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A Perturbed Risk Model with Dependence Based on a Generalized Farlie-Gumbel-Morgenstern Copula *

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Abstract: In this paper, we consider a perturbed compound Poisson risk model with dependence, where the dependence structure for the claim size and the inter-claim time is modeled by a generalized Farlie-Gumbel-Morgenstern copula. The integro equations, the Laplace transforms and the defective renewal equations for the Gerber-Shiu functions are obtained. For exponential claims, some explicit expressions are obtained, and some numerical examples for the ruin probabilities are also provided.

Keywords: time-dependent claims; Gerber-Shiu function; Laplace transform; defective renewal equation; ruin probability

2010 Mathematics Subject Classification: 60G51; 62P05; 91B30

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§1. Introduction

The actuarial ruin model perturbed by a diffusion process was first put forword in [1]. By then, it has received remarkable attention in insurance mathematics, see e.g. [2–6]. However, in all the aforementioned papers, it is depended on a postulation of independence between the claim size and the inter-claim time. Although such an assumption indeed simplifies the study of many risk problems, it has been proved to be very restrictive and inadequate in some applications. To overcome the drawback of the independence hypothesis, the ruin model with dependence has received considerable critical attention in

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actuarial mathematics. The ruin model with dependence but without diffusion has been studied by many authors, see e.g. [7–10]. The need for extensions to the perturbed actuarial ruin model has led to several studies on the modeling of dependence. Among them, Zhou and Cai [11] considered a perturbed risk model with dependence between premium rate and claim size. Zhang and Yang [12] examined several properties for a ruin model perturbed by a Brownian motion with dependence based on a Farlie-Gumbel-Morgenstern (FGM) copula. Zhang et al. [13] studied the Gerber-Shiu function for a renewal ruin model perturbed by a jump-diffusion process with dependence by using a q-potential measure.

In this paper, we consider the actuarial ruin model perturbed by a Brownian motion with dependence based on a generalized FGM copula. The surplus process has the form

$$U(t) = u + pt - \sum_{i=1}^{N(t)} X_i + \sigma W(t), \qquad t \geqslant 0, \tag{1}$$

where $u \geqslant 0$ is the initial principal and $p > 0$ is the premium rate. The counting process $\{N(t) : t \geqslant 0\}$ is a Poisson process defined by $N(t) = \max\{n : V_1 + V_2 + \cdots + V_n \leqslant t\}$, where the inter-claim times $\{V_j,\ j = 1, 2, \ldots\}$ are a sequence of exponential random variables (r.v.'s) distributed like a generic variable V with probability density function (p.d.f.) $k(t) = \lambda \mathrm{e}^{-\lambda t}$ for $\lambda > 0$, cumulative distribution function (c.d.f.) $K(t) = 1 - \mathrm{e}^{-\lambda t}$ and Laplace transform (L.T.) $\widehat{k}(s) = \lambda/(\lambda + s)$. The individual claim sizes $\{X_i,\ i = 1, 2, \ldots\}$ are assumed to be a sequence of strictly positive r.v.'s distributed as a generic variable X with p.d.f. $f(x)$, c.d.f. $F(x)$ and L.T. $\widehat{f}(s)$. We suppose that the claim size and the inter-claim time are dependent. Finally, $W(t)$ independent of the aggregate claims process is a standard Brownian motion starting from zero, and $\sigma > 0$ is the diffusion volatility.

The aim of this paper is to study the Gerber-Shiu function. The risk model studied in this paper extends that of [12] by using a generalized FGM copula instead of a FGM copula. It is important to point out that the key difference between [12] and ours is that they solve their problems through the integro-differential equation approach, while we gain our results by applying Laplace transform method. Their approach should assume that the claim size has a continuous density function and the Gerber-Shiu function is twice continuously differentiable, but our method only needs the claim size with Laplace transform and the Gerber-Shiu function with inversion of Laplace transform. Finally, our results extend the range of the dependence parameter θ.

The rest of the paper is organized as follows. In Section 2, we briefly describe the dependence structure based on a generalized FGM copula and introduce some ruin measures.

We discuss in Section 3 a generalized Lundberg equation. The Laplace transforms for the Gerber-Shiu functions are obtained in Section 4. In Section 5, we derive the defective renewal equations for the Gerber-Shiu functions, and accordingly, the analytic expressions are also obtained. Finally, explicit expressions of the Gerber-Shiu functions and numerical results of the ruin probabilities are provided for exponential claims in Section 6.

§2. Dependence Structure and Ruin Measures

Motivated by [9], we suppose that the joint distribution of (X, V) is based on a generalized FGM copula, which belongs to the family of copulas introduced and studied by [14]. The copula is denoted by

$$C(u, v) = uv + \theta h(u)g(v), \qquad 0 \leqslant u, v \leqslant 1, \tag{2}$$

where $h(u) = u^a(1-u)^b$ and $g(v) = v^c(1-v)^d$ with $a, b, c, d \geqslant 1$. It is an extension to the classical FGM copulas

$$C(u, v) = uv + \theta uv(1-u)(1-v), \qquad 0 \leqslant u, v \leqslant 1,$$

where $-1 \leqslant \theta \leqslant 1$. One motivation of these extensions is to improve the range of dependence association between the components of (X, V). Rodriguez-Lallena and Ubena-Flores [14] show that the admissible range for θ increases with the values of a, b, c and d. For example, if $a = b = c = d = 2$, then $-27 \leqslant \theta \leqslant 27$.

As in [9], the pairs $\{(X_i, V_i),\ i \in \mathbb{N}\}$ form a sequence of independent and identically distributed random vectors distributed as the generic random vector (X, V), in which the components may be dependent. The joint p.d.f. and the joint c.d.f. of (X, V) are denoted by $f_{X,V}(x, t)$ and $F_{X,V}(x, t)$ respectively.

The p.d.f. associated to (2) is given by

$$c(u, v) = 1 + \theta h'(u)g'(v). \tag{3}$$

From (2), the bivariate c.d.f. of $F_{X,V}(x, t)$ is defined by

$$F_{X,V}(x, t) = C(F(x), K(t)) = F(x)K(t) + \theta F(x)^a[1 - F(x)]^b K(t)^c[1 - K(t)]^d, \tag{4}$$

where $F(x)$ and $K(t)$ are the distributions of the marginals of (X, V) respectively.

Accordingly (3), the joint p.d.f. of $f_{X,V}(x, t)$ is given by

$$f_{X,V}(x, t) = c(F(x), K(t))f(x)k(t) = f(x)k(t) + \theta h'(F(x))g'(K(t))f(x)k(t). \tag{5}$$

For simplicity of presentation, let

$$g_X(x) = h'(F(x))f(x), \tag{6}$$

and

$$h_V(t) = g'(K(t))k(t), \tag{7}$$

where the Laplace transforms of $g_X(x)$ and $h_V(t)$ are denoted by $\widehat{g}_X(s)$ and $\widehat{h}_V(s)$ respectively. Then by these notations, the p.d.f. of (X, V) could be written as

$$f_{X,V}(x,t) = c(F(x), K(t))f(x)k(t) = f(x)k(t) + \theta g_X(x)h_V(t). \tag{8}$$

In particular, we know from (5) and (6) that the conditional p.d.f. of the claim size is given by

$$f_{X|V=t}(x) = f(x) + \theta g'(K(t))g_X(x). \tag{9}$$

Let $T = \inf\{t : U(t) \leqslant 0\}$ or ∞ otherwise, be the ruin time associated with risk model (1), and denote the ultimate ruin probability by

$$\psi(u) = \mathsf{P}(T < \infty \,|\, U(0) = u).$$

By examining the sample paths of the process $U(t)$, it shows that ruin can be caused either by the oscillation of the Brownian motion or a claim. Similar to [2], we can split the ruin probability into two parts:

$$\psi(u) = \psi_s(u) + \psi_w(u),$$

where $\psi_s(u)$ is the ruin probability caused by a claim, and $\psi_w(u)$ is the ruin probability due to oscillation. The requirement of a positive security loading is that the following net profit condition holds

$$\mathsf{E}(pV - X) > 0.$$

Let $w(x_1, x_2)$, $x_1 \geqslant 0$, $x_2 \geqslant 0$, be a nonnegative function. For $\delta > 0$, we define the Gerber-Shiu function at ruin by

$$\phi(u) = \mathsf{E}[\mathrm{e}^{-\delta T} w(U(T-), |U(T)|) I(T < \infty) \,|\, U(0) = u],$$

where $I(\cdot)$ is an indicator function. The quantity $w(U(T-), |U(T)|)$ is viewed as a penalty at the ruin time for the surplus immediately before ruin $U(T-)$ and the deficit at ruin $|U(T)|$. It provides a unified approach to study ruin theory in different risk models. Similarly, we can also decompose $\phi(u)$ as follows, i.e.

$$\phi(u) = \phi_s(u) + \phi_w(u),$$

where

$$\phi_s(u) = \mathsf{E}[\mathrm{e}^{-\delta T} w(U(T-), |U(T)|) I(T < \infty, U(T) < 0) \,|\, U(0) = u]$$

is Gerber-Shiu function at ruin that is due to a claim, and

$$\begin{aligned}\phi_w(u) &= \mathsf{E}[\mathrm{e}^{-\delta T} w(U(T-), |U(T)|) I(T < \infty, U(T) = 0) \,|\, U(0) = u] \\ &= w(0,0)\mathsf{E}[\mathrm{e}^{-\delta T} I(T < \infty, U(T) = 0) \,|\, U(0) = u]\end{aligned}$$

is Gerber-Shiu function at ruin that is caused by oscillation. Without loss of generality, we suppose that $w(0,0) = 1$. Note the cases $\delta = 0$ and $w(x_1, x_2) = 1$, $\phi_s(u)$ and $\phi_w(u)$ correspond to the ruin probabilities $\psi_s(u)$ and $\psi_w(u)$.

Throughout this paper, we assume that $a, b \geqslant 1$, $c \in \{2, 3, \ldots\}$ and $d > 1$.

§3. Analysis of a Generalized Lundberg Equation

In this section, the chief aim focuses on deriving the roots of a generalized Lundberg equation associated with the ruin model studied in this paper. Throughout the paper, the Laplace transform of a function is defined by adding a cap to the corresponding letter.

First, we introduce the following lemma, which provides analytic expressions for $h_V(t)$ and $\widehat{h}_V(s)$.

Lemma 1 The function $h_V(t)$ defined in equation (7) could be expressed as

$$h_V(t) = \sum_{i=1}^{c+1} \alpha_i \mathrm{e}^{-\lambda_i t}$$

and its associate L.T. is given by

$$\begin{aligned}\widehat{h}_V(s) &= \frac{c!\lambda^c s}{\prod\limits_{i=1}^{c+1}(s + \lambda_i)} &(10)\\ &= \sum_{i=1}^{c+1} \frac{\alpha_i}{s + \lambda_i}, &(11)\end{aligned}$$

where

$$\lambda_i = \lambda(d + i - 1), \tag{12}$$

$$\alpha_i = \frac{c!\lambda^c(-\lambda_i)}{\prod\limits_{j=1,\, j\neq i}^{c+1}(-\lambda_i + \lambda_j)} = \frac{c!\lambda^c(-1)^{c-1}\lambda_i}{\tau'_{c+1}(\lambda_i)}, \tag{13}$$

and

$$\tau_{c+1}(s) = \prod_{j=1}^{c+1}(s - \lambda_i). \tag{14}$$

Proof See formula (19) on page 446 of [9]. □

In order to deduce the generalized Lundberg equation, we consider the discrete-time process embedded in the continuous-time surplus process $\{U(t) : t \geqslant 0\}$. Set $U_0 = 0$, and for $n \in \mathbb{N}^+$, we denote U_n to be the surplus instantaneous after the nth claim, i.e.

$$\begin{aligned} U_n &= u + \sum_{i=1}^{n}(pV_i - X_i) + \sigma W\Big(\sum_{i=1}^{n} V_i\Big) \\ &\triangleq u + \sum_{i=1}^{n}[pV_i - X_i + \sigma W(V_i)], \end{aligned}$$

where $\triangleq$ signifies equality in distribution. Now we look for a number s such that the process $\{\mathrm{e}^{-\delta V_n + sU_n},\ n = 0, 1, 2, \ldots\}$ is a martingale. This process is a martingale if and only if

$$\mathsf{E}(\mathrm{e}^{-\delta V + s[pV - X + \sigma B(V)]}) = 1, \tag{15}$$

which is the generalized Lundberg equation associated with the risk model (1). By (8), the left-hand side of (15) can be written as

$$\begin{aligned} \mathsf{E}(\mathrm{e}^{-\delta V + s[pV - X + \sigma B(V)]}) &= \int_0^{\infty}\int_0^{\infty} f_{X,V}(x,t)\mathsf{E}(\mathrm{e}^{-\delta t + s[pt - X + \sigma B(t)]})\mathrm{d}x\mathrm{d}t \\ &= \int_0^{\infty}\int_0^{\infty} f(x)k(t)\mathrm{e}^{-sx + (\sigma^2 s^2/2 + ps - \delta)t}\mathrm{d}x\mathrm{d}t \\ &\quad + \theta\int_0^{\infty}\int_0^{\infty} g_X(x)h_V(t)\mathrm{e}^{-sx + (\sigma^2 s^2/2 + ps - \delta)t}\mathrm{d}x\mathrm{d}t \\ &= \widehat{f}(s)\widehat{k}\Big(\delta - ps - \frac{\sigma^2}{2}s^2\Big) + \theta\widehat{g}_X(s)\widehat{h}_V\Big(\delta - ps - \frac{\sigma^2}{2}s^2\Big). \end{aligned} \tag{16}$$

for $\mathrm{Re}(s) \geqslant 0$ and $\mathrm{Re}(\sigma^2 s^2/2 + ps) < \lambda + \delta$, where $\mathrm{Re}(\cdot)$ represents the real part of a number. Substituting (10) and (16) into (15) and using $\widehat{k}(s) = \lambda/(\lambda + s)$, the generalized Lundberg equation (15) reduces to

$$\frac{\lambda}{\lambda + \delta - ps - \sigma^2 s^2/2}\widehat{f}(s) + \theta\frac{c!\lambda^c(\delta - ps - \sigma^2 s^2/2)}{\prod\limits_{i=1}^{c+1}(\lambda_i + \delta - ps - \sigma^2 s^2/2)}\widehat{g}_X(s) = 1. \tag{17}$$

When $\sigma = 0$, equation (17) equals to (20) in [9].

We declare without proof of the following proposition which can be readily proved by the (generalized) Rouché's theorem.

Proposition 2 For $\delta > 0$ and $\theta \neq 0$, Eq. (17) has exactly $c+2$ roots, say $\rho_1(\delta), \rho_2(\delta), \ldots, \rho_{c+2}(\delta)$, with $\mathrm{Re}(\rho_i(\delta)) > 0$, $i = 1, 2, \ldots, c+2$. For $\delta = 0$ and $\theta \neq 0$, Eq. (17) has exactly $c + 1$ roots, say $\rho_1(0), \rho_2(0), \ldots, \rho_{c+1}(0)$, with $\mathrm{Re}(\rho_i(0)) > 0$, $i = 1, 2, \ldots, c + 1$ and one root, $\rho_{c+2}(0)$, equal to zero. In the sequel, for simplicity we write ρ_i for $\rho_i(\delta)$.

Remark 3 As mentioned at the end of Section 2, we suppose that $d > 1$ in this paper. When $d = 1$, Eq. (17) has exactly $c+1$ roots, say $\{\rho_i(\delta),\ i = 1, 2, \ldots, c+1\}$ in the right-half-complex plane with $\mathrm{Re}(\rho_i(\delta)) > 0$ for $\delta > 0$ and $\theta \neq 0$. For $\delta = 0$ and $\theta \neq 0$, Eq. (17) has exactly c roots, say $\{\rho_i(0),\ i = 1, 2, \ldots, c\}$ with $\mathrm{Re}(\rho_i(0)) > 0$, and one root $\rho_{c+1}(0)$ equal to zero.

In the following sections, we consider the case $d > 1$ and the roots of Eq. (17) are distinct.

§4. Laplace Transforms

The emphasis of this section lies in deriving the Laplace transforms of the Gerber-Shiu functions $\phi_s(u)$ and $\phi_w(u)$.

First, we give analytic expressions of $g'(K(t))$ and $\widehat{g'}(K(t))$, which play an important role in analyzing the Gerber-Shiu functions.

Lemma 4 The function $g'(K(t))$ could be expressed as

$$g'(K(t)) = \sum_{i=1}^{c+1} \beta_i \mathrm{e}^{-(\lambda_i-\lambda)t}, \tag{18}$$

and its associate L.T. is given by

$$\widehat{g'}(K(t)) = \frac{c!\lambda^{c-1}(s-\lambda)}{\prod\limits_{i=1}^{c+1}(s-\lambda+\lambda_i)} \tag{19}$$

$$= \sum_{i=1}^{c+1} \frac{\beta_i}{s-\lambda+\lambda_i}, \tag{20}$$

where

$$\beta_i = \frac{c!\lambda^{c-1}(-\lambda_i)}{\prod\limits_{j=1,\, j\neq i}^{c+1}(-\lambda_i+\lambda_j)} = \frac{\alpha_i}{\lambda}. \tag{21}$$

Proof Taking Laplace transforms on $g'(K(t))$, and using integration by parts and (10), we obtain

$$\begin{aligned}\widehat{g'}(K(t)) &= \int_0^\infty \mathrm{e}^{-st} g'(K(t))\mathrm{d}t = \frac{1}{\lambda}\int_0^\infty \mathrm{e}^{-(s-\lambda)t} k(t) g'(K(t))\mathrm{d}t \\ &= \frac{1}{\lambda}\int_0^\infty \mathrm{e}^{-(s-\lambda)t} h_V(t)\mathrm{d}t = \frac{1}{\lambda}\widehat{h}_V(s-\lambda) \\ &= \frac{c!\lambda^{c-1}(s-\lambda)}{\prod\limits_{i=1}^{c+1}(s-\lambda+\lambda_i)}. \end{aligned} \tag{22}$$

Using partial fractions, it follows that

$$\widehat{g}'(K(t)) = \sum_{i=1}^{c+1} \frac{\beta_i}{s-\lambda+\lambda_i}. \tag{23}$$

Therefore, by inverting the Laplace transform in (23), we obtain (18). □

Now we introduce some auxiliary results. Let $\overline{B}(t) = \sup\limits_{0\leqslant s\leqslant t} B(s)$ be the running spermium of $B(t)$, where $B(t) = -pt - \sigma W(t)$ is a Brownian motion starting from zero with drift $-p$ and variance σ^2. Denote by $\tau_u = \inf\{t \geqslant 0 : B(t) = u\}$ the first hitting time of the value $u > 0$. From Eq. (2.0.1) in [15; page 295], we derive for $\delta \geqslant 0$,

$$\mathsf{E}(\mathrm{e}^{-\delta\tau_u}) = \mathrm{e}^{-\upsilon u} \tag{24}$$

with $\upsilon = p/\sigma^2 + \sqrt{2\delta/\sigma^2 + p^2/\sigma^4}$.

For $q > 0$, we denote $\boldsymbol{e}_q$ to be an exponential random variable with mean $1/q$, and introduce the following measure

$$\mathscr{U}_q(u, \mathrm{d}y) = \mathsf{P}(\overline{B}(\boldsymbol{e}_q) < u,\ B(\boldsymbol{e}_q) \in \mathrm{d}y), \qquad u > 0,\ u > y,$$

whose explicit expression is provided in the following well-known lemma.

Lemma 5 Suppose that $\boldsymbol{e}_q$ and $\{B(t)\}$ are independent, we know that the following variables

$$\overline{B}(\boldsymbol{e}_q); \qquad \overline{B}(\boldsymbol{e}_q) - B(\boldsymbol{e}_q)$$

are independent and exponentially distributed with rates

$$\nu_1 = \frac{p}{\sigma^2} + \sqrt{\frac{2q}{\sigma^2} + \frac{p^2}{\sigma^4}}, \qquad \nu_2 = -\frac{p}{\sigma^2} + \sqrt{\frac{2q}{\sigma^2} + \frac{p^2}{\sigma^4}},$$

respectively. Then, we obtain for $0 \leqslant y < u$,

$$\mathscr{U}_q(u, \mathrm{d}y) = \frac{\nu_1\nu_2}{\nu_1+\nu_2}(\mathrm{e}^{-\nu_1 y} - \mathrm{e}^{-(\nu_1+\nu_2)u+\nu_2 y})\mathrm{d}y, \tag{25}$$

and for $y < 0$,

$$\mathscr{U}_q(u, \mathrm{d}y) = \frac{\nu_1\nu_2}{\nu_1+\nu_2}(\mathrm{e}^{\nu_2 y} - \mathrm{e}^{-(\nu_1+\nu_2)u+\nu_2 y})\mathrm{d}y. \tag{26}$$

Proof See Lemma 2 in [12]. □

From (25) and (26), we know that the measure $\mathscr{U}_q(u, \mathrm{d}y)$ is absolutely continuous with respect to Lebesgue measure for $u > 0$.

In order to derive the Laplace transforms of $\phi_s(u)$ and $\phi_w(u)$, we define the following potential measure

$$\mathscr{P}(u, \mathrm{d}y, \mathrm{d}x) = \mathsf{E}[\mathrm{e}^{-\delta V} I(\overline{B}(V) < u, B(V) \in \mathrm{d}y, X \in \mathrm{d}x)], \qquad u, x > 0,\ y < u, \tag{27}$$

where $\delta \geqslant 0$. Using Lemma 4, Lemma 5 and taking equation (9) into consideration, we obtain the following lemma.

Lemma 6 The measure $\mathscr{P}(u, \mathrm{d}y, \mathrm{d}x)$ has a density given by

$$p(u,y,x) = \sum_{i=1}^{c+1} \frac{\theta\alpha_i\varsigma_{i1}\varsigma_{i2}}{(\lambda_i+\delta)(\varsigma_{i1}+\varsigma_{i2})}(\mathrm{e}^{-\varsigma_{i1}y} - \mathrm{e}^{-(\varsigma_{i1}+\varsigma_{i2})u+\varsigma_{i2}y})g_X(x) + \frac{\lambda\varrho_1\varrho_2}{(\lambda+\delta)(\varrho_1+\varrho_2)}(\mathrm{e}^{-\varrho_1 y} - \mathrm{e}^{-(\varrho_1+\varrho_2)u+\varrho_2 y})f(x) \tag{28}$$

for $0 \leqslant y < u$, and

$$p(u,y,x) = \sum_{i=1}^{c+1} \frac{\theta\alpha_i\varsigma_{i1}\varsigma_{i2}}{(\lambda_i+\delta)(\varsigma_{i1}+\varsigma_{i2})}(\mathrm{e}^{\varsigma_{i2}y} - \mathrm{e}^{-(\varsigma_{i1}+\varsigma_{i2})u+\varsigma_{i2}y})g_X(x) + \frac{\lambda\varrho_1\varrho_2}{(\lambda+\delta)(\varrho_1+\varrho_2)}(\mathrm{e}^{\varrho_2 y} - \mathrm{e}^{-(\varrho_1+\varrho_2)u+\varrho_2 y})f(x) \tag{29}$$

for $y < 0$, where

$$\varrho_1 = \frac{p}{\sigma^2} + \sqrt{\frac{2(\lambda+\delta)}{\sigma^2} + \frac{p^2}{\sigma^4}}, \qquad \varrho_2 = -\frac{p}{\sigma^2} + \sqrt{\frac{2(\lambda+\delta)}{\sigma^2} + \frac{p^2}{\sigma^4}}, \tag{30}$$

and for $i = 1, 2, \ldots, c+1$,

$$\varsigma_{i1} = \frac{p}{\sigma^2} + \sqrt{\frac{2(\lambda_i+\delta)}{\sigma^2} + \frac{p^2}{\sigma^4}}, \qquad \varsigma_{i2} = -\frac{p}{\sigma^2} + \sqrt{\frac{2(\lambda_i+\delta)}{\sigma^2} + \frac{p^2}{\sigma^4}}. \tag{31}$$

Proof Conditioning on the value of V, and using Lemma 4 and (9), we have

$$\begin{aligned}
\mathscr{P}(u, \mathrm{d}y, \mathrm{d}x) &= \int_0^\infty \lambda \mathrm{e}^{-(\lambda+\delta)} f_{X|V=t}(x)\mathsf{P}(\overline{B}(t) < u, B(t) \in \mathrm{d}y)\mathrm{d}x\mathrm{d}t \\
&= \int_0^\infty \lambda \mathrm{e}^{-(\lambda+\delta)} f(x)\mathsf{P}(\overline{B}(t) < u, B(t) \in \mathrm{d}y)\mathrm{d}x\mathrm{d}t \\
&\quad + \lambda\theta \int_0^\infty \sum_{i=1}^{c+1} \beta_i \mathrm{e}^{-(\lambda_i+\delta)t} g_X(x)\mathsf{P}(\overline{B}(t) < u, B(t) \in \mathrm{d}y)\mathrm{d}x\mathrm{d}t \\
&= \frac{\lambda}{\lambda+\delta} f(x)\mathscr{U}_{\lambda+\delta}(u, \mathrm{d}y)\mathrm{d}x + g_X(x)\sum_{i=1}^{c+1} \frac{\lambda\theta\beta_i}{\lambda_i+\delta}\mathscr{U}_{\lambda_i+\delta}(u, \mathrm{d}y)\mathrm{d}x \\
&= \frac{\lambda}{\lambda+\delta} f(x)\mathscr{U}_{\lambda+\delta}(u, \mathrm{d}y)\mathrm{d}x + g_X(x)\sum_{i=1}^{c+1} \frac{\theta\alpha_i}{\lambda_i+\delta}\mathscr{U}_{\lambda_i+\delta}(u, \mathrm{d}y)\mathrm{d}x
\end{aligned} \tag{32}$$

which together with Lemma 5 gives the desired results. $\square$

In the following, we will derive the Laplace transforms for the Gerber-Shiu functions $\phi_s(u)$ and $\phi_w(u)$.

For $\phi_s(u)$, by conditioning on the time and amount of first claim, and using the definition of $p(u, y, x)$, one obtains

$$\phi_s(u) = \int_{t\in(0,\infty)} \int_{y\in(-\infty,u)} \int_{x\in(0,u-y]} \mathrm{e}^{-\delta t}\mathsf{P}(\overline{B}(t) < u, B(t) \in \mathrm{d}y)$$

$$
\begin{aligned}
&\times \phi_s(u-y-x) f_{X,V}(x,t)\mathrm{d}x\mathrm{d}t \\
&+ \int_{t\in(0,\infty)}\int_{y\in(-\infty,u)}\int_{x\in(u-y,\infty)} \mathrm{e}^{-\delta t}\mathsf{P}(\overline{B}(t)<u, B(t)\in\mathrm{d}y) \\
&\times w(u-y, x-(u-y)) f_{X,V}(x,t)\mathrm{d}x\mathrm{d}t \\
= &\int_{-\infty}^{u}\int_{0}^{u-y} \phi_s(u-y-x)p(u,y,x)\mathrm{d}x\mathrm{d}y \\
&+ \int_{-\infty}^{u}\int_{u-y}^{\infty} w(u-y, x-(u-y))p(u,y,x)\mathrm{d}x\mathrm{d}y. \qquad (33)
\end{aligned}
$$

Using (28) and (29), (33) can be rewritten as

$$
\begin{aligned}
\phi_s(u) = &\sum_{i=1}^{c+1} \frac{\theta\alpha_i\varsigma_{i1}\varsigma_{i2}}{(\lambda_i+\delta)(\varsigma_{i1}+\varsigma_{i2})}\int_0^u (\mathrm{e}^{-\varsigma_{i1}y} - \mathrm{e}^{-(\varsigma_{i1}+\varsigma_{i2})u+\varsigma_{i2}y})\sigma_{s,1}(u-y)\mathrm{d}y \\
&+ \frac{\lambda\varrho_1\varrho_2}{(\lambda+\delta)(\varrho_1+\varrho_2)}\int_0^u (\mathrm{e}^{-\varrho_1 y} - \mathrm{e}^{-(\varrho_1+\varrho_2)u+\varrho_2 y})\sigma_{s,2}(u-y)\mathrm{d}y \\
&+ \sum_{i=1}^{c+1} \frac{\theta\alpha_i\varsigma_{i1}\varsigma_{i2}}{(\lambda_i+\delta)(\varsigma_{i1}+\varsigma_{i2})}\int_{-\infty}^0 (\mathrm{e}^{\varsigma_{i2}y} - \mathrm{e}^{-(\varsigma_{i1}+\varsigma_{i2})u+\varsigma_{i2}y})\sigma_{s,1}(u-y)\mathrm{d}y \\
&+ \frac{\lambda\varrho_1\varrho_2}{(\lambda+\delta)(\varrho_1+\varrho_2)}\int_{-\infty}^0 (\mathrm{e}^{\varrho_2 y} - \mathrm{e}^{-(\varrho_1+\varrho_2)u+\varrho_2 y})\sigma_{s,2}(u-y)\mathrm{d}y, \qquad (34)
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_{s,1}(u) &= \int_0^u \phi_s(u-x)g_X(x)\mathrm{d}x + \omega_1(u), \\
\sigma_{s,2}(u) &= \int_0^u \phi_s(u-x)f(x)\mathrm{d}x + \omega_2(u), \\
\omega_1(u) &= \int_u^\infty w(u, x-u)g_X(x)\mathrm{d}x, \\
\omega_2(u) &= \int_u^\infty w(u, x-u)f(x)\mathrm{d}x.
\end{aligned}
$$

Taking a change of variable $z=u-y$, (34) turns to

$$
\begin{aligned}
\phi_s(u) = &\sum_{i=1}^{c+1} \frac{\theta\alpha_i\varsigma_{i1}\varsigma_{i2}}{(\lambda_i+\delta)(\varsigma_{i1}+\varsigma_{i2})}\Big[\int_0^u \mathrm{e}^{-\varsigma_{i1}(u-z)}\sigma_{s,1}(z)\mathrm{d}z + \int_u^\infty \mathrm{e}^{\varsigma_{i2}(u-z)}\sigma_{s,1}(z)\mathrm{d}z \\
&- \int_0^\infty \mathrm{e}^{-\varsigma_{i1}u-\varsigma_{i2}z}\sigma_{s,1}(z)\mathrm{d}z\Big] + \frac{\lambda\varrho_1\varrho_2}{(\lambda+\delta)(\varrho_1+\varrho_2)}\Big[\int_0^u \mathrm{e}^{-\varrho_1(u-z)}\sigma_{s,2}(z)\mathrm{d}z \\
&+ \int_u^\infty \mathrm{e}^{\varrho_2(u-z)}\sigma_{s,2}(z)\mathrm{d}z - \int_0^\infty \mathrm{e}^{-\varrho_1 u-\varrho_2 z}\sigma_{s,2}(z)\mathrm{d}z\Big]. \qquad (35)
\end{aligned}
$$

In what follows, we introduce the Dickson-Hipp operator T_s and some of its properties. The operator T_s on a function h is defined by

$$
T_s h(x) = \int_x^\infty \mathrm{e}^{-s(y-x)}h(y)\mathrm{d}y, \qquad x>0,
$$

where h is a real-valued integrable function and s is a nonnegative real number (or a complex number with nonnegative real part). Some useful properties of the operator T_s needed in this paper are listed below:

(i) $T_s h(0) = \int_0^\infty \mathrm{e}^{-sy} h(y)\mathrm{d}y = \widehat{h}(s)$, $s \in \mathbb{C}$.

(ii) $T_s T_r h(x) = T_r T_s h(x) = \dfrac{T_s h(x) - T_r h(x)}{r - s}$, $x \geqslant 0$, $r \neq s \in \mathbb{C}$.

For more properties of T_s, see [16].

Adopting the Dickson-Hipp operator T_s with property (i) brings (35) into

$$\begin{aligned}\phi_s(u) = \sum_{i=1}^{c+1} \frac{\theta\alpha_i\varsigma_{i1}\varsigma_{i2}}{(\lambda_i+\delta)(\varsigma_{i1}+\varsigma_{i2})}\Big[\int_0^u \mathrm{e}^{-\varsigma_{i1}(u-z)}\sigma_{s,1}(z)\mathrm{d}z + T_{\varsigma_{i2}}\sigma_{s,1}(u) \\ - \mathrm{e}^{-\varsigma_{i1}u} T_{\varsigma_{i2}}\sigma_{s,1}(0)\Big] + \frac{\lambda\varrho_1\varrho_2}{(\lambda+\delta)(\varrho_1+\varrho_2)}\Big[\int_0^u \mathrm{e}^{-\varrho_1(u-z)}\sigma_{s,2}(z)\mathrm{d}z \\ + T_{\varrho_2}\sigma_{s,2}(u) - \mathrm{e}^{-\varrho_1 u} T_{\varrho_2}\sigma_{s,2}(0)\Big].\end{aligned} \tag{36}$$

Taking Laplace transform of (36) and using the properties of Laplace transform, we have

$$\begin{aligned}\widehat{\phi}_s(s) = \sum_{i=1}^{c+1} \frac{\theta\alpha_i\varsigma_{i1}\varsigma_{i2}}{(\lambda_i+\delta)(\varsigma_{i1}+\varsigma_{i2})}\Big[\frac{\widehat{\sigma}_{s,1}(s) - \widehat{\sigma}_{s,1}(\varsigma_{i2})}{s+\varsigma_{i1}} + \frac{\widehat{\sigma}_{s,1}(\varsigma_{i2}) - \widehat{\sigma}_{s,1}(s)}{s-\varsigma_{i2}}\Big] \\ + \frac{\lambda\varrho_1\varrho_2}{(\lambda+\delta)(\varrho_1+\varrho_2)}\Big[\frac{\widehat{\sigma}_{s,2}(s) - \widehat{\sigma}_{s,2}(\varrho_2)}{s+\varrho_1} + \frac{\widehat{\sigma}_{s,2}(\varrho_2) - \widehat{\sigma}_{s,2}(s)}{s-\varrho_2}\Big],\end{aligned} \tag{37}$$

where

$$\widehat{\sigma}_{s,1}(s) = \widehat{\phi}_s(s)\widehat{g}_X(s) + \widehat{\omega}_1(s), \qquad \widehat{\sigma}_{s,2}(s) = \widehat{\phi}_s(s)\widehat{f}(s) + \widehat{\omega}_2(s). \tag{38}$$

Substituting (30) and (31) into (37) and by careful calculations, we obtain

$$\widehat{\phi}_s(s) = \frac{\theta\widehat{h}_V(\delta - ps - \sigma^2 s^2/2)\widehat{\omega}_1(s) + \widehat{k}(\delta - ps - \sigma^2 s^2/2)\widehat{\omega}_2(s) - \widehat{\eta}(s)}{1 - \theta\widehat{h}_V(\delta - ps - \sigma^2 s^2/2)\widehat{g}_X(s) - \widehat{k}(\delta - ps - \sigma^2 s^2/2)\widehat{f}(s)}, \tag{39}$$

where

$$\widehat{\eta}(s) = \theta\sum_{i=1}^{c+1} \frac{\alpha_i\widehat{\sigma}_{s,1}(\varsigma_{i2})}{\lambda_i + \delta - ps - \sigma^2 s^2/2} + \frac{\lambda\widehat{\sigma}_{s,2}(\varrho_2)}{\lambda + \delta - ps - \sigma^2 s^2/2}.$$

From (39), we can derive the following theorem.

Theorem 7 The Laplace transform $\widehat{\phi}_s(s)$ can be expressed as

$$\widehat{\phi}_s(s) = \frac{\widehat{\gamma}_{1,s}(s) - \widehat{\gamma}_{2,s}(s)}{\widehat{h}_1(s) - \widehat{h}_2(s)}, \tag{40}$$

where

$$\widehat{h}_1(s) = \Big(\lambda + \delta - ps - \frac{\sigma^2}{2}s^2\Big)\prod_{i=1}^{c+1}\Big(\lambda_i + \delta - ps - \frac{\sigma^2}{2}s^2\Big),$$

$$\widehat{h}_2(s)=\lambda\prod_{i=1}^{c+1}\Big(\lambda_i+\delta-ps-\frac{\sigma^2}{2}s^2\Big)\widehat{f}(s)+\theta c!\lambda^c\Big(\delta-ps-\frac{\sigma^2}{2}s^2\Big)\Big(\lambda+\delta-ps-\frac{\sigma^2}{2}s^2\Big)\widehat{g}_X(s),$$

$$\widehat{\gamma}_{1,s}(s)=\theta c!\lambda^c\Big(\delta-ps-\frac{\sigma^2}{2}s^2\Big)\Big(\lambda+\delta-ps-\frac{\sigma^2}{2}s^2\Big)\widehat{\omega}_1(s)+\lambda\prod_{i=1}^{c+1}\Big(\lambda_i+\delta-ps-\frac{\sigma^2}{2}s^2\Big)\widehat{\omega}_2(s),$$

and $\widehat{\gamma}_{2,s}(s)$ in terms of $\pi(s)=\sigma^2s^2/2+ps$ is a polynomial of degree $c+1$ or less, with

$$\widehat{\gamma}_{2,s}(s)=\sum_{j=1}^{c+2}\widehat{\gamma}_{1,s}(\rho_j)\prod_{k=1,\,k\neq j}^{c+2}\frac{\pi(\rho_k)-\pi(s)}{\pi(\rho_k)-\pi(\rho_j)}.$$

Proof Multiplying both the numerator and denominator of (39) by $(\lambda+\delta-ps-\sigma^2s^2/2)\prod_{j=1}^{c+1}(\lambda_j+\delta-ps-\sigma^2s^2/2)$ yields (40), with

$$\begin{aligned}\widehat{\gamma}_{2,s}(s)&=\Big(\lambda+\delta-ps-\frac{\sigma^2}{2}s^2\Big)\prod_{j=1}^{c+1}\Big(\lambda_j+\delta-ps-\frac{\sigma^2}{2}s^2\Big)\widehat{\eta}(s)\\&=\lambda\prod_{j=1}^{c+1}\Big(\lambda_j+\delta-ps-\frac{\sigma^2}{2}s^2\Big)\widehat{\sigma}_{s,2}(\varrho_2)\\&\quad+\theta\Big(\lambda+\delta-ps-\frac{\sigma^2}{2}s^2\Big)\prod_{j=1}^{c+1}\Big(\lambda_j+\delta-ps-\frac{\sigma^2}{2}s^2\Big)\sum_{i=1}^{c+1}\frac{\alpha_i\widehat{\sigma}_{s,1}(\varsigma_{i2})}{\lambda_i+\delta-ps-\sigma^2s^2/2}\\&=\lambda\prod_{j=1}^{c+1}\Big(\lambda_j+\delta-ps-\frac{\sigma^2}{2}s^2\Big)\widehat{\sigma}_{s,2}(\varrho_2)\\&\quad+\theta\Big(\lambda+\delta-ps-\frac{\sigma^2}{2}s^2\Big)\sum_{i=1}^{c+1}\alpha_i\Big[\prod_{j=1,\,j\neq i}^{c+1}\Big(\lambda_j+\delta-ps-\frac{\sigma^2}{2}s^2\Big)\Big]\widehat{\sigma}_{s,1}(\varsigma_{i2}),\end{aligned}$$

which in terms of $\sigma^2s^2/2+ps$ is a polynomial of degree $c+1$ or less. It is simple to check that the generalized Lundberg equation (17) can also be written as $\widehat{h}_1(s)-\widehat{h}_2(s)=0$, which means that $\rho_j's$, $j=1,2,\ldots,c+2$ are roots of the denominator in (40). These roots must also be the roots of the numerator in (40) since $\widehat{\phi}_s(s)$ is analytic for $\mathrm{Re}(s)\geqslant 0$, and thus $\widehat{\gamma}_{1,s}(\rho_j)=\widehat{\gamma}_{2,s}(\rho_j)$, $j=1,2,\ldots,c+2$. By the Lagrange interpolation formula, we obtain the desired result. $\square$

For $\phi_w(u)$, conditioning on whether or not ruin arises owing to oscillation before the first claim, we obtain

$$\begin{aligned}\phi_w(u)=\int_{t\in(0,\infty)}\int_{y\in(-\infty,u)}\int_{x\in(0,u-y]}&\mathrm{e}^{-\delta t}\mathsf{P}(\overline{B}(t)<u,B(t)\in\mathrm{d}y)\\&\times\phi_w(u-y-x)f_{X,V}(x,t)\mathrm{d}x\mathrm{d}t+\mathsf{E}[\mathrm{e}^{-\delta\tau_u}I(\tau_u<V)],\end{aligned}\tag{41}$$

where τ_u is the first hitting time defined at the beginning of this section.

Since V independent of $\{B(t)\}$ is a exponential random variable with parameter λ, we have

$$\mathsf{E}[\mathrm{e}^{-\delta\tau_u}I(\tau_u<V)]=\mathsf{E}\{\mathsf{E}[\mathrm{e}^{-\delta\tau_u}I(\tau_u<V)]\,|\,\{B(t)\}\}=\mathsf{E}(\mathrm{e}^{-(\lambda+\delta)\tau_u})=\mathrm{e}^{-\varrho_1 u}\tag{42}$$

thanks to (24). Therefore, from (42) and Lemma 6, (41) can be rewritten as

$$\phi_w(u) = \int_{-\infty}^{u} \int_{0}^{u-y} \phi_w(u-y-x) p(u,y,x) \mathrm{d}x\mathrm{d}y + \mathrm{e}^{-\varrho_1 u}. \tag{43}$$

Proceeding as in the proof of Theorem 7 for the rest of proof, we obtain the following theorem:

Theorem 8 The Laplace transform $\widehat{\phi}_w(s)$ can be expressed as

$$\widehat{\phi}_w(s) = \frac{\widehat{\gamma}_{1,w}(s) - \widehat{\gamma}_{2,w}(s)}{\widehat{h}_1(s) - \widehat{h}_2(s)}, \tag{44}$$

where

$$\widehat{\gamma}_{1,w}(s) = \frac{\sigma^2}{2}(\varrho_2 - s) \prod_{i=1}^{c+1} \Big(\lambda_i + \delta - ps - \frac{\sigma^2}{2} s^2 \Big),$$

and $\widehat{\gamma}_{2,w}(s)$ in terms of $\sigma^2 s^2/2 + ps$ is a polynomial of degree $c+1$ or less, with

$$\widehat{\gamma}_{2,w}(s) = \sum_{j=1}^{c+2} \widehat{\gamma}_{1,w}(\rho_j) \prod_{k=1,\, k\neq j}^{c+2} \frac{\pi(\rho_k) - \pi(s)}{\pi(\rho_k) - \pi(\rho_j)}.$$

Here $\widehat{h}_1(s)$ and $\widehat{h}_2(s)$ are defined in Theorem 7.

Remark 9 Because the surplus process $U(t)$ defined in this paper only regenerates itself at the claim epoches, the common approach that consider whether or not there is a claim during the infinitesimal time interval $[0, \mathrm{d}t]$ can't be applied to study the risk model of this paper. So we apply a potential measure method in [12] to research the Gerber-Shiu function for the extended insurance risk model.

§5. Defective Renewal Equations

In this section, we derive some defective renewal equations for the Gerber-Shiu functions by using the roots of the generalized Lundberg equation.

Now we consider the common denominator of (40) and (44). For convenience, let

$$\Gamma(\pi(s)) = \prod_{j=1}^{c+2} [\pi(\rho_j) - \pi(s)], \qquad \Gamma'(\pi(\rho_j)) = \prod_{k=1,\, k\neq j}^{c+2} [\pi(\rho_k) - \pi(\rho_j)].$$

Using Lemma 1, we know that $\widehat{h}_1(s) - \Gamma(\pi(s))$ which is a polynomial function of $\sigma^2 s^2/2 + ps$ with degree $c+1$ satisfies

$$\widehat{h}_1(\rho_j) - \Gamma(\pi(\rho_j)) = \widehat{h}_2(\rho_j)$$

for $j = 1, 2, \ldots, c+2$. Then, using the Lagrange interpolating formula one obtains

$$\widehat{h}_1(s) = \Gamma(\pi(s)) + \sum_{j=1}^{c+2} \widehat{h}_2(\rho_j) \prod_{k=1,\, k\neq j}^{c+2} \frac{\pi(\rho_k) - \pi(s)}{\pi(\rho_k) - \pi(\rho_j)}.$$

Thus, we have

$$\begin{aligned}
\widehat{h}_1(s)-\widehat{h}_2(s) &= \Gamma(\pi(s))+\sum_{j=1}^{c+2}\widehat{h}_2(\rho_j)\prod_{k=1,\,k\neq j}^{c+2}\frac{\pi(\rho_k)-\pi(s)}{\pi(\rho_k)-\pi(\rho_j)}-\widehat{h}_2(s)\\
&= \Gamma(\pi(s))+\sum_{j=1}^{c+2}[\widehat{h}_2(\rho_j)-\widehat{h}_2(s)]\prod_{k=1,\,k\neq j}^{c+2}\frac{\pi(\rho_k)-\pi(s)}{\pi(\rho_k)-\pi(\rho_j)}\\
&= \Gamma(\pi(s))\Big\{1-\sum_{j=1}^{c+2}\frac{\widehat{h}_2(s)-\widehat{h}_2(\rho_j)}{[\pi(\rho_j)-\pi(s)]\Gamma'(\pi(\rho_j))}\Big\}\\
&= \Gamma(\pi(s))\Bigg\{1-\lambda\sum_{j=1}^{c+2}\frac{\prod\limits_{i=1}^{c+1}[\lambda_i+\delta-\pi(s)]-\prod\limits_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{[\pi(\rho_j)-\pi(s)]\Gamma'(\pi(\rho_j))}\widehat{f}(s)\\
&\quad -\lambda\sum_{j=1}^{c+2}\frac{\prod\limits_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{[\pi(\rho_j)-\pi(s)]\Gamma'(\pi(\rho_j))}[\widehat{f}(s)-\widehat{f}(\rho_j)]\\
&\quad -\theta c!\lambda^c\sum_{j=1}^{c+2}\frac{[\delta-\pi(s)][\lambda+\delta-\pi(s)]-[\delta-\pi(\rho_j)][\lambda+\delta-\pi(\rho_j)]}{[\pi(\rho_j)-\pi(s)]\Gamma'(\pi(\rho_j))}\widehat{g}_X(s)\\
&\quad -\theta c!\lambda^c\sum_{j=1}^{c+2}\frac{[\delta-\pi(\rho_j)][\lambda+\delta-\pi(\rho_j)]}{[\pi(\rho_j)-\pi(s)]\Gamma'(\pi(\rho_j))}[\widehat{g}_X(s)-\widehat{g}_X(\rho_j)]\Bigg\}.
\end{aligned}\tag{45}$$

Since $\prod\limits_{i=1}^{c+1}[\lambda_i+\delta-\pi(s)]$ and $[\delta-\pi(s)][\lambda+\delta-\pi(s)]$ are two polynomials in function of $\pi(s)$ with degree $c+1$ and 2 respectively,

$$\frac{\prod\limits_{i=1}^{c+1}[\lambda_i+\delta-\pi(s)]-\prod\limits_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{\pi(s)-\pi(\rho_j)}$$

and

$$\frac{[\delta-\pi(s)][\lambda+\delta-\pi(s)]-[\delta-\pi(\rho_j)][\lambda+\delta-\pi(\rho_j)]}{\pi(s)-\pi(\rho_j)}$$

are two polynomials in function of $\pi(s)$ with degree c and 1 respectively, using the following formula

$$\sum_{i=1}^{n}\frac{(s_i-s)^k}{\prod\limits_{j=1,\,j\neq i}^{n}(s_i-s_j)}=\begin{cases}1, & k=n-1;\\ 0, & k=0,1,2,\ldots,n-2;\\ -\dfrac{1}{\prod\limits_{i=1}^{n}(s-s_i)}, & k=-1.\end{cases}$$

Eq. (45) can be rewritten as

$$\widehat{h}_1(s)-\widehat{h}_2(s)=\Gamma(\pi(s))\Bigg\{1-\sum_{j=1}^{c+2}\frac{\lambda\prod\limits_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{(\sigma^2/2)(s+\rho_j+2p/\sigma^2)\Gamma'(\pi(\rho_j))}\frac{\widehat{f}(s)-\widehat{f}(\rho_j)}{\rho_j-s}$$

$$-\sum_{j=1}^{c+2}\frac{\theta c!\lambda^c[\delta-\pi(\rho_j)][\lambda+\delta-\pi(\rho_j)]}{(\sigma^2/2)(s+\rho_j+2p/\sigma^2)\Gamma'(\pi(\rho_j))}\frac{\widehat{g}_X(s)-\widehat{g}_X(\rho_j)}{\rho_j-s}\Bigg\}. \tag{46}$$

Similarly, we obtain

$$\begin{aligned}\widehat{\gamma}_{1,s}(s)-\widehat{\gamma}_{2,s}(s)&=\widehat{\gamma}_{1,s}(s)-\sum_{j=1}^{c+2}\widehat{\gamma}_{1,s}(\rho_j)\prod_{k=1,\,k\neq j}^{c+2}\frac{\pi(\rho_k)-\pi(s)}{\pi(\rho_k)-\pi(\rho_j)}\\&=\Gamma(\pi(s))\sum_{j=1}^{c+2}\frac{\widehat{\gamma}_{1,s}(s)-\widehat{\gamma}_{1,s}(\rho_j)}{[\pi(\rho_j)-\pi(s)]\Gamma'(\pi(\rho_j))}\\&=\Gamma(\pi(s))\Bigg\{\sum_{j=1}^{c+2}\frac{\theta c!\lambda^c[\delta-\pi(\rho_j)][\lambda+\delta-\pi(\rho_j)]}{(\sigma^2/2)(s+\rho_j+2p/\sigma^2)\Gamma'(\pi(\rho_j))}\frac{\widehat{\omega}_1(s)-\widehat{\omega}_1(\rho_j)}{\rho_j-s}\\&\quad+\sum_{j=1}^{c+2}\frac{\lambda\prod_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{(\sigma^2/2)(s+\rho_j+2p/\sigma^2)\Gamma'(\pi(\rho_j))}\frac{\widehat{\omega}_2(s)-\widehat{\omega}_2(\rho_j)}{\rho_j-s}\Bigg\}.\end{aligned} \tag{47}$$

Also, we can derive

$$\begin{aligned}\widehat{\gamma}_{1,w}(s)-\widehat{\gamma}_{2,w}(s)&=\widehat{\gamma}_{1,w}(s)-\sum_{j=1}^{c+2}\widehat{\gamma}_{1,w}(\rho_j)\prod_{k=1,\,k\neq j}^{c+2}\frac{\pi(\rho_k)-\pi(s)}{\pi(\rho_k)-\pi(\rho_j)}\\&=\Gamma(\pi(s))\sum_{j=1}^{c+2}\frac{\widehat{\gamma}_{1,w}(s)-\widehat{\gamma}_{1,w}(\rho_j)}{[\pi(\rho_j)-\pi(s)]\Gamma'(\pi(\rho_j))}\\&=\Gamma(\pi(s))\sum_{j=1}^{c+2}\frac{\prod_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{(s+\rho_j+2p/\sigma^2)\Gamma'(\pi(\rho_j))}.\end{aligned} \tag{48}$$

Plugging (46) and (47) back into (40), and plugging (46) and (48) back into (44), respectively, then using the property (ii) of T_s, we have

$$\begin{aligned}\widehat{\phi}_s(s)=&\Bigg\{\sum_{j=1}^{c+2}\Bigg\{\frac{\theta c!\lambda^c[\delta-\pi(\rho_j)][\lambda+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}\frac{T_sT_{\rho_j}\omega_1(0)}{s+\rho_j+2p/\sigma^2}\\&+\frac{\lambda\prod_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}\frac{T_sT_{\rho_j}\omega_2(0)}{s+\rho_j+2p/\sigma^2}\Bigg\}\Bigg\}\\&\Bigg/\Bigg\{1-\sum_{j=1}^{c+2}\Bigg\{\frac{\theta c!\lambda^c[\delta-\pi(\rho_j)][\lambda+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}\frac{T_sT_{\rho_j}g_X(0)}{s+\rho_j+2p/\sigma^2}\\&+\frac{\lambda\prod_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}\frac{T_sT_{\rho_j}f(0)}{s+\rho_j+2p/\sigma^2}\Bigg\}\Bigg\},\end{aligned} \tag{49}$$

and

$$\widehat{\phi}_w(s)=\Bigg\{\sum_{j=1}^{c+2}\frac{\prod_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{(s+\rho_j+2p/\sigma^2)\Gamma'(\pi(\rho_j))}\Bigg\}$$

$$\Big/\Bigg\{1-\sum_{j=1}^{c+2}\Bigg\{\frac{\theta c!\lambda^c[\delta-\pi(\rho_j)][\lambda+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}\frac{T_sT_{\rho_j}g_X(0)}{s+\rho_j+2p/\sigma^2}$$
$$+\frac{\lambda\prod\limits_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}\frac{T_sT_{\rho_j}f(0)}{s+\rho_j+2p/\sigma^2}\Bigg\}\Bigg\}. \tag{50}$$

Rewriting (49) and (50) as

$$\widehat{\phi}_s(s)=\widehat{\phi}_s(s)\times\sum_{j=1}^{c+2}\Bigg\{\frac{\theta c!\lambda^c[\delta-\pi(\rho_j)][\lambda+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}\frac{T_sT_{\rho_j}g_X(0)}{s+\rho_j+2p/\sigma^2}$$
$$+\frac{\lambda\prod\limits_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}\frac{T_sT_{\rho_j}f(0)}{s+\rho_j+2p/\sigma^2}\Bigg\}$$
$$+\sum_{j=1}^{c+2}\Bigg\{\frac{\theta c!\lambda^c[\delta-\pi(\rho_j)][\lambda+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}\frac{T_sT_{\rho_j}\omega_1(0)}{s+\rho_j+2p/\sigma^2}$$
$$+\frac{\lambda\prod\limits_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}\frac{T_sT_{\rho_j}\omega_2(0)}{s+\rho_j+2p/\sigma^2}\Bigg\}, \tag{51}$$

and

$$\widehat{\phi}_w(s)=\widehat{\phi}_w(s)\times\sum_{j=1}^{c+2}\Bigg\{\frac{\theta c!\lambda^c[\delta-\pi(\rho_j)][\lambda+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}\frac{T_sT_{\rho_j}g_X(0)}{s+\rho_j+2p/\sigma^2}$$
$$+\frac{\lambda\prod\limits_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}\frac{T_sT_{\rho_j}f(0)}{s+\rho_j+2p/\sigma^2}\Bigg\}$$
$$+\sum_{j=1}^{c+2}\frac{\prod\limits_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{(s+\rho_j+2p/\sigma^2)\Gamma'(\pi(\rho_j))}. \tag{52}$$

By inverting the Laplace transforms of (51) and (52), we obtain the following results.

Theorem 10 The Gerber-Shiu functions $\phi_s(u)$ and $\phi_w(u)$ satisfy the following defective renewal functions

$$\phi_s(u)=\int_0^u\phi_s(u-x)g(x)\mathrm{d}x+H_s(u), \tag{53}$$
$$\phi_w(u)=\int_0^u\phi_w(u-x)g(x)\mathrm{d}x+H_w(u), \tag{54}$$

where

$$g(x)=\sum_{j=1}^{c+2}\Bigg\{\frac{\theta c!\lambda^c[\delta-\pi(\rho_j)][\lambda+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}\mathrm{e}^{-(\rho_j+2p/\sigma^2)x}*T_{\rho_j}g_X(x)$$

$$\left. + \frac{\lambda \prod_{i=1}^{c+1} [\lambda_i + \delta - \pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))} \mathrm{e}^{-(\rho_j + 2p/\sigma^2)x} * T_{\rho_j} f(x) \right\},$$

$$H_s(u) = \sum_{j=1}^{c+2} \left\{ \frac{\theta c! \lambda^c [\delta - \pi(\rho_j)][\lambda + \delta - \pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))} \mathrm{e}^{-(\rho_j + 2p/\sigma^2)u} * T_{\rho_j} \omega_1(u) \right.$$

$$\left. + \frac{\lambda \prod_{i=1}^{c+1} [\lambda_i + \delta - \pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))} \mathrm{e}^{-(\rho_j + 2p/\sigma^2)u} * T_{\rho_j} \omega_2(u) \right\},$$

$$H_w(u) = \sum_{j=1}^{c+2} \frac{\prod_{i=1}^{c+1} [\lambda_i + \delta - \pi(\rho_j)]}{\Gamma'(\pi(\rho_j))} \mathrm{e}^{-(\rho_j + 2p/\sigma^2)u},$$

and $*$ is the convolution operator.

Proof For (53) and (54) to be defective renewal equations, we need to prove that $\int_0^\infty g(x)\mathrm{d}x < 1$ or equivalently $\widehat{g}(0) < 1$.

By (46), we obtain

$$\widehat{g}(s) = 1 - \frac{\widehat{h}_1(s) - \widehat{h}_2(s)}{\Gamma(\pi(s))}.$$

Then for $\delta > 0$, we have

$$\int_0^\infty g(x)\mathrm{d}x = \widehat{g}(0) = 1 - \frac{\widehat{h}_1(0) - \widehat{h}_2(0)}{\Gamma(\pi(0))} = 1 - \frac{\delta \prod_{i=1}^{c+1} (\lambda_i + \delta)}{\prod_{j=1}^{c+2} (\sigma^2 \rho_j^2/2 + p\rho_j)} < 1. \tag{55}$$

As for $\delta = 0$, setting $s = \rho_{c+2}(\delta)$ in (15) and then taking derivatives w.r.t. δ yields

$$\rho'_{c+2}(0) = \frac{\mathsf{E}(V)}{\mathsf{E}(pV - X)} > 0$$

due to the net profit condition. Finally, taking the limit $\delta \to 0$ in (55) and using L'Hôspital's rule, we obtain

$$\int_0^\infty g(x)\mathrm{d}x = 1 - \frac{\prod_{i=1}^{c+1} \lambda_i}{\prod_{j=1}^{c+1} [(\sigma^2/2)\rho_j^2(0) + p\rho_j(0)]} \times \lim_{\delta \to 0} \frac{\delta}{(\sigma^2/2)\rho_{c+2}^2(\delta) + p\rho_{c+2}(\delta)}$$

$$= 1 - \frac{\prod_{i=1}^{c+1} \lambda_i \mathsf{E}(pV - X)}{\prod_{j=1}^{c+1} [(\sigma^2/2)\rho_j^2(0) + p\rho_j(0)] p\mathsf{E}(V)} < 1.$$

The proof is completed. $\square$

Now setting $(1+k)^{-1} = \widehat{g}(0) < 1$ and $G(x) = (1+k)\int_0^x g(y)\mathrm{d}y$, then $G(x)$ is a proper distribution. Define the following compound geometric tail

$$\overline{A}(x) = 1 - A(x) = \sum_{n=1}^{\infty} \frac{k}{1+k}\Big(\frac{1}{1+k}\Big)^n \overline{G^{*n}}(x), \qquad x \geqslant 0,$$

where $\overline{G^{*n}}(x)$ is the tail of the n-fold convolution of G with itself. Then by Theorem 2.1 of [17], we know that the general solutions to (53) and (54) are

$$\phi_s(u) = \frac{1+k}{k}\int_0^u H_s(u-x)\mathrm{d}A(x) + H_s(u), \tag{56}$$

$$\phi_d(u) = \frac{1+k}{k}\int_0^u H_d(u-x)\mathrm{d}A(x) + H_d(u). \tag{57}$$

For general claim distribution F, the expression of compound geometric distribution function G is rather complicated. Thus, in the next section, we consider a special case that the claim sizes are exponentially distributed. Then, explicit expressions for the Gerber-Shiu functions $\phi_s(u)$ and $\phi_d(u)$ are given.

Remark 11 From (56) and (57), we know that the general solutions of the Gerber-Shiu functions $\phi_s(u)$ and $\phi_d(u)$ can be expressed via a compound geometric distribution function G. It should be point out that the expression of G just requires the claim size distribution F with Laplace transform.

§6. Exponential Claim Amounts

In this section, we assume that the parameter a is a strictly positive integer and the claim size X follows an exponential distribution with p.d.f. $f(x) = \mu\mathrm{e}^{-\mu x}$ and L.T.

$$\widehat{f}(s) = \frac{\mu}{s+\mu}. \tag{58}$$

From the assumption of X and the definitions of h (with $a \in \{1, 2, \ldots\}$) and g (with $c \in \{2, 3, \ldots\}$) in the generalized FGM copula, $g_X(x)$, denoted by (6), has the same form as $h_V(t)$. Therefore, we obtain from Lemma 1

$$g_X(x) = \sum_{i=1}^{a+1} \xi_i \mathrm{e}^{-\mu_i x}, \tag{59}$$

and

$$\widehat{g}_X(s) = \sum_{i=1}^{a+1} \frac{\xi_i}{s+\mu_i} = \frac{a!\mu^a s}{\prod\limits_{i=1}^{a+1}(s+\mu_i)}, \tag{60}$$

where

$$\mu_i = \mu(b+i-1) \qquad (i=1,\ldots,a+1),$$

and

$$\xi_i = \frac{a!\mu^a(-\mu_i)}{\prod_{j=1,\,j\neq i}^{a+1}(-\mu_i+\mu_j)}.$$

It is simple to check that

$$T_{\rho_j}g_X(x) = \sum_{i=1}^{a+1}\frac{\xi_i\mathrm{e}^{-\mu_i x}}{\rho_j+\mu_i}, \qquad T_{\rho_j}f(x) = \frac{\mathrm{e}^{-\mu x}}{\rho_j+\mu}. \tag{61}$$

Then

$$T_sT_{\rho_j}g_X(0) = \sum_{i=1}^{a+1}\frac{\xi_i}{(\rho_j+\mu_i)(s+\mu_i)}, \qquad T_sT_{\rho_j}f(0) = \frac{\mu}{(\rho_j+\mu)(s+\mu)}. \tag{62}$$

Plugging the above results back into (49) and (50), then multiplying both the denominators and the numerators in (49) and (50) by $(s+\mu)\prod_{j=1}^{c+2}(s+\rho_j+2p/\sigma^2)\prod_{i=1}^{a+1}(s+\mu_i)$ yields

$$\widehat{\phi}_s(s) = \sum_{j=1}^{c+2}\frac{r_{s,1,j}(s)T_sT_{\rho_j}\omega_1(0)+r_{s,2,j}(s)T_sT_{\rho_j}\omega_2(0)}{p(s)-q(s)}, \tag{63}$$

and

$$\widehat{\phi}_w(s) = \sum_{j=1}^{c+2}\frac{r_{w,j}(s)}{p(s)-q(s)}, \tag{64}$$

where

$$p(s) = (s+\mu)\prod_{j=1}^{c+2}\left(s+\rho_j+\frac{2p}{\sigma^2}\right)\prod_{i=1}^{a+1}(s+\mu_i),$$

$$\begin{aligned}q(s) = \sum_{j=1}^{c+2}\Bigg\{&\frac{\theta c!\lambda^c[\delta-\pi(\rho_j)][\lambda+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}\sum_{i=1}^{a+1}\frac{\xi_i(s+\mu)}{\rho_j+\mu_i}\prod_{k=1,\,k\neq j}^{c+2}\left(s+\rho_k+\frac{2p}{\sigma^2}\right)\\ &\times\prod_{l=1,\,l\neq i}^{a+1}(s+\mu_l)+\frac{\lambda\prod_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}\frac{\mu}{\rho_j+\mu}\prod_{k=1,\,k\neq j}^{c+2}\left(s+\rho_k+\frac{2p}{\sigma^2}\right)\prod_{i=1}^{a+1}(s+\mu_i)\Bigg\},\end{aligned}$$

$$r_{s,1,j}(s) = \frac{\theta c!\lambda^c[\delta-\pi(\rho_j)][\lambda+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}(s+\mu)\prod_{k=1,\,k\neq j}^{c+2}\left(s+\rho_k+\frac{2p}{\sigma^2}\right)\prod_{i=1}^{a+1}(s+\mu_i),$$

$$r_{s,2,j}(s) = \frac{\lambda\prod_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{(\sigma^2/2)\Gamma'(\pi(\rho_j))}(s+\mu)\prod_{k=1,\,k\neq j}^{c+2}\left(s+\rho_k+\frac{2p}{\sigma^2}\right)\prod_{i=1}^{a+1}(s+\mu_i),$$

$$r_{w,j}(s) = \frac{\prod_{i=1}^{c+1}[\lambda_i+\delta-\pi(\rho_j)]}{\Gamma'(\pi(\rho_j))}(s+\mu)\prod_{k=1,\,k\neq j}^{c+2}\left(s+\rho_k+\frac{2p}{\sigma^2}\right)\prod_{i=1}^{a+1}(s+\mu_i).$$

It is readily seen that $p(s)-q(s)$ in (63) and (64) is a polynomial with degree $a+c+4$ with leading coefficient 1. In particular, by Proposition 2, $p(s)-q(s)$ has no zeros with nonnegative real part, then it can be expressed as

$$p(s)-q(s)=\prod_{n=1}^{a+c+4}(s+R_n),$$

where all $R_n's$ have positive real parts. If these $a+c+4$ roots are all distinct, by performing partial fractions, we have

$$\frac{r_{s,1,j}(s)}{p(s)-q(s)}=\sum_{n=1}^{a+c+4}\sum_{j=1}^{c+2}\frac{a_{n,j}}{s+R_n},\qquad \frac{r_{s,2,j}(s)}{p(s)-q(s)}=\sum_{n=1}^{a+c+4}\sum_{j=1}^{c+2}\frac{b_{n,j}}{s+R_n},\tag{65}$$

$$\frac{r_{w,j}(s)}{p(s)-q(s)}=\sum_{n=1}^{a+c+4}\sum_{j=1}^{c+2}\frac{c_{n,j}}{s+R_n},\tag{66}$$

where

$$a_{n,j}=\frac{r_{s,1,j}(-R_n)}{\prod\limits_{m=1,\,m\neq n}^{a+c+4}(R_m-R_n)},$$

$$b_{n,j}=\frac{r_{s,2,j}(-R_n)}{\prod\limits_{m=1,\,m\neq n}^{a+c+4}(R_m-R_n)},$$

$$c_{n,j}=\frac{r_{w,j}(-R_n)}{\prod\limits_{m=1,\,m\neq n}^{a+c+4}(R_m-R_n)}.$$

Then plugging (65) and (66) into (63) and (64), respectively, one obtains

$$\widehat{\phi}_s(s)=\sum_{n=1}^{a+c+4}\sum_{j=1}^{c+2}\Big[\frac{a_{n,j}T_sT_{\rho_j}\omega_1(0)}{s+R_n}+\frac{b_{n,j}T_sT_{\rho_j}\omega_2(0)}{s+R_n}\Big],\tag{67}$$

and

$$\widehat{\phi}_w(s)=\sum_{n=1}^{a+c+4}\sum_{j=1}^{c+2}\frac{c_{n,j}}{s+R_n}.\tag{68}$$

By inverting the Laplace transforms of (67) and (68), we have

Theorem 12 Suppose that $f(x)=\mu\mathrm{e}^{-\mu x}$ for $\mu>0$, and the $R_n's$ are distinct. Then the Gerber-Shiu functions satisfy

$$\phi_s(u)=\sum_{n=1}^{a+c+4}\sum_{j=1}^{c+2}\big[a_{n,j}\mathrm{e}^{-R_nu}*T_{\rho_j}\omega_1(u)+b_{n,j}\mathrm{e}^{-R_nu}*T_{\rho_j}\omega_2(u)\big],\tag{69}$$

and

$$\phi_w(u)=\sum_{n=1}^{a+c+4}\sum_{j=1}^{c+2}c_{n,j}\mathrm{e}^{-R_nu}.\tag{70}$$

Example 13 (The ruin probability) For the numerical results, we assume that $\delta = 0$ and $w(x_1, x_2) = 1$, then the Gerber-Shiu functions reduce to $\psi_s(u)$ and $\psi_w(u)$. By formula (70) we derive

$$\psi_w(u) = \sum_{n=1}^{a+c+4} \sum_{j=1}^{c+2} c_{n,j} \mathrm{e}^{-R_n u}. \tag{71}$$

Next, we consider $\psi_s(u)$. Note that $\omega_1(u) = \sum\limits_{i=1}^{a+1} (\xi_i/\mu_i)\mathrm{e}^{-\mu_i u}$, $\omega_2(u) = \mathrm{e}^{-\mu u}$. Thus, by formula (69) we have

$$\begin{aligned}\psi_s(u) = \sum_{n=1}^{a+c+4} \sum_{j=1}^{c+2} \Big[&\sum_{i=1}^{a+1} \frac{a_{n,j}\xi_i}{\mu_i(\mu_i + \rho_j)(R_n - \mu_i)} (\mathrm{e}^{-\mu_i u} - \mathrm{e}^{-R_n u}) \\ &+ \frac{b_{n,j}}{(\mu + \rho_j)(R_n - \mu)} (\mathrm{e}^{-\mu u} - \mathrm{e}^{-R_n u}) \Big].\end{aligned} \tag{72}$$

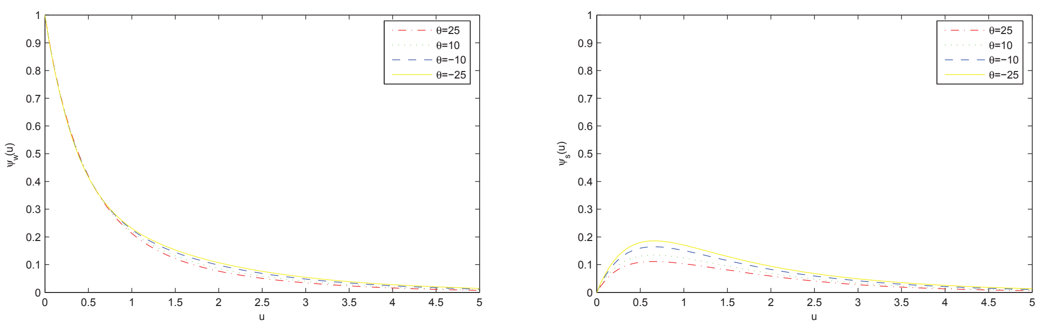
(i) Set $\mu = 1.5$, $\lambda = 1$, $p = 2$, $\sigma^2/2 = 1$, $a = b = c = d = 2$ and $\theta = 25, 10, -10, -25$. From (71) and (72), we obtain

$$\psi_w(u) = \sum_{n=1}^{8} \sum_{j=1}^{4} c_{n,j} \mathrm{e}^{-R_n u}, \tag{73}$$

and

$$\begin{aligned}\psi_s(u) = \sum_{n=1}^{8} \sum_{j=1}^{4} \Big[&\sum_{i=1}^{3} \frac{a_{n,j}\xi_i}{\mu_i(\mu_i + \rho_j)(R_n - \mu_i)} (\mathrm{e}^{-\mu_i u} - \mathrm{e}^{-R_n u}) \\ &+ \frac{b_{n,j}}{(\mu + \rho_j)(R_n - \mu)} (\mathrm{e}^{-\mu u} - \mathrm{e}^{-R_n u}) \Big].\end{aligned} \tag{74}$$

Figure 1 (a) and (b) show the behaviors of $\psi_w(u)$ and $\psi_s(u)$ for different dependent parameters $\theta = 25, 10, -10, -25$. It is clearly see the impact of the dependence parameter θ on the ruin probabilities $\psi_w(u)$ and $\psi_s(u)$ from Figure 1 (a) and (b).



(a) (b)

Figure 1 **Influence of the parameter θ. (a) Ruin probabilities due to oscillation $\psi_w(u)$. (b) Ruin probabilities caused by a claim $\psi_s(u)$.**

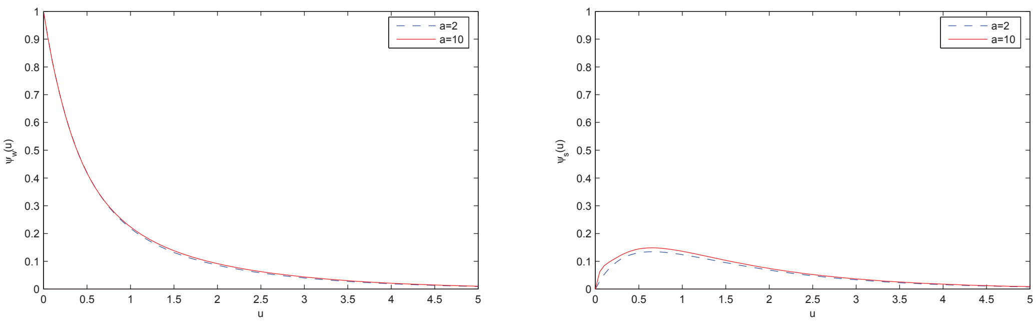
(ii) Set $\mu=1.5$, $\lambda=1$, $p=2$, $\sigma^2/2=1$, $b=c=d=2$, $\theta=10$ and $a=2,10$. From (71) and (72), we have

$$\psi_w(u)=\sum_{n=1}^{a+6}\sum_{j=1}^{4}c_{n,j}\mathrm{e}^{-R_n u}, \tag{75}$$

and

$$\begin{aligned}\psi_s(u)=&\sum_{n=1}^{a+6}\sum_{j=1}^{4}\Big[\sum_{i=1}^{a+1}\frac{a_{n,j}\xi_i}{\mu_i(\mu_i+\rho_j)(R_n-\mu_i)}(\mathrm{e}^{-\mu_i u}-\mathrm{e}^{-R_n u})\\&+\frac{b_{n,j}}{(\mu+\rho_j)(R_n-\mu)}(\mathrm{e}^{-\mu u}-\mathrm{e}^{-R_n u})\Big].\end{aligned} \tag{76}$$

Figure 2 (a) and (b) show the behaviors of $\psi_w(u)$ and $\psi_s(u)$ for different parameters $a=2,10$. It is clearly see the impact of the parameter a on the ruin probabilities $\psi_w(u)$ and $\psi_s(u)$ from Figure 2 (a) and (b).



(a) **(b)**

Figure 2 Influence of the parameter a. (a) Ruin probabilities due to oscillation $\psi_w(u)$. (b) Ruin probabilities caused by a claim $\psi_s(u)$.

(iii) Set $\mu=1.5$, $\lambda=1$, $p=2$, $\sigma^2/2=1$, $a=b=c=2$, $\theta=10$ and $d=2,15$. From (71) and (72), we have

$$\psi_w(u)=\sum_{n=1}^{8}\sum_{j=1}^{4}c_{n,j}\mathrm{e}^{-R_n u}, \tag{77}$$

and

$$\begin{aligned}\psi_s(u)=&\sum_{n=1}^{8}\sum_{j=1}^{4}\Big[\sum_{i=1}^{3}\frac{a_{n,j}\xi_i}{\mu_i(\mu_i+\rho_j)(R_n-\mu_i)}(\mathrm{e}^{-\mu_i u}-\mathrm{e}^{-R_n u})\\&+\frac{b_{n,j}}{(\mu+\rho_j)(R_n-\mu)}(\mathrm{e}^{-\mu u}-\mathrm{e}^{-R_n u})\Big].\end{aligned} \tag{78}$$

Figure 3 (a) and (b) show the behaviors of $\psi_w(u)$ and $\psi_s(u)$ for different parameters $d=2,15$. It is clearly see the impact of the parameter d on the ruin probabilities $\psi_w(u)$ and $\psi_s(u)$ from Figure 3 (a) and (b).

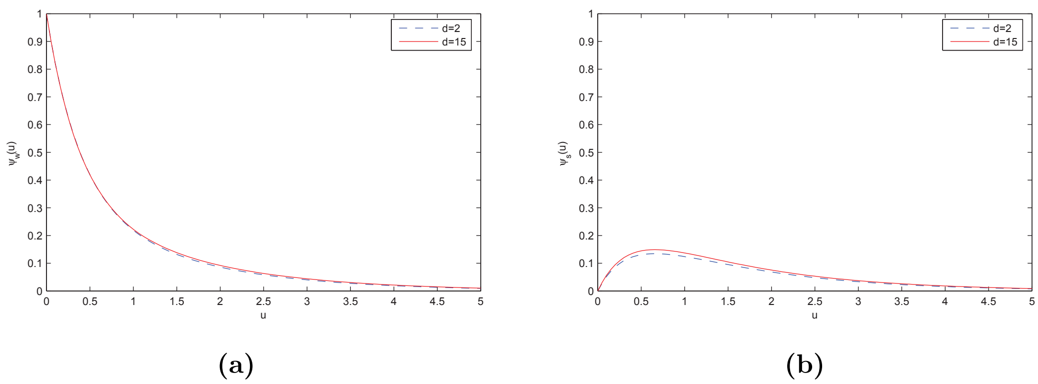


Figure 3 Influence of the parameter d. (a) Ruin probabilities due to oscillation $\psi_w(u)$. (b) Ruin probabilities caused by a claim $\psi_s(u)$.

References

[1] GERBER H U. An extension of the renewal equation and its application in the collective theory of risk [J]. *Scand Actuar J*, 1970, **1970(3-4)**: 205–210.

[2] DUFRESNE F, GERBER H U. Risk theory for the compound Poisson process that is perturbed by diffusion [J]. *Insurance Math Econom*, 1991, **10(1)**: 51–59.

[3] TSAI C C L. On the expectations of the present values of the time of ruin perturbed by diffusion [J]. *Insurance Math Econom*, 2003, **32(3)**: 413–429.

[4] TSAI C C L, WILLMOT G E. A generalized defective renewal equation for the surplus process perturbed by diffusion [J]. *Insurance Math Econom*, 2002, **30(1)**: 51–66.

[5] CHI Y C. Analysis of the expected discounted penalty function for a general jump-diffusion risk model and applications in finance [J]. *Insurance Math Econom*, 2010, **46(2)**: 385–396.

[6] CHI Y C, LIN X S. On the threshold dividend strategy for a generalized jump-diffusion risk model [J]. *Insurance Math Econom*, 2011, **48(3)**: 326–337.

[7] BOUDREAULT M, COSSETTE H, LANDRIAULT D, et al. On a risk model with dependence between interclaim arrivals and claim sizes [J]. *Scand Actuar J*, 2006, **2006(5)**: 265–285.

[8] ALBRECHER H, TEUGELS J L. Exponential behavior in the presence of dependence in risk theory [J]. *J Appl Probab*, 2006, **43(1)**: 257–273.

[9] COSSETTE H, MARCEAU E, MARRI F. On the compound Poisson risk model with dependence based on a generalized Farlie-Gumbel-Morgenstern copula [J]. *Insurance Math Econom*, 2008, **43(3)**: 444–455.

[10] COSSETTE H, MARCEAU E, MARRI F. Analysis of ruin measures for the classical compound Poisson risk model with dependence [J]. *Scand Actuar J*, 2010, **2010(3)**: 221–245.

[11] ZHOU M, CAI J. A perturbed risk model with dependence between premium rates and claim sizes [J]. *Insurance Math Econom*, 2009, **45(3)**: 382–392.

[12] ZHANG Z M, YANG H. Gerber-Shiu analysis in a perturbed risk model with dependence between claim sizes and interclaim times [J]. *J Comput Appl Math*, 2011, **235(5)**: 1189–1204.

[13] ZHANG Z M, YANG H L, YANG H. On a Sparre Andersen risk model with time-dependent claim sizes and jump-diffusion perturbation [J]. *Methodol Comput Appl Probab*, 2012, **14(4)**: 973–995.

[14] RODRÍGUEZ-LALLENA J A, ÚBEDA-FLORES M. A new class of bivariate copulas [J]. *Statist Probab Lett*, 2004, **66(3)**: 315–325.

[15] BORODIN A N, SALMINEN P. *Handbook of Brownian Motion - Facts and Formulae* [M]. 2nd ed. Basel: Birkhäuser, 2002.

[16] DICKSON D C M, HIPP C. On the time to ruin for Erlang(2) risk processes [J]. *Insurance Math Econom*, 2001, **29(3)**: 333–344.

[17] LIN X S, WILLMOT G E. Analysis of a defective renewal equation arising in ruin theory [J]. *Insurance Math Econom*, 1999, **25(1)**: 63–84.

基于广义 FGM Copula 的相依和扰动风险模型下的 Gerber-Shiu 函数分析

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摘　要: 该文考虑了带扰动的相依风险模型, 并以一类广义的 Farlie-Gumbel-Morgenstern copula 定义了索赔额和索赔时间间隔之间的相依结构. 首先, 该模型下期望折扣罚金函数所满足的积分方程、拉普拉斯变换和瑕疵更新方程被给出. 最后当索赔额分布为指数分布时, 给出了期望折扣罚金函数所满足的解析解和破产概率的数值实例.

关键词: 时间相依索赔额; 期望折扣罚金函数; 拉普拉斯变换; 瑕疵更新方程; 破产概率

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