应用概率统计 第 34 卷
第 4 期 2018 年 8 月

Chinese Journal of Applied Probability and Statistics
Aug., 2018, Vol. 34, No. 4, pp. 331-344
doi: 10.3969/j.issn.1001-4268.2018.04.001

Statistical Inference on Competing Risks Model from Exponentiated Weibull Distribution under Type-I Progressively Hybrid Censoring Data *

ZHANG Chunfang⋆
(*School of Mathematics and Statistics, Xidian University, Xi'an, 710126, China*)
SHI Yimin
(*School of Natural and Applied Sciences, Northwestern Polytechnical University, Xi'an, 710072, China*)
WU Min
(*School of Economics & Management, Shanghai Maritime University, Shanghai, 201306, China*)

Abstract: In this paper, we investigate a competing risks model based on exponentiated Weibull distribution under Type-I progressively hybrid censoring scheme. To estimate the unknown parameters and reliability function, the maximum likelihood estimators and asymptotic confidence intervals are derived. Since Bayesian posterior density functions cannot be given in closed forms, we adopt Markov chain Monte Carlo method to calculate approximate Bayes estimators and highest posterior density credible intervals. To illustrate the estimation methods, a simulation study is carried out with numerical results. It is concluded that the maximum likelihood estimation and Bayesian estimation can be used for statistical inference in competing risks model under Type-I progressively hybrid censoring scheme.

Keywords: progressively hybrid censoring; competing risks; maximum likelihood estimation; Bayesian estimation; Monte Carlo method

2010 Mathematics Subject Classification: 62F10; 62F15; 62N05

Citation: ZHANG C F, SHI Y M, WU M. Statistical inference on competing risks model from exponentiated Weibull distribution under Type-I progressively hybrid censoring data [J]. Chinese J Appl Probab Statist, 2018, 34(4): 331–344.

§1. Introduction

The two-parameter Weibull distribution is popular as the lifetime distribution in life test, but it still has its drawback. Its monotone hazard function is unavailable to

*The project was supported by the National Natural Science Foundation (Grant Nos. 71171164; 71401134; 71571144; 11701406), the Natural Science Basic Research Program of Shaanxi Province (Grant No. 2015JM1003).

⋆Corresponding author, E-mail: cfzhang917@xidian.edu.cn.

Received June 11, 2015. Revised November 13, 2016.

accommodate nonmonotone (especially bathtub shaped) hazard rates which often occur in practical applications. To fit nonmonotone hazard functions, Mudholkar and Srivastava [1] proposed the exponentiated Weibull distribution (EWD) as a generalization of Weibull distribution, which was widely applied in many fields [2]. For example, Ahmad et al. [3–7] put it into use in the accelerated life tests and software reliability.

In fact, a failure product that takes place in a life-testing experiment results from many factors including internal structure of the product and external conditions, like temperature and humidity, and these factors cannot be ignored in analyzing failure data. There has been some literature on statistical inference in the presence of competing risks based on complete failure data [8–10], Type-II progressive censoring [11], and Type-II progressively hybrid censoring [12], Type-I progressively hybrid censoring [13], generalized progressive hybrid censoring [14], adaptive progressively hybrid censoring [15]. However, in the competing risks model, EWD was not considered among these lifetime distributions referred in the references.

Considering the EWD as a lifetime distribution, there are some major references under different censored schemes. Singh et al. gave Bayesian estimates based on squared loss function and LINEX loss function under complete failure data [16], and Type-II censoring data [17,18]. Shi and Hu studied empirical Bayesian estimation of the shape parameter of two-parameter EWD [19]. Kim et al. [20] obtained Bayesian estimation for the exponentiated Weibull model under Type-II progressive censoring data.

Since complete failure data in the life tests are obtained with high expenses and unexpected termination time, the censoring schemes are necessary to be used. For the termination time, Type-I censoring schemes are applicable with the known testing time. Considering failure number, Type-I hybrid censoring schemes cost less time comparing with Type-I censoring schemes. But many times, the surviving tested units are removed for other studies and tests. At this time, Type-I progressively hybrid censoring scheme (PHCS), which was put forward by Childs et al. [21], can be available to meet these needs. So far, there is few literature referred to statistical inference on competing risks from EWD under Type-I PHCS.

In this paper, we mainly study statistical inference under Type-I PHCS in the presence of independent competing risks from EWD. The remainder of this paper is arranged as follows. Section 2 shows the assumptions and likelihood function based on the constructed model. Maximum likelihood estimation and asymptotic confidence intervals of shape parameters are presented in Section 3. In Section 4, Bayesian estimates and highest posterior density credible intervals are obtained by the Markov chain Monte Carlo (M-

CMC) method. Further, numerical results are presented to illustrate our methodology in Section 5. Ultimately, conclusions are given in Section 6.

§2. Model Assumptions and Likelihood Function

In this section, we construct a competing risks model and obtain the likelihood function based on Type-I progressively hybrid censoring data (PHCD).

2.1 Model Description

Suppose that n units are put in the life testing. Type-I PHCS can be described as follows. Failure number m, removal vector $(R_1, R_2, \ldots, R_m)$ and censoring time T_0 are fixed in advance, where $1 \leqslant m \leqslant n$, $T_0 \in (0, \infty)$, $0 \leqslant R_i < n$, $i = 1, 2, \ldots, m$. When the i-th failure takes place, the time is recorded as $t_{i:m:n}$ and R_i units are progressively removed from the remaining survived units. The experiment under Type-I PHCS is terminated at $\min(t_{m:m:n}, T_0)$. There are two cases to be denoted:

Case I : $t_{1:m:n} < t_{2:m:n} < \cdots < t_{D:m:n} < T_0 < \cdots < t_{m:m:n}$, if $T_0 < t_{m:m:n}$;

Case II: $t_{1:m:n} < t_{2:m:n} < \cdots < t_{m:m:n} \leqslant T_0$, if $t_{m:m:n} \leqslant T_0$.

Finally, the observed failure data and removal vector can be given as $(t_{1:m:n}, t_{2:m:n}, \ldots, t_{D:m:n}, T_0)$ and $(R_1, R_2, \ldots, R_D, R^*_{D+1})$, where $1 \leqslant D \leqslant m$, $R^*_{D+1} = n - \sum\limits_{i=1}^{D}(R_i + 1)$. In Case I, $D = 1, 2, \ldots, m-1$. Otherwise, $D = m$ in Case II.

Consider p independent competing risks from EWD in the life testing under Type-I PHCS. To make statistical inference on competing risks model, the basic assumptions are given.

1) There is just one cause leading to the failure in the life testing.

2) Let X_{ij} denote the i-th failure time from type-j failure cause under Type-I PHCS and $X_{ij} \sim \mathrm{EWD}(x; \alpha_j, \theta_j)$, where the shape parameters $\alpha_j > 0$ and $\theta_j > 0$ for $j = 1, 2, \ldots, p$.

3) $X_{i1}, X_{i2}, \ldots, X_{ip}$ are independent with each other. The probability density function (pdf), cumulative distribution function (cdf) and reliability function with shape parameters (α_j, θ_j) of X_{ij} are expressed as, respectively,

$$f_j(x) = \alpha_j \theta_j x^{\alpha_j - 1} \mathrm{e}^{-x^{\alpha_j}} (1 - \mathrm{e}^{-x^{\alpha_j}})^{\theta_j - 1}, \qquad x > 0, \tag{1}$$

$$F_j(x) = (1 - \mathrm{e}^{-x^{\alpha_j}})^{\theta_j}, \qquad x > 0, \tag{2}$$

$$\overline{F}_j(x) = 1 - F_j(x), \qquad x > 0. \tag{3}$$

4) The i-th competing failure time is $T_{i:m:n} = \min(X_{i1}, X_{i2}, \ldots, X_{ip})$ for $i = 1, 2, \ldots, m$ with the indicator vector $\delta_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{ip})$ of the i-th failure time, which is defined as

$$\delta_{ij} = \begin{cases} 1, & \text{if } T_{i:m:n} = X_{ij}; \\ 0, & \text{if } T_{i:m:n} \neq X_{ij}. \end{cases}$$

Based on above assumptions, the reliability function of $T_{i:m:n}$ can be given by

$$R(t) = \prod_{j=1}^{p} [1 - (1 - \mathrm{e}^{-t^{\alpha_j}})^{\theta_j}], \qquad t > 0;\ \alpha_j > 0,\ \theta_j > 0. \tag{4}$$

Finally, the competing risks data under Type-I PHCS can be formed as $\widetilde{T} = \{(T_{i:m:n}, \delta_i), T_0 : i = 1, 2, \ldots, D\}$.

2.2 Likelihood Function

Given above assumptions, the competing risks data $\widetilde{T} = \{(T_{i:m:n}, \delta_i), T_0 : i = 1, 2, \ldots, D\}$, and removal vector $(R_1, R_2, \ldots, R_D, R^*_{D+1})$, the likelihood function of the type-j failure cause can be given as follows

$$l_j(\alpha_j, \theta_j \,|\, \widetilde{t}\,) \propto \Big\{ \prod_{i=1}^{D} f_j(t_i)^{\delta_{ij}} \cdot \overline{F}_j(t_i)^{1-\delta_{ij}} \cdot [\overline{F}_j(t_i)]^{R_i} \Big\} \cdot [\overline{F}_j(t_0)]^{R^*_{D+1}}, \tag{5}$$

where $\widetilde{t} = (\widetilde{t}_1, \widetilde{t}_2, \ldots, \widetilde{t}_D, t_0)$ is a sample of $\widetilde{T}$. By substituting Equations (1), (3) into Equation (5), the log-likelihood function of all the failure causes with parameters $\boldsymbol{\Theta} = (\alpha_1, \theta_1, \alpha_2, \theta_2, \ldots, \alpha_p, \theta_p)$ is proportional to

$$L = \sum_{j=1}^{p} \ln l_j(\alpha_j, \theta_j \,|\, \widetilde{t}\,) \propto \sum_{j=1}^{p} [d_j(\ln \alpha_j + \ln \theta_j) - W_j + V_j], \tag{6}$$

where, for $i = 0, 1, 2, \ldots, D$, $j = 1, 2, \ldots, p$,

$$\begin{aligned} & u_{ij} \triangleq u(t_i, \alpha_j) = 1 - \mathrm{e}^{-t_i^{\alpha_j}}, \qquad d_j = \sum_{i=1}^{D} \delta_{ij}, \\ & W_j \triangleq W(\alpha_j) = \sum_{i=1}^{D} [\delta_{ij} t_i^{\alpha_j} + \delta_{ij} \ln u_{ij} - \delta_{ij}(\alpha_j - 1) \ln t_i], \\ & V_j \triangleq V(\alpha_j, \theta_j) = \sum_{i=1}^{D} [\theta_j \delta_{ij} \ln u_{ij} + (1 - \delta_{ij} + R_i) \ln(1 - u_{ij}^{\theta_j})] + R^*_{D+1} \ln(1 - u_{0j}^{\theta_j}). \end{aligned} \tag{7}$$

Note that W_j and u_{ij} only have the parameter α_j.

§3. Maximum Likelihood Estimation

Under the Type-I progressively hybrid censoring scheme, the maximum likelihood estimates (MLEs) and asymptotic confidence intervals are presented based on competing risks model in this section.

3.1 Point Estimation

Let $\mathbf{\Theta} = (\alpha_1, \theta_1, \alpha_2, \theta_2, \ldots, \alpha_p, \theta_p)$. The likelihood equations are obtained by differentiating (6) with respect to α_j and θ_j for $j = 1, 2, \ldots, p$ and equating the results to zero, which are expressed as

$$\frac{\partial L}{\partial \alpha_j} = \frac{d_j}{\alpha_j} - R^*_{D+1} m_0(\alpha_j, \theta_j) + \sum_{i=1}^{D} [\delta_{ij} x_i(\alpha_j) + \delta_{ij}(\theta_j - 1) y_i(\alpha_j) - (1 - \delta_{ij} + R_i) m_i(\alpha_j, \theta_j)] = 0, \tag{8}$$

$$\frac{\partial L}{\partial \theta_j} = \frac{d_j}{\theta_j} - R^*_{D+1} z_0(\alpha_j, \theta_j) + \sum_{i=1}^{D} [\delta_{ij} \ln u_{ij} - (1 - \delta_{ij} + R_i) z_i(\alpha_j, \theta_j)] = 0, \tag{9}$$

where $i = 0, 1, 2, \ldots, D$,

$$x_i(\alpha_j) = (1 - t_i^{\alpha_j}) \ln t_i, \qquad y_i(\alpha_j) = t_i^{\alpha_j} u_{ij}^{-1} (1 - u_{ij}) \ln t_i,$$

$$z_i(\alpha_j, \theta_j) = u_{ij}^{\theta_j} (1 - u_{ij}^{\theta_j})^{-1} \ln u_{ij}, \qquad m_i(\alpha_j, \theta_j) = \frac{\theta_j t_i^{\alpha_j} u_{ij}^{\theta_j - 1} (1 - u_{ij}) \ln t_i}{1 - u_{ij}^{\theta_j}}.$$

The MLEs $\widehat{\alpha}_{jM}$ and $\widehat{\theta}_{jM}$ of the parameters α_j and θ_j can be computed by solving Equations (8) and (9). As the closed forms for $\widehat{\alpha}_{jM}$ and $\widehat{\theta}_{jM}$ cannot be given, the Newton–Raphson method is employed to obtain MLEs $\widehat{\mathbf{\Theta}}_M = (\widehat{\alpha}_{1M}, \widehat{\theta}_{1M}, \widehat{\alpha}_{2M}, \widehat{\theta}_{2M}, \ldots, \widehat{\alpha}_{pM}, \widehat{\theta}_{pM})$. The MLE $\widehat{R}_M$ of the reliability $R(t)$ in Equation (4) can be given by

$$\widehat{R}_M \triangleq R(t; \widehat{\alpha}_{jM}, \widehat{\theta}_{jM}) = \prod_{j=1}^{p} [1 - (1 - \mathrm{e}^{-t^{\widehat{\alpha}_{jM}}})^{\widehat{\theta}_{jM}}].$$

3.2 Interval Estimation

The confidence intervals are developed based on the asymptotic normal distribution of the MLEs $\widehat{\mathbf{\Theta}}_M$. The asymptotic distribution is given by

$$\widehat{\mathbf{\Theta}}_M - \mathbf{\Theta} \to \mathrm{N}(\mathbf{0}, I^{-1}(\mathbf{\Theta})).$$

The inverse Fisher information matrix $I^{-1}(\mathbf{\Theta})$ is the asymptotic variance-covariance matrix of the MLEs for the parameters $\mathbf{\Theta}$. Based on the independence among competing failure causes, the inverse Fisher information matrix is expressed as

$$I^{-1}(\boldsymbol{\Theta})=\begin{bmatrix} I_1^{-1}(\alpha_1,\theta_1) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & I_2^{-1}(\alpha_2,\theta_2) & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & I_p^{-1}(\alpha_p,\theta_p) \end{bmatrix},$$

where

$$I_j^{-1}(\alpha_j,\theta_j)=\begin{bmatrix} -\mathsf{E}\Big[\dfrac{\partial^2 L(\boldsymbol{\Theta}\,|\,\widetilde{t}\,)}{\partial\alpha_j^2}\Big] & -\mathsf{E}\Big[\dfrac{\partial^2 L(\boldsymbol{\Theta}\,|\,\widetilde{t}\,)}{\partial\alpha_j\partial\theta_j}\Big] \\ -\mathsf{E}\Big[\dfrac{\partial^2 L(\boldsymbol{\Theta}\,|\,\widetilde{t}\,)}{\partial\theta_j\partial\alpha_j}\Big] & -\mathsf{E}\Big[\dfrac{\partial^2 L(\boldsymbol{\Theta}\,|\,\widetilde{t}\,)}{\partial\theta_j^2}\Big] \end{bmatrix}, \qquad j=1,2,\ldots,p. \tag{10}$$

In fact, the explicit expressions for the expectation (10) cannot be given in the closed forms. Then, the approximate asymptotic variance-covariance matrix of $I_j^{-1}(\alpha_j,\theta_j)$ can be given by

$$\widehat{I}_j^{-1}(\alpha_j,\theta_j)=\begin{bmatrix} -\dfrac{\partial^2 L(\boldsymbol{\Theta}\,|\,\widetilde{t}\,)}{\partial\alpha_j^2} & -\dfrac{\partial^2 L(\boldsymbol{\Theta}\,|\,\widetilde{t}\,)}{\partial\alpha_j\partial\theta_j} \\ -\dfrac{\partial^2 L(\boldsymbol{\Theta}\,|\,\widetilde{t}\,)}{\partial\theta_j\partial\alpha_j} & -\dfrac{\partial^2 L(\boldsymbol{\Theta}\,|\,\widetilde{t}\,)}{\partial\theta_j^2} \end{bmatrix}^{-1} \downarrow(\widehat{\alpha}_{jM},\widehat{\theta}_{jM})=\begin{bmatrix} \widehat{V}_{j_{11}} & \widehat{V}_{j_{12}} \\ \widehat{V}_{j_{21}} & \widehat{V}_{j_{22}} \end{bmatrix}.$$

The second partial derivatives of the log-likelihood function with respect to α_j and θ_j in Equation (10) are given as follows:

$$\begin{aligned}
\frac{\partial^2 L(\boldsymbol{\Theta}\,|\,\widetilde{t}\,)}{\partial\alpha_j^2} &= -\frac{d_j}{\alpha_j^2}+\sum_{i=1}^{D}(\ln t_i)^2(1-u_{ij})t_i^{\alpha_j}[(\theta_j-1)(u_{ij}^{-1}-t_i^{\alpha_j}u_{ij}^{-2})-\delta_{ij}(1-u_{ij})^{-1}] \\
&\quad -\sum_{i=0}^{D}\alpha_j t_i^{\alpha_j}(\ln t_i)^2 u_{ij}^{\theta_j-1}(1-u_{ij}^{\theta_j})^{-1}(1-u_{ij}) \\
&\quad \times[(1-\delta_{ij}+R_i)I(i\neq 0)+R_{D+1}^{*}I(i=0)] \\
&\quad \times[1+(\theta_j-1)t_i^{\alpha_j}(1-u_{ij})+\theta_j t_i^{\alpha_j}u_{ij}^{\theta_j-1}(1-u_{ij}^{\theta_j})^{-1}(1-u_{ij})-t_i^{\alpha_j}], \\
\frac{\partial^2 L(\boldsymbol{\Theta}\,|\,\widetilde{t}\,)}{\partial\theta_j^2} &= -\frac{d_j}{\theta_j^2}-\sum_{i=0}^{D}[(1-\delta_{ij}+R_i)I(i\neq 0)+R_{D+1}^{*}I(i=0)](1-u_{ij}^{\theta_j})^{-2}u_{ij}(\ln u_{ij})^2, \\
\frac{\partial^2 L(\boldsymbol{\Theta}\,|\,\widetilde{t}\,)}{\partial\alpha_j\partial\theta_j} &= \sum_{i=1}^{D}\delta_{ij}(1-u_{ij})u_{ij}^{-1}t_i^{\alpha_j}\ln t_i-\sum_{i=0}^{D}t_i^{\alpha_j}\ln t_i(1-u_{ij})u_{ij}^{\theta_j-1}(1-u_{ij}^{\theta_j})^{-1} \\
&\quad \times[(1-\delta_{ij}+R_i)I(i\neq 0)+R_{D+1}^{*}I(i=0)][1-\theta_j\ln u_{ij}(1-u_{ij}^{\theta_j})^{-1}].
\end{aligned}$$

The approximate $100(1-\gamma)\%$ confidence intervals for the parameters α_j and θ_j are, respectively, for $j=1,2,\ldots,p$, expressed as,

$$\Big(\widehat{\alpha}_{jM}-\mu_{\gamma/2}\sqrt{\widehat{V}_{j_{11}}},\ \widehat{\alpha}_{jM}+\mu_{\gamma/2}\sqrt{\widehat{V}_{j_{11}}}\Big), \qquad \Big(\widehat{\theta}_{jM}-\mu_{\gamma/2}\sqrt{\widehat{V}_{j_{22}}},\ \widehat{\theta}_{jM}+\mu_{\gamma/2}\sqrt{\widehat{V}_{j_{22}}}\Big)$$

where $\mu_{\gamma/2}$ is the upper $(\gamma/2)$th percentile of the standard normal distribution.

Discussing the monotone property of the reliability $R(t;\mathbf{\Theta})$, for $j=1,2,\ldots,p$, we have

$$\frac{\partial R(t;\mathbf{\Theta})}{\partial\theta_j}=-(1-\mathrm{e}^{-t_j^{\alpha}})^{\theta_j}\ln(1-\mathrm{e}^{-t^{\alpha_j}})>0$$

and

$$\frac{\partial R(t;\mathbf{\Theta})}{\partial\alpha_j}=-\theta_j(1-\mathrm{e}^{-t^{\alpha_j}})^{\theta_j-1}\mathrm{e}^{-t^{\alpha_j}}t^{\alpha_j}\ln t,$$

which satisfies

$$\begin{cases}\dfrac{\partial R(t;\mathbf{\Theta})}{\partial\alpha_j}>0, & 0<t<1;\\ \dfrac{\partial R(t;\mathbf{\Theta})}{\partial\alpha_j}\leqslant 0, & t\geqslant 1.\end{cases}$$

Therefore, the approximate $100(1-\gamma)\%$ confidence intervals for the reliability $R(t)$ is given by

$$\begin{cases}\Big(R\Big(t;\widehat{\alpha}_{jM}+\mu_{\gamma/2}\sqrt{\widehat{V}_{j11}},\widehat{\theta}_{jM}-\mu_{\gamma/2}\sqrt{\widehat{V}_{j22}}\Big),R\Big(t;\widehat{\alpha}_{jM}-\mu_{\gamma/2}\sqrt{\widehat{V}_{j11}},\widehat{\theta}_{jM}+\mu_{\gamma/2}\sqrt{\widehat{V}_{j22}}\Big)\Big),\\ \hfill t\geqslant 1;\\ \Big(R\Big(t;\widehat{\alpha}_{jM}-\mu_{\gamma/2}\sqrt{\widehat{V}_{j11}},\widehat{\theta}_{jM}-\mu_{\gamma/2}\sqrt{\widehat{V}_{j22}}\Big),R\Big(t;\widehat{\alpha}_{jM}+\mu_{\gamma/2}\sqrt{\widehat{V}_{j11}},\widehat{\theta}_{jM}+\mu_{\gamma/2}\sqrt{\widehat{V}_{j22}}\Big)\Big),\\ \hfill \text{others.}\end{cases}$$

§4. Bayesian Inference

Bayesian estimation is presented based on the squared loss function in this section. Since the bivariate parameters (α_j,θ_j), $j=1,2,\ldots,p$ are independent with each other, we choose the bivariate priors of (α_j,θ_j) with the following form

$$\pi_j(\alpha_j,\theta_j)=\pi_{j1}(\theta_j\,|\,\alpha_j)\pi_{j2}(\alpha_j),\qquad \alpha_j>0,\ \theta_j>0. \tag{11}$$

Therefore, the joint prior density function of $\mathbf{\Theta}=(\alpha_1,\theta_1,\alpha_2,\theta_2,\ldots,\alpha_p,\theta_p)$ is formed as

$$\pi(\mathbf{\Theta})=\prod_{j=1}^{p}\pi_j(\alpha_j,\theta_j).$$

From Equations (5) and (11), the joint posterior density function is proportional to

$$\pi(\mathbf{\Theta}\,|\,\widetilde{t})\propto\prod_{j=1}^{p}\alpha_j^{d_j}\theta_j^{d_j}\exp(-W_j+V_j)\cdot\pi_j(\alpha_j,\theta_j), \tag{12}$$

where d_j, W_j and V_j for $j=1,2,\ldots,p$ are given in Equation (7). The joint posterior density function (12) indicates that the bivariate parameters (α_j,θ_j) are independent with each other after considering the prior information and the sample information from the density function of type-j failure cause. Thus, we have

$$\pi(\alpha_j,\theta_j\,|\,\widetilde{t}\,)\propto\alpha_j^{d_j}\theta_j^{d_j}\exp(-W_j+V_j)\cdot\pi_j(\alpha_j,\theta_j). \tag{13}$$

As for $\pi_j(\alpha_j,\theta_j)$ in Equation (13), Jaheen and Harbi [22] suggested a bivariate prior to obtain Bayesian estimates of the parameters for the exponentiated Weibull model based on the generalized order statistics from the Type-II progressive censoring samples. The bivariate prior was initially chosen by Nassar and Eissa [23] and applied to disscuss Bayesian estimation by Kim et al. [20], where the prior of θ_j was the gamma prior conditionally when α_j was known and the prior of α_j was the exponential prior. Jaheen and Harbi [22] explained that the prior belief of the experimenters could be covered by the gamma prior, then they took the gamma prior $\pi_{j2}(\alpha_j)$ instead of the exponential prior of α_j.

According to above discussions, the prior distributions in the competing risks model are taken as follows

$$\begin{aligned}\pi_{j1}(\theta_j\,|\,\alpha_j)&=\frac{\alpha_j^{-v}}{\Gamma(v)}\theta_j^{v-1}\mathrm{e}^{-\theta_j/\alpha_j},\qquad \theta_j>0,\\ \pi_{j2}(\alpha_j)&=\frac{b^{-d}}{\Gamma(d)}\alpha_j^{d-1}\mathrm{e}^{-\alpha_j/b},\qquad \alpha_j>0,\end{aligned}$$

where v, b and d are assumed to be known. Therefore, the joint prior of α_j and θ_j can be expressed as

$$\pi_j(\alpha_j,\theta_j)=\frac{b^{-d}}{\Gamma(v)\Gamma(d)}\theta_j^{v-1}\alpha_j^{d-v-1}\exp\Big[-\Big(\frac{\theta_j}{\alpha_j}+\frac{\alpha_j}{b}\Big)\Big],\qquad \theta_j>0,\ \alpha_j>0. \tag{14}$$

By substituting Equation (14) into Equation (13), we get

$$\pi(\alpha_j,\theta_j\,|\,\widetilde{t}\,)\propto\alpha_j^{d_j+d-v-1}\theta_j^{d_j+v-1}\exp\Big[-\Big(W_j-V_j+\frac{\theta_j}{\alpha_j}+\frac{\alpha_j}{b}\Big)\Big].$$

Under the squared loss function, the Bayesian estimate of a function $g_j=g(\alpha_j,\theta_j)$, for $j=1,2,\ldots,p$, can be given by

$$\widehat{g}_{jB}=\mathsf{E}_{\alpha_j,\theta_j\,|\,\widetilde{t}}[g(\alpha_j,\theta_j)]=\frac{\displaystyle\int_0^\infty\int_0^\infty g(\alpha_j,\theta_j)\pi(\alpha_j,\theta_j\,|\,\widetilde{t}\,)\mathrm{d}\alpha_j\mathrm{d}\theta_j}{\displaystyle\int_0^\infty\int_0^\infty \pi(\alpha_j,\theta_j\,|\,\widetilde{t}\,)\mathrm{d}\alpha_j\mathrm{d}\theta_j}. \tag{15}$$

Note that the Bayes estimate of $g_j=g(\alpha_j,\theta_j)$ in Equation (15) cannot be computed with closed forms. Therefore, we employ MCMC method, which was introduced by Chen and Shao [24], to obtain the Bayes estimates (BEs) and highest posterior density (HPD) credible intervals of unknown parameters. The main steps of computation are given as follows.

Step 1: Set $i=1$ and an initial value $\boldsymbol{\omega}^{(i)}=(\alpha_j^{(i)},\theta_j^{(i)})$;

Step 2: Let $i=i+1$. Generate a proposal $\boldsymbol{\omega}^*$ from a bivariate proposal distribution $q(\boldsymbol{\omega}\,|\,\boldsymbol{\omega}^{(i-1)})$ and a u from a Uniform$(0,1)$ distribution. Evaluate the acceptance probability

$$\beta=\min\Big(1,\frac{\pi(\boldsymbol{\omega}^*\,|\,\widetilde{t}\,)}{\pi(\boldsymbol{\omega}^{i-1}\,|\,\widetilde{t}\,)}\frac{q(\boldsymbol{\omega}^{i-1}\,|\,\boldsymbol{\omega}^*)}{q(\boldsymbol{\omega}^*\,|\,\boldsymbol{\omega}^{i-1})}\Big).$$

If $u\leqslant\beta$, accept the proposal $\boldsymbol{\omega}^i=\boldsymbol{\omega}^*$. Otherwise, $\boldsymbol{\omega}^i=\boldsymbol{\omega}^{i-1}$;

Step 3: Repeat Step 2 until $i=M$. Obtain $(\alpha_j^{(1)},\theta_j^{(1)}),(\alpha_j^{(2)},\theta_j^{(2)}),\ldots,(\alpha_j^{(M)},\theta_j^{(M)})$ and $\{g_j^{(i)}=g(\alpha_j^{(i)},\theta_j^{(i)}),\ i=1,2,\ldots,M\}$;

Step 4: The Bayes estimates $\widehat{\alpha}_{jB}$, $\widehat{\theta}_{jB}$ and $\widehat{g}_{jB}$ can be approximated by

$$\widehat{\alpha}_{jB}=\frac{\sum\limits_{i=M_0}^{M}\alpha_j^{(i)}}{M-M_0},\qquad \widehat{\theta}_{jB}=\frac{\sum\limits_{i=M_0}^{M}\theta_j^{(i)}}{M-M_0},\qquad \widehat{g}_{jB}=\frac{\sum\limits_{i=M_0}^{M}g_j^{(i)}}{M-M_0},$$

and their posterior variances are, respectively,

$$\frac{\sum\limits_{i=M_0}^{M}(\alpha_j^{(i)}-\widehat{\alpha}_{jB})^2}{M-M_0},\qquad \frac{\sum\limits_{i=M_0}^{M}(\theta_j^{(i)}-\widehat{\theta}_{jB})^2}{M-M_0},\qquad \frac{\sum\limits_{i=M_0}^{M}(g_j^{(i)}-\widehat{g}_{jB})^2}{M-M_0},$$

where M_0 is the burn-in period;

Step 5: Order $g_j^{(i)}$, $M_0\leqslant i\leqslant M$, that is,

$$g_j^{(M_0+1)}\leqslant g_j^{(M_0+2)}\leqslant\cdots\leqslant g_j^{(M)}.$$

The $100(1-\gamma)\%$ HPD credible intervals of g_j is given by $(g_j^{(i^*)},g_j^{(i^*+(1-\gamma)M)})$, where i^* satisfies

$$g_j^{(i^*+(1-\gamma)M)}-g_j^{(i^*)}=\min_{M_0\leqslant i\leqslant\gamma M}(g_j^{(i+(1-\gamma)M)}-g_j^{(i)}).$$

Similarly, we can obtain HPD credible intervals of α_j and θ_j.

When $g_j=\overline{F}_j(t;\alpha_j,\theta_j)$, the Bayes estimates of $\overline{F}_j(t;\alpha_j,\theta_j)$ is $\widehat{g}_{jB}$. Thus, the reliability $R(t)$ in Equation (4) can be estimated by $\widehat{R}_M=\prod\limits_{j=1}^{p}\widehat{g}_{jB}$.

§5. Numerical Analysis

In this section, we present the numerical results of MLEs and BEs of the shape parameters and reliability function based on the competing risks model under Type-I PHCS.

Meanwhile, mean squared errors (MSEs) and interval lengths (ILs) of 95% asymptotic confidence intervals and HPD credible intervals are shown to discuss the performance of the MLEs and BEs.

Consider two competing risks, that is, $p = 2$. Set the values of shape parameters $(\alpha_1, \theta_1, \alpha_2, \theta_2) = (2.0,\ 3.0,\ 3.0,\ 2.0)$, and the values of the hyper-parameters from prior density functions $(d, v, b) = (3.0,\ 2.0,\ 1.0)$. Removal vector $(R_1, R_2, \ldots, R_m)$ in Type-I PHCS satisfies that $R_1 = R_2 = \cdots = R_{m-1} = [n/m] - 1$ and $R_m = n - m - \sum\limits_{i=1}^{m-1} R_i$, where $\{[x], x > 0\}$ presents the lower nearest integer of x. Using MCMC method to compute BEs and HPD credible intervals, the sample size M of Markov chain is equal to 5 000 with the burn-in period $M_0 = 500$. The numerical results of MLEs and BEs for parameters and reliability function at $t = 0.5$ are reported in Tables 1−4.

Tabel 1 MLEs, MSEs (between brackets) and ILs of shape parameters

$(n, m/n)$	Parameters	$T_0 = 1.0$	$T_0 = 1.4$	$T_0 = 2.0$
$(30, 0.2)$	α_1	2.3683 (0.1590), 4.0939	2.3889 (0.2019), 4.1494	2.5433 (0.3356), 4.4651
	α_2	4.0852 (1.6561), 5.7009	4.1578 (1.7120), 5.7109	3.9204 (1.2750), 5.3980
	θ_1	2.1401 (0.7393), 3.1526	2.1446 (0.7315), 3.2894	2.1360 (0.7360), 3.2880
	θ_2	1.6210 (0.1436), 2.5080	1.5710 (0.1840), 2.4539	1.5615 (0.1853), 2.4815
$(45, 0.2)$	α_1	2.4071 (0.1927), 3.5103	2.4124 (0.2237), 3.5465	2.6226 (0.4391), 4.0189
	α_2	4.2069 (1.6506), 4.8467	4.1785 (1.8456), 4.7850	4.2981 (2.0111), 4.9307
	θ_1	2.2501 (0.5622), 2.8325	2.1814 (0.6699), 2.8616	2.2367 (0.5825), 2.8792
	θ_2	1.5649 (0.1892), 1.9453	1.5932 (0.1654), 2.0684	1.5590 (0.1944), 1.9886
$(60, 0.2)$	α_1	2.2056 (0.0510), 2.7600	2.7747 (0.7305), 4.0681	2.1933 (0.0374), 2.7966
	α_2	3.8448 (0.9332), 3.6121	4.0027 (1.0055), 3.8058	4.6396 (3.0962), 4.6405
	θ_1	2.2436 (0.5721), 2.4403	2.4396 (0.3140), 2.9763	2.1766 (0.6780), 2.4817
	θ_2	1.5272 (0.2235), 1.6017	1.4221 (0.3113), 1.4974	1.6150 (0.1482), 1.7956
$(60, 0.4)$	α_1	2.1093 (0.0167), 3.0461	1.5778 (0.1783), 2.2613	1.6728 (0.1070), 2.3680
	α_2	4.0419 (1.2812), 3.7121	4.8172 (3.4122), 3.8257	4.3952 (2.0312), 3.4120
	θ_1	1.9567 (1.0886), 2.4946	1.3078 (2.5670), 1.8400	1.3110 (2.3685), 1.8644
	θ_2	1.3604 (0.4090), 1.3792	1.2582 (0.5502), 1.1891	1.2218 (0.6057), 1.1485
$(100, 0.4)$	α_1	1.9371 (0.0018), 2.3596	1.5415 (0.2103), 1.8731	1.4607 (0.2908), 1.7930
	α_2	3.9486 (1.0186), 2.5894	4.2682 (1.6083), 2.5265	4.3961 (1.9491), 2.5977
	θ_1	2.0832 (0.8404), 2.1557	1.3415 (2.3670), 1.5609	1.3061 (2.3836), 1.5055
	θ_2	1.3141 (0.4704), 0.9697	1.1782 (0.6754), 0.8492	1.2055 (0.6313), 0.8585

In Tables 1−2, we can find that MSEs of MLEs and BEs decrease as n increases for fixed m/n and T_0. The ILs of asymptotic confidence intervals and HPD credible intervals

Tabel 2 BEs, MSEs (between brackets) and ILs of shape parameters

$(n, m/n)$	Parameters	$T_0 = 1.0$	$T_0 = 1.4$	$T_0 = 2.0$
$(30, 0.2)$	α_1	$2.3124\,(0.1242), 2.8446$	$2.3221\,(0.1522), 2.8086$	$2.5077\,(0.3084), 2.9966$
	α_2	$3.9842\,(1.4622), 4.3061$	$4.0626\,(1.4996), 4.3830$	$3.8599\,(1.1539), 4.1868$
	θ_1	$2.2110\,(0.6225), 2.4967$	$2.1884\,(0.6586), 2.4825$	$2.1915\,(0.6441), 2.4963$
	θ_2	$1.7357\,(0.0699), 2.2706$	$1.6902\,(0.0960), 2.1650$	$1.6625\,(0.1113), 2.1302$
$(45, 0.2)$	α_1	$2.2613\,(0.0859), 2.7884$	$2.2390\,(0.1018), 2.7528$	$2.4987\,(0.2902), 2.9045$
	α_2	$3.9525\,(1.0977), 4.3715$	$4.0256\,(1.4534), 4.3570$	$4.0984\,(1.4844), 4.4942$
	θ_1	$2.2628\,(0.5435), 2.5887$	$2.1530\,(0.7174), 2.3563$	$2.2600\,(0.5477), 2.5614$
	θ_2	$1.6187\,(0.1454), 2.0417$	$1.6438\,(0.1269), 2.1563$	$1.5883\,(0.1695), 1.9865$
$(60, 0.2)$	α_1	$2.2056\,(0.0510), 2.7600$	$2.7747\,(0.7305), 4.0681$	$2.1933\,(0.0374), 2.7966$
	α_2	$3.8448\,(0.9332), 3.6121$	$4.0027\,(1.0055), 3.8058$	$4.6396\,(3.0962), 4.6405$
	θ_1	$2.2436\,(0.5721), 2.4403$	$2.4396\,(0.3140), 2.9763$	$2.1766\,(0.6780), 2.4817$
	θ_2	$1.5272\,(0.2235), 1.6017$	$1.4221\,(0.3113), 1.4974$	$1.6150\,(0.1482), 1.7956$
$(60, 0.4)$	α_1	$2.1093\,(0.0167), 3.0461$	$1.5778\,(0.1783), 2.2613$	$1.6728\,(0.1070), 2.3680$
	α_2	$4.0419\,(1.2812), 3.7121$	$4.8172\,(3.4122), 3.8257$	$4.3952\,(2.0312), 3.4120$
	θ_1	$1.9567\,(1.0886), 2.4946$	$1.3078\,(2.5670), 1.8400$	$1.3110\,(2.3685), 1.8644$
	θ_2	$1.3604\,(0.4090), 1.3792$	$1.2582\,(0.5502), 1.1891$	$1.2218\,(0.6057), 1.1485$
$(100, 0.4)$	α_1	$1.9371\,(0.0018), 2.3596$	$1.5415\,(0.2103), 1.8731$	$1.4607\,(0.2908), 1.7930$
	α_2	$3.9486\,(1.0186), 2.5894$	$4.2682\,(1.6083), 2.5265$	$4.3961\,(1.9491), 2.5977$
	θ_1	$2.0832\,(0.8404), 2.1557$	$1.3415\,(2.3670), 1.5609$	$1.3061\,(2.3836), 1.5055$
	θ_2	$1.3141\,(0.4704), 0.9697$	$1.1782\,(0.6754), 0.8492$	$1.2055\,(0.6313), 0.8585$

Tabel 3 MLEs, MSEs (between brackets) and ILs of reliability function at $t = 0.5$

$(n, m/n)$	Reliability	$T_0 = 1.0$	$T_0 = 1.4$	$T_0 = 2.0$
$(30, 0.2)$	$R(0.5)$	$0.8764\,(0.0112), 0.8859$	$0.8638\,(0.7462), 0.8978$	$0.8595\,(0.7387), 0.8965$
$(45, 0.2)$	$R(0.5)$	$0.8870\,(0.0136), 0.7326$	$0.8733\,(0.0106), 0.7563$	$0.8731\,(0.0105), 0.7535$
$(60, 0.2)$	$R(0.5)$	$0.8951\,(0.0155), 0.6100$	$0.8750\,(0.0109), 0.6489$	$0.8861\,(0.0134), 0.6160$
$(60, 0.4)$	$R(0.5)$	$0.8357\,(0.0043), 0.7245$	$0.6971\,(0.0054), 0.7540$	$0.6801\,(0.0082), 0.7367$
$(100, 0.4)$	$R(0.5)$	$0.8479\,(0.0060), 0.5478$	$0.6889\,(0.0066), 0.6341$	$0.6757\,(0.0090), 0.6207$

Tabel 4 BEs, MSEs (between brackets) and ILs of reliability function at $t = 0.5$

$(n, m/n)$	Reliability	$T_0 = 1.0$	$T_0 = 1.4$	$T_0 = 2.0$
$(30, 0.2)$	$R(0.5)$	$0.8865\,(0.0135), 0.0973$	$0.8722\,(0.0104), 0.0937$	$0.8706\,(0.0100), 0.0970$
$(45, 0.2)$	$R(0.5)$	$0.8796\,(0.0119), 0.0948$	$0.8683\,(0.0096), 0.0924$	$0.8674\,(0.0094), 0.0884$
$(60, 0.2)$	$R(0.5)$	$0.8787\,(0.0117), 0.1014$	$0.8530\,(0.0068), 0.0806$	$0.8652\,(0.0090), 0.0876$
$(60, 0.4)$	$R(0.5)$	$0.8145\,(0.0019), 0.0882$	$0.6809\,(0.0080), 0.0943$	$0.6605\,(0.0121), 0.0964$
$(100, 0.4)$	$R(0.5)$	$0.8017\,(0.0010), 0.0640$	$0.6479\,(0.0150), 0.0679$	$0.6422\,(0.0164), 0.0589$

become smaller when n increases. From Tables $3-4$, it can be found that ILs decrease with the increasing n for fixed m/n and T_0 except for some cases. As the estimates of parameters and reliability index are obtained under Type-I PHCS, the final termination time is flexible. Thus, these cases in Tables $3-4$ are admissible. In summary, the maximum likelihood estimation and Bayesian estimation have a good performance.

§6. Conclusion

We construct an exponentiated-Weibull competing risks model based on Type-I progressively hybrid censoring scheme in this paper. Maximum likelihood estimation and Bayesian estimation are discussed to analyze our model. By simulation study, it is concluded that the proposed model is available to analyze the failure data with nonmonotone hazard rate. The maximum likelihood estimation and Bayesian estimation have a good performance. In future work, the model can be applied into accelerated life tests for products with high reliability.

References

[1] MUDHOLKAR G S, SRIVASTAVA D K. Exponentiated Weibull family for analyzing bathtub failure-rate data [J]. *IEEE Trans Reliab*, 1993, **42(2)**: 299–302.

[2] NADARAJAH S, CORDEIRO G M, ORTEGA E M M. The exponentiated Weibull distribution: a survey [J]. *Statist Papers*, 2013, **54(3)**: 839–877.

[3] AHMAD N, ISLAM A, SALAM A. Analysis of optimal accelerated life test plans for periodic inspection: the case of exponentiated Weibull failure model [J]. *Int J Qual Reliab Manag*, 2006, **23(8)**: 1019–1046.

[4] AHMAD N, BOKHARI M U, QUADRI S M K, et al. The exponentiated Weibull software reliability growth model with various testing-efforts and optimal release policy: a performance analysis [J]. *Int J Qual Reliab Manag*, 2008, **25(2)**: 211–235.

[5] AHMAD N, KHAN M G M, QUADRI S M K, et al. Modelling and analysis of software reliability with Burr type X testing-effort and release-time determination [J]. *J Model Manag*, 2009, **4(1)**: 28–54.

[6] AHMAD N, KHAN M G M, RAFI L S. A study of testing-effort dependent inflection S-shaped software reliability growth models with imperfect debugging [J]. *Int J Qual Reliab Manag*, 2010, **27(1)**: 89–110.

[7] AHMAD N, QUADRI S M K, MOHD R. Comparison of predictive capability of software reliability growth models with exponentiated Weibull distribution [J]. *Int J Comput Appl*, 2011, **15(6)**: 40–43.

[8] BADARINATHI R, TIWARI R C. Hierarchical Bayesian approach to reliability estimation under competing risk [J]. *Microelectron Reliab*, 1992, **32(1-2)**: 249–258.

[9] WANG C P, GHOSH M. Bayesian analysis of bivariate competing risks models [J]. *Sankhyā Ser B*, 2000, **62(3)**: 388–401.

[10] HU W H, LI G, LI N. A Bayesian approach to joint analysis of longitudinal measurements and competing risks failure time data [J]. *Stat Med*, 2009, **28(11)**: 1601–1619.

[11] KUNDU D, PRADHAN B. Bayesian analysis of progressively censored competing risks data [J]. *Sankhya B*, 2011, **73(2)**: 276–296.

[12] KUNDU D, JOARDER A. Analysis of type-II progressively hybrid censored competing risks data [J]. *JMASM*, 2006, **5(1)**: 152–170.

[13] ZHANG C F, SHI Y M, WU M. Statistical inference for competing risks model in step-stress partially accelerated life tests with progressively type-I hybrid censored Weibull life data [J]. *J Comput Appl Math*, 2016, **297**: 65–74.

[14] ZHANG C F, SHI Y M. Statistical prediction of failure times under generalized progressive hybrid censoring in a simple step-stress accelerated competing risks model [J]. *J Syst Eng Electron*, 2017, **28(2)**: 282–291.

[15] ZHANG C F, SHI Y M, BAI X C, et al. Inference for constant-stress accelerated life tests with dependent competing risks from bivariate Birnbaum-Saunders distribution based on adaptive progressively hybrid censoring [J]. *IEEE Trans Reliab*, 2017, **66(1)**: 111–122.

[16] SINGH U, GUPTA P K, UPADHYAY S K. Estimation of exponentiated Weibull shape parameters under linex loss function [J]. *Comm Statist Simulation Comput*, 2002, **31(4)**: 523–537.

[17] SINGH U, GUPTA P K, UPADHYAY S K. Estimation of three-parameter exponentiated-Weibull distribution under type-II censoring [J]. *J Statist Plann Inference*, 2005, **134(2)**: 350–372.

[18] SINGH U, GUPTA P K, UPADHYAY S K. Estimation of parameters for exponentiated-Weibull family under type-II censoring scheme [J]. *Comput Statist Data Anal*, 2005, **48(3)**: 509–523.

[19] SHI J H, WU H X. Empirical Bayes estimation of the shape parameter of two-parameter exponentiated Weibull distribution [J]. *Math Pract Theory*, 2009, **39(3)**: 201–208. (in Chinese)

[20] KIM C, JUNG J, CHUNG Y. Bayesian estimation for the exponentiated Weibull model under type II progressive censoring [J]. *Statist Papers*, 2011, **52(1)**: 53–70.

[21] CHILDS A, CHANDRASEKAR B, BALAKRISHNAN N. Exact likelihood inference for an exponential parameter under progressive hybrid censoring schemes [M] // VONTA F, NIKULIN M, LIMNIOS N, et al. (eds) *Statistical Models and Methods for Biomedical and Technical Systems*. Boston: Birkhäuser, 2008: 319–330.

[22] JAHEEN Z F, HARBI M M A. Bayesian estimation for the exponentiated Weibull model via Markov chain Monte Carlo simulation [J]. *Comm Statist Simulation Comput*, 2011, **40(4)**: 532–543.

[23] NASSAR M M, EISSA F H. Bayesian estimation for the exponentiated Weibull model [J]. *Comm Statist Theory Methods*, 2005, **33(10)**: 2343–2362.

[24] CHEN M H, SHAO Q M. Monte Carlo estimation of Bayesian credible and HPD intervals [J]. *J Comput Graph Statist*, 1999, **8(1)**: 69–92.

逐步 I 型混合截尾下指数–威布尔分布竞争失效模型的统计分析

张春芳
(西安电子科技大学数学与统计学院, 西安, 710126)

师义民
(西北工业大学理学院, 西安, 710072)

吴 敏
(上海海事大学经济管理学院, 上海, 201306)

摘 要: 本文基于指数–威布尔分布研究逐步 I 型混合截尾竞争失效模型的统计推断问题. 根据模型假设和竞争失效数据, 推导出未知参数和产品可靠度的极大似然估计; 考虑极大似然估计的渐近正态性质, 计算出观测 Fisher 信息阵, 从而获得未知参数和可靠度的渐近置信区间. 由于贝叶斯后验密度函数不具有封闭形式, 利用 MCMC 方法给出未知参数和可靠度的近似贝叶斯估计以及最大后验密度可信区间. 最后通过模拟研究对估计方法作出解释并给出数值结果. 结果表明极大似然方法和贝叶斯方法可以对逐步 I 型混合截尾竞争失效模型进行统计推断.

关键词: 逐步混合截尾; 竞争失效; 最大似然估计; 贝叶斯估计; 蒙特卡洛方法
中图分类号: O213.2