emptyset(T) = 2) \leqslant 2p_2$. Similarly we have $\mathsf{P}(T \in \mathbb{T}_1, k_\emptyset(T) = n) \leqslant np_n$. Finally we see that

$$\mathsf{P}(T \in \mathbb{T}_1) = \sum_n \mathsf{P}(T \in \mathbb{T}_1, k_\emptyset(T) = n) \leqslant \sum_n np_n < 1,$$

so the statement a.s. $T \in \mathbb{T}_1$ does not hold. $\square$

Proposition 3 Consider a critical offspring distribution $p = (p_0, p_1, p_2, \ldots)$ with bounded support. We also assume that $p_0 + p_1 < 1$. Then for the GW tree τ with the offspring distribution p, as $n \to \infty$ along the subsequence $\{n : \mathsf{P}(W(\tau) = n) > 0\}$,

$$\mathrm{dist}\,(\tau \,|\, W(\tau) = n) \to \mathrm{dist}\,(\tau^*(p)).$$

Proof First of all, all limits in this proof are understood along the subsequence $\{n : \mathsf{P}(W(\tau) = n) > 0\}$. As in the second paragraph of the proof of Theorem 2, we have for every $\boldsymbol{t} \in \mathbb{T}_0$ and every $x \in \mathscr{L}(\boldsymbol{t})$,

$$\lim_{n\to\infty} \mathsf{P}(\tau \in \mathbb{T}(\boldsymbol{t}, x) \,|\, W(\tau) = n) = \frac{1}{p_0}\mathsf{P}(\tau = \boldsymbol{t}).$$

Note that p is critical, so $\mu = 1$. Then (6) asserts that

$$\lim_{n\to\infty} \mathsf{P}(\tau \in \mathbb{T}(\boldsymbol{t}, x) \,|\, W(\tau) = n) = \mathsf{P}(\tau^*(p) \in \mathbb{T}(\boldsymbol{t}, x)).$$

As in the third paragraph of the proof of Theorem 2, we have for every $\boldsymbol{t} \in \mathbb{T}_0$,

$$\lim_{n\to\infty} \mathsf{P}(\tau = \boldsymbol{t} \,|\, W(\tau) = n) = 0 = \mathsf{P}(\tau^*(p) = \boldsymbol{t}).$$

Finally apply Lemma 1 to finish the proof. □

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Galton–Watson 树的局部极限的一个反例

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摘 要: 本文考虑后代分布具有有界支撑的 Galton–Watson 树. 我们证明在具有大的宽度这一条件概率分布下, Galton–Watson 树不会局部收敛到任何具有唯一一条无穷脊柱的随机树.

关键词: 随机树; 条件概率分布; 宽度; 不死树

中图分类号: O211.65