Hájek-Rényi-Type Inequality and Strong Law of Large Numbers for Associated Sequences^{*}

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Abstract: In this paper, a new Hájek-Rényi-type inequality for mean zero associated random variables is obtained, which generalizes and improves the result of Theorem 2.2 of [9]. In addition, a Brunk-Prokhorov-type strong law of large numbers is also given.

Keywords: associated random variables; demimartingales; Hájek-Rényi-type inequality; strong law of large numbers

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§1. Introduction

We use the following notations. Let X_1, X_2, \ldots or S_1, S_2, \ldots denote a sequence of random variables defined on a fixed probability space $(\Omega, \mathscr{F}, \mathsf{P})$. $S_0 \doteq 0, I(A)$ is an indicator function of set A.

The concept of (positively) associated random variables was introduced by Esary et al.^[1].

Definition 1 A finite sequence X_1, X_2, \ldots, X_n is said to be associated if for any componentwise nondecreasing functions f and g on \mathbb{R}^n ,

$$Cov(f(X_1, X_2, ..., X_n), g(X_1, X_2, ..., X_n)) \ge 0,$$

assuming of course that the covariance exists. An infinite sequence $\{X_n, n \ge 1\}$ is said to be associated if every finite sub-family is associated.

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It is easy to see that if $\{X_n, n \ge 1\}$ is a sequence of associated random variables, then the covariance is nonnegative. Let us recall that the sequences of independent random variables are associated and nondecreasing functions of associated random variables are also associated.

Hájek and Rényi^[2] proved the following important inequality. If $\{X_i, i \ge 1\}$ is a sequence of independent random variables with mean zero and finite second moments, and $\{b_i, i \ge 1\}$ is a sequence of positive nondecreasing real numbers, then, for any $\varepsilon > 0$ and any positive integer $m \le n$,

$$P\Big(\max_{m\leqslant k\leqslant n} \left|\frac{1}{b_k}\sum_{j=1}^k X_j\right| > \varepsilon\Big) \leqslant \varepsilon^{-2}\Big(\sum_{j=m+1}^n \frac{\mathsf{E}X_j^2}{b_j^2} + \frac{1}{b_m^2}\sum_{j=1}^m \mathsf{E}X_j^2\Big).$$

The inequality was studied by many authors (see [3-6]). Prakasa Rao^[7] extended the Hájek-Rényi inequality to associated random variables. Sung^[8] improved the inequality of [7] and gave some applications for associated random variables. Recently, Hu et al.^[9] improved the result of [8] by using the demimartingale's method that is different from Sung's.

Definition 2 Let S_1, S_2, \ldots be an L^1 sequence of random variables. Assume that for $j = 1, 2, \ldots$,

$$\mathsf{E}((S_{j+1} - S_j)f(S_1, S_2, \dots, S_j)) \ge 0 \tag{1}$$

for all componentwise nondecreasing functions f whenever the expectation is defined. Then $\{S_j, j \ge 1\}$ is called a demimartingale. Moreover, if f is assumed to be nonnegative, the sequence $\{S_j, j \ge 1\}$ is called a demisubmartingale.

Newman and Wright^[10] showed that the partial sum of sequence of mean zero associated random variables is a demimartingale.

It is easily seen that if the function f is not required to be nondecreasing, then (1) is equivalent to the condition that $\{S_j, j \ge 1\}$ is a martingale with the natural choice of σ -fields. Similarly, if f is assumed to be nonnegative and not necessarily nondecreasing, then (1) is equivalent to the condition that $\{S_j, j \ge 1\}$ is a submartingale with the natural choice of σ -fields.

Definition 2 is due to [10], [11] extended various results including Chow's maximal inequality for (sub)martingales to the case of demi(sub)martingales and obtained a strong law of large numbers for demimartingales. Hu et al.^[9] extended the Hájek-Rényi inequality to the case of demimartingales and obtained the Hájek-Rényi-type inequality, a strong law of large numbers and strong growth rate for associated random variables.

In this paper, we prove a new Hájek-Rényi-type inequality for mean zero associated random variables, which improves the result of [9]. Using this result, we obtain the integrability of supremum and strong law of large numbers for associated random variables.

Lemma 3 (cf. [12; Theorem 2.1]) Let $\{S_n, n \ge 1\}$ be a demimartingale. Let g be a nonnegative convex function on R with g(0) = 0 and $\{c_n, n \ge 1\}$ be a nonincreasing sequence of positive numbers, define $S_n^* = \max(c_1g(S_1), c_2g(S_2), \ldots, c_ng(S_n))$ with $S_0^* = 0$. Then for $\forall \varepsilon > 0$,

$$\varepsilon \mathsf{P}(S_n^* \ge \varepsilon) \leqslant \sum_{i=1}^n c_i \mathsf{E}[(g(S_i) - g(S_{i-1}))I\{S_n^* \ge \varepsilon\}].$$
(2)

Corollary 4 Let $\{S_n, n \ge 1\}$ be a demimartingale. Let g be a nonnegative convex function on R with g(0) = 0 and $\{c_n, n \ge 1\}$ be a nonincreasing sequence of positive numbers, define $S_n^* = \max(c_1g(S_1), c_2g(S_2), \ldots, c_ng(S_n))$ with $S_0^* = 0$. Then for $\forall \varepsilon > 0$,

$$\varepsilon \mathsf{P}(S_n^* \ge \varepsilon) \le c_1 \mathsf{E}g(S_1) + \sum_{i=2}^n c_i \mathsf{E}[g(S_i) - g(S_{i-1})].$$
(3)

Proof By Lemma 3, we can see that

$$\begin{split} \varepsilon \mathsf{P}(S_n^* \ge \varepsilon) &\leqslant \sum_{i=1}^n c_i \mathsf{E}[(g(S_i) - g(S_{i-1}))I\{S_n^* \ge \varepsilon\}] \\ &= \sum_{i=1}^{n-1} (c_i - c_{i+1}) \mathsf{E}[g(S_i)I\{S_n^* \ge \varepsilon\}] + c_n \mathsf{E}[g(S_n)I\{S_n^* \ge \varepsilon\}] \\ &\leqslant \sum_{i=1}^{n-1} (c_i - c_{i+1}) \mathsf{E}g(S_i) + c_n \mathsf{E}g(S_n) \\ &= c_1 \mathsf{E}g(S_1) + \sum_{i=2}^n c_i \mathsf{E}[g(S_i) - g(S_{i-1})]. \end{split}$$

Lemma 5 (cf. [13; Theorem 3.1]) Let $\{S_n, n \ge 1\}$ be a demimartingale sequence, $\{b_n, n \ge 1\}$ be a nondecreasing unbounded sequence of positive real numbers. Let q > 1/2 and $\mathsf{E}|S_k|^{2q} < \infty$ for each k. Assume that

$$\sum_{k=1}^{\infty} \frac{\mathsf{E}(|S_k|^{2q} - |S_{k-1}|^{2q})}{b_k^{2q}} < \infty, \tag{4}$$

then

$$\lim_{n \to \infty} \frac{S_n}{b_n} = 0 \qquad \text{a.s.},\tag{5}$$

and with the growth rate

$$\frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) \qquad \text{a.s.},\tag{6}$$

where

$$\beta_n = \max_{1 \le k \le n} b_k v_k^{\delta/2q}, \qquad \forall \, 0 < \delta < 1,$$

$$v_{n} = \sum_{k=n}^{\infty} \frac{\alpha_{k}}{b_{k}^{2q}}, \qquad \lim_{n \to \infty} \frac{\beta_{n}}{b_{n}} = 0,$$

$$\alpha_{k} = \left(\frac{2q}{2q-1}\right)^{2q} (\mathsf{E}|S_{k}|^{2q} - \mathsf{E}|S_{k-1}|^{2q}), \tag{7}$$

and

$$\mathsf{E}\Big(\sup_{k\ge 1}\Big|\frac{S_k}{b_k}\Big|^{2q}\Big) \leqslant 4\Big(\frac{2q}{2q-1}\Big)^{2q}\sum_{k=1}^{\infty}\frac{\mathsf{E}(|S_k|^{2q}-|S_{k-1}|^{2q})}{b_k^{2q}} < \infty.$$
(8)

§2. Hájek-Rényi-Type Inequalities

Firstly, we give a Hájek-Rényi-type inequality for demimartingales.

Theorem 6 (Hájek-Rényi-type inequality for demimartingales) Let $\{S_k, k \ge 1\}$ be a demimartingale and $\{c_k, k \ge 1\}$ be a nonincreasing sequence of positive numbers. Let $\mathsf{E}S_k^2 < \infty$ for each k, then for every $\varepsilon > 0$ and $1 \le m \le n$,

$$\mathsf{P}\Big(\max_{m\leqslant k\leqslant n} c_k |S_k| \geqslant \varepsilon\Big) \leqslant \frac{1}{\varepsilon^2} \Big(c_m^2 \mathsf{E} S_m^2 + \sum_{k=m+1}^n c_k^2 \mathsf{E} (S_k^2 - S_{k-1}^2) \Big).$$
(9)

Proof Since $\{S_k, k \ge 1\}$ is a demimartingale, by the definition of demimartingale, it is easily seen that $\{S_i, i \ge n\}$ is also a demimartingale for fixed n. Taking $g(x) = x^2$, so it is a nonnegative convex function, thus by Corollary 4 we have

$$\begin{split} \mathsf{P}\Big(\max_{m\leqslant k\leqslant n}c_k|S_k|\geqslant \varepsilon\Big) &= \mathsf{P}\Big(\max_{m\leqslant k\leqslant n}c_k^2S_k^2\geqslant \varepsilon^2\Big)\\ &\leqslant \frac{1}{\varepsilon^2}\Big(c_m^2\mathsf{E}S_m^2 + \sum_{k=m+1}^n c_k^2\mathsf{E}(S_k^2 - S_{k-1}^2)\Big). \end{split} \ \Box$$

Theorem 7 (Hájek-Rényi-type inequality for associated sequences) Let $\{X_n, n \ge 1\}$ be a sequence of associated random variables with $EX_n = 0$ and $EX_n^2 < \infty$, $n \ge 1$. Let $\{b_n, n \ge 1\}$ be a nondecreasing sequence of positive real numbers. Then, for any $\varepsilon > 0$ and any positive integer $m \le n$,

$$\mathsf{P}\Big(\max_{1\leqslant k\leqslant n} \left|\frac{1}{b_k}\sum_{j=1}^k X_j\right| \geqslant \varepsilon\Big) \leqslant \frac{1}{\varepsilon^2} \Big(\sum_{j=1}^n \frac{\mathsf{E}X_j^2}{b_j^2} + 2\sum_{j=1}^n \frac{\mathsf{Cov}\left(X_j, S_{j-1}\right)}{b_j^2}\Big),\tag{10}$$

$$\mathsf{P}\Big(\max_{m\leqslant k\leqslant n} \left|\frac{1}{b_k}\sum_{j=1}^k X_j\right| \ge \varepsilon\Big) \leqslant \frac{2}{\varepsilon^2} \Big(\frac{1}{b_m^2}\sum_{j=1}^m \mathsf{Cov}\left(X_j, S_j\right) + \sum_{j=m+1}^n \frac{\mathsf{Cov}\left(X_j, S_j\right)}{b_j^2}\Big).$$
(11)

Proof Let $S_n = \sum_{i=1}^n X_i$, then $\{S_n, n \ge 1\}$ is a deminartingale with $\mathsf{E}S_n = 0, n \ge 1$. If we take $c_n = 1/b_n$ for $n \ge 1$, then for every $\varepsilon > 0$ and $1 \le m \le n$, we have by Theorem 6,

$$\mathsf{P}\Big(\max_{m\leqslant k\leqslant n} \left|\frac{1}{b_k}\sum_{j=1}^k X_j\right| \ge \varepsilon\Big) \leqslant \frac{1}{\varepsilon^2} \Big(\frac{1}{b_m^2} \mathsf{E} S_m^2 + \sum_{k=m+1}^n \frac{1}{b_k^2} \mathsf{E} (S_k^2 - S_{k-1}^2)\Big).$$
(12)

Since

No. 2

$$E(S_k^2 - S_{k-1}^2) = E[(S_k - S_{k-1})(S_k + S_{k-1})]$$

= $E[X_k(2S_{k-1} + X_k)]$
= $2E(X_kS_{k-1}) + E(X_k^2),$

then if we take m = 1 in (12), we have

$$\mathsf{P}\Big(\max_{1\leqslant k\leqslant n} \left|\frac{1}{b_k}\sum_{j=1}^k X_j\right| \geqslant \varepsilon \Big) \leqslant \frac{1}{\varepsilon^2} \Big(\frac{\mathsf{E}X_1^2}{b_1^2} + \sum_{k=2}^n \frac{2\mathsf{E}(X_k S_{k-1}) + \mathsf{E}(X_k^2)}{b_k^2} \Big) \\ \leqslant \frac{1}{\varepsilon^2} \Big(\sum_{j=1}^n \frac{\mathsf{E}X_j^2}{b_j^2} + 2\sum_{j=1}^n \frac{\mathsf{Cov}\left(X_j, S_{j-1}\right)}{b_j^2} \Big).$$

Furthermore, we can obtain

$$\begin{split} \mathsf{P}\Big(\max_{m\leqslant k\leqslant n} \left|\frac{1}{b_k}\sum_{j=1}^k X_j\right| \geqslant \varepsilon\Big) \leqslant \frac{1}{\varepsilon^2} \Big(\frac{\mathsf{E}S_m^2}{b_m^2} + \sum_{k=m+1}^n \frac{2\mathsf{E}(X_k S_{k-1}) + \mathsf{E}(X_k^2)}{b_k^2}\Big) \\ \leqslant \frac{2}{\varepsilon^2} \Big(\frac{1}{b_m^2}\sum_{j=1}^m \mathsf{Cov}\left(X_j, S_j\right) + \sum_{j=m+1}^n \frac{\mathsf{Cov}\left(X_j, S_j\right)}{b_j^2}\Big). \end{split}$$

If we take $c_k = 1$ and m = 1 in Theorem 6, then we have the following result.

Corollary 8 Let $\{S_n, n \ge 1\}$ be a demimartingale sequence, then for every $\varepsilon > 0$ and $n \ge 1$,

$$\mathsf{P}\Big(\max_{1\leqslant k\leqslant n}|S_k|\geqslant \varepsilon\Big)\leqslant \frac{1}{\varepsilon^2}\mathsf{E}S_n^2.$$
(13)

§3. A Strong Law of Large Numbers and a Strong Growth Rate for Associated Sequences

In this section, we will give the integrability of supremum, a strong law of large numbers and a strong growth rate for associated sequences by using the results of [13].

Theorem 9 Let $\{b_n, n \ge 1\}$ be a nondecreasing unbounded sequence of positive real numbers. Let $\{X_n, n \ge 1\}$ be a sequence of mean zero associated random variables satisfying $\sum_{i=1}^{\infty} \text{Cov}(X_j, S_j)/b_j^2 < \infty$. Then

$$\mathsf{E}\Big(\sup_{n \ge 1} \Big|\frac{S_n}{b_n}\Big|^r\Big) \leqslant 1 + \frac{2r}{2-r} \sum_{j=1}^{\infty} \frac{\mathsf{Cov}\left(X_j, S_j\right)}{b_j^2} < \infty, \qquad \forall \, 0 < r < 2, \tag{14}$$

$$\mathsf{E}\Big(\sup_{n\ge 1}\Big|\frac{S_n}{b_n}\Big|^2\Big) \leqslant 16\sum_{j=1}^{\infty}\frac{\mathsf{E}(S_j^2 - S_{j-1}^2)}{b_j^2} \leqslant 32\sum_{j=1}^{\infty}\frac{\mathsf{Cov}\,(X_j, S_j)}{b_j^2} < \infty,\tag{15}$$

$$\lim_{n \to \infty} \frac{S_n}{b_n} = 0 \qquad \text{a.s.},\tag{16}$$

and with the growth rate

$$\frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) \qquad \text{a.s.},\tag{17}$$

where

$$\beta_n = \max_{1 \le k \le n} b_k v_k^{\delta/2}, \qquad \forall 0 < \delta < 1,$$

$$v_n = \sum_{k=n}^{\infty} \frac{\alpha_k}{b_k^2}, \qquad \lim_{n \to \infty} \frac{\beta_n}{b_n} = 0,$$

$$\alpha_k = 4\mathsf{E}(S_k^2 - S_{k-1}^2) = 8\mathsf{E}(X_k S_{k-1}) + 4\mathsf{E}X_k^2, \qquad k \ge 1.$$
(18)

Proof For $\forall 0 < r < 2$ and t > 0, by Theorem 7,

$$\begin{split} \mathsf{P}\Big(\sup_{n\geqslant 1}\Big|\frac{S_n}{b_n}\Big|^r > t\Big) &\leqslant \lim_{N\to\infty} \mathsf{P}\Big(\max_{1\leqslant n\leqslant N}\Big|\frac{S_n}{b_n}\Big| > t^{1/r}\Big) \\ &\leqslant \lim_{N\to\infty}\frac{1}{t^{2/r}}\Big(\sum_{j=1}^N\frac{\mathsf{E}X_j^2}{b_j^2} + 2\sum_{j=1}^N\frac{\mathsf{Cov}\left(X_j,S_{j-1}\right)}{b_j^2}\Big) \\ &\leqslant \frac{2}{t^{2/r}}\sum_{j=1}^\infty\frac{\mathsf{Cov}\left(X_j,S_j\right)}{b_j^2}, \end{split}$$

thus

$$\begin{split} \mathsf{E}\Big(\sup_{n\geqslant 1}\Big|\frac{S_n}{b_n}\Big|^r\Big) &= \int_0^\infty \mathsf{P}\Big(\sup_{n\geqslant 1}\Big|\frac{S_n}{b_n}\Big|^r > t\Big)\mathrm{d}t\\ &= \int_0^1 \mathsf{P}\Big(\sup_{n\geqslant 1}\Big|\frac{S_n}{b_n}\Big|^r > t\Big)\mathrm{d}t + \int_1^\infty \mathsf{P}\Big(\sup_{n\geqslant 1}\Big|\frac{S_n}{b_n}\Big|^r > t\Big)\mathrm{d}t\\ &\leqslant 1 + 2\sum_{j=1}^\infty \frac{\mathsf{Cov}\,(X_j,S_j)}{b_j^2}\int_1^\infty t^{-2/r}\mathrm{d}t\\ &\leqslant 1 + \frac{2r}{2-r}\sum_{j=1}^\infty \frac{\mathsf{Cov}\,(X_j,S_j)}{b_j^2} < \infty. \end{split}$$

Since the partial sum of mean zero associated random variables is a demimartingale, and

$$\begin{split} \sum_{k=1}^{\infty} \frac{\mathsf{E}(S_k^2 - S_{k-1}^2)}{b_k^2} &= \sum_{k=1}^{\infty} \frac{\mathsf{E}(X_k S_{k-1}) + \mathsf{E} X_k^2}{b_k^2} \\ &\leqslant 2 \sum_{j=1}^{\infty} \frac{\mathsf{Cov}\left(X_j, S_j\right)}{b_j^2} < \infty, \end{split}$$

then (15)-(18) follow from lemma 5 (take q = 1).

Remark 10 For associated sequences, Hu et al. ^[9] obtained the following result (Theorem 2.2 in [9]). No. 2

$$\begin{split} & \mathsf{P}\Big(\max_{1\leqslant k\leqslant n} \Big|\frac{1}{b_k}\sum_{j=1}^k X_j\Big| \geqslant \varepsilon\Big) \leqslant \frac{2}{\varepsilon^2} \Big(\sum_{j=1}^n \frac{\mathsf{E}X_j^2}{b_j^2} + 2\sum_{j=1}^n \frac{\mathsf{Cov}\left(X_j, S_{j-1}\right)}{b_j^2}\Big), \\ & \mathsf{P}\Big(\max_{m\leqslant k\leqslant n} \Big|\frac{1}{b_k}\sum_{j=1}^k X_j\Big| \geqslant \varepsilon\Big) \leqslant \frac{4}{\varepsilon^2} \Big(\frac{1}{b_m^2}\sum_{j=1}^m \mathsf{Cov}\left(X_j, S_j\right) + \sum_{j=m+1}^n \frac{\mathsf{Cov}\left(X_j, S_j\right)}{b_j^2}\Big). \end{split}$$

In Theorem 7 of this article we improve the results as follows: the factors 2, 4 in Theorem 2.2 of [9] are improved to 1, 2, respectively. Under the condition $\sum_{j=1}^{\infty} \text{Cov}(X_j, S_j)/b_j^2 < \infty$, in [9] the author obtained $S_n/b_n \to 0$ a.s. and for 0 < r < 2,

$$\mathsf{E}\Big(\sup_{n\geqslant 1}\Big|\frac{S_n}{b_n}\Big|^r\Big)\leqslant 1+\frac{4r}{2-r}\sum_{j=1}^\infty\frac{\mathsf{Cov}\,(X_j,S_j)}{b_j^2}<\infty,$$

(see Theorem 3.1 in [9]). We get a sharper result in Theorem 9 under the same conditions as that in [9]. Corollary 8 improves the result of Lemma 2.1 in [8].

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PA 序列的 Hájek-Rényi 型不等式及强大数定律

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摘要:本文给出了零均值 PA 序列的一个新的 Hájek-Rényi 型不等式,该不等式推广了文献 [9] 中的结果.
此外,本文还得到了零均值 PA 序列的一个 Brunk-Prokhorov 型强大数定律.
关键词:相协随机变量;弱鞅; Hájek-Rényi 型不等式;强大数定律
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