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On Occupation Times for Compound Poisson Risk Model with Two-Step Premium Rate*

ZHANG Aili

(*School of Statistics and Mathematics, Nanjing Audit University, Nanjing, 211815, China*)

LIU Zhang⋆

(*School of Computer and Information Engineering, Jiangxi Agricultural University, Nanchang, 330045, China*)

Abstract: In this paper, we consider the classical compound Poisson risk model with two-step premium rate. Using an alternative approach, we find the explicit expressions for the Laplace transforms of joint occupation times over disjoint intervals for this model. The Laplace transforms are expressed in terms of scale functions of Lévy processes.

Keywords: compound Poisson risk model; joint occupation times; two-step premium rate; scale functions

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§1. Introduction

In the past several years, the researches on the quantity of occupation time and relevant derivatives become interesting topics in mathematical finance and risk theory. The quantity can be used to simulate the amount of time that a stochastic process stays within a certain range. One of the objectives for such topics is to study Laplace transforms of occupation times. Using different approaches, many explicit expressions on Laplace transforms of occupation times have been obtained for various of risk models. For example, Pitman and Yor [1] considered a general diffusion risk model and derived some results on occupation times by applying excursion theory; afterwards, Li and Zhou [2] considered the same risk model and studied the joint Laplace transforms of occupation times by an

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⋆Corresponding author, E-mail: liuzhang1006@163.com.

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alternative perturbation approach. In addition, many results related to occupation times have also been obtained for a class of risk models with jumps. For instance, the analysis of the occupation time of the negative surplus has been considered by Reis [3] for the classical compound Poisson risk model, and by Zhang and Wu [4] for a compound Poisson risk model with diffusion. For other respective investigations on risk models with jumps, see, e.g., [5–10]. For more applications of occupation times in mathematical finance and risk theory, see [11] and [12], etc.

All of the papers mentioned above assumed that the premium rate in risk model is a constant. However, in practice, the insurance company (such as the catastrophic insurance or traffic accident insurance) may adopt a low premium rate to attract customers to bring more profits, when it faces a very good operation environment or sufficient cash flow. Hence in recent years, more and more attention has been paid to two-step premium rate (or variable premium rate) for risk models, which can be used as risk indicator for an insurance portfolio. Asmussen[13; Chapter VII] investigated the risk model with barrier or threshold strategy which may be regarded as a special case of two-step premium rate. Zhang et al. [14] considered the classical risk model with a two-step premium rate. Karnaukh [15] studied the risk process with stochastic income and two-step premium rate. For more financial interpretation and application of the two-step premium rate, see [16, 17], etc.

Recently, there are also many researches involving in the problem of controlled markets, that is, the problem of finding optimal barriers for a class of one-dimensional reflected stochastic differential equations. See, for instance, [18–20], etc. Note that there are many similarities between the studies of these problems and those of occupation times, because both of them concern the exit (time) or boundary problem. As argued in [19], the reflected stochastic differential equation model arises as an important approximating process in a regulated financial market system. In a regulated market, the government would like to implement its macro interventions on the prices of major commodities and services, as well as the domestic interest rates and the foreign exchange rates, so the resulting price dynamics are controlled by the price interval $[a, b]$. On the other hand, there are also some differences in the study of occupation times and controlled markets. The studies of occupation times usually focus on the total amount of time that the surplus of the risk process falls within the range $[0, b]$ so as to monitor the risk of an insurance company, while controlled markets are usually studied to determine the optimal pricing barriers a and b so as to minimize the risk and maximize the expected return in financial market systems. For more practical examples related to regulated markets and their applications, see [21–23] and the references therein.

More recently, Li and Zhou [24] studied the joint Laplace transforms of occupation times of the spectrally negative Lévy processes by adopting a fairly new approach. In their model, the joint Laplace transform is identified with the probability that two independent sequences of Poisson arrival times avoid the time durations when the spectrally negative Lévy processes takes values from interval $(0, a)$ and interval (a, b), respectively, before the corresponding exit time. So inspired by [24] and [19], we consider the compound Poisson risk model with two-step premium rate in this paper, and our objective is to derive the explicit expressions of the joint Laplace transform of occupation times for the model.

The rest of the paper is organized as follows. In Section 2, we present the details of our model and some preliminary outcomes. In Section 3 we then adopt a method similar to [24] to find Laplace transforms of joint occupation times over disjoint intervals for the compound Poisson process with two-step premium rate. Some briefly reviews on the scale functions of Lévy processes and relevant identities for the compound Poisson process are respectively presented in Appendix.

§2. Model Specification and Some Preliminary Outcomes

Consider the classical compound Poisson risk model with two-step premium rate, that is, the process U is given by the following dynamic equations

$$\mathrm{d}U_t = \begin{cases} c_1\mathrm{d}t - \mathrm{d}\Big(\sum\limits_{k=1}^{N(t)} Y_k\Big), & \text{when } U_t \in (-\infty, a]; \\ c_2\mathrm{d}t - \mathrm{d}\Big(\sum\limits_{k=1}^{N(t)} Y_k\Big), & \text{when } U_t \in (a, +\infty), \end{cases} \tag{1}$$

with initial surplus $U_0 = x$, $c_1 > 0$ and $c_2 > 0$ being the premium rates when the dynamic surplus are less than a and greater than a, respectively; $N(t)$ being a Poisson process representing the number of claims up to time t, and $\{Y_k\}_{k\geqslant 1}$ being the sequence of claim amounts with exponential distribution and parameter $\mu > 0$. Note that there are two dynamic parts on the right hand side of (1). The first part $c_1\mathrm{d}t - \mathrm{d}\Big(\sum\limits_{k=1}^{N(t)} Y_k\Big)$ means that process U evolves linearly at rate c_1 between successive claim arrival times when it is below the threshold level a, while the second dynamic part $c_2\mathrm{d}t - \mathrm{d}\Big(\sum\limits_{k=1}^{N(t)} Y_k\Big)$ means that the process U evolves linearly at rate c_2 between successive claim arrival times when it is above the level a. As discussed in Section 1, the second dynamic evolution rate of the process U is not equal to the first one due to different operation environment or cash flow of an insurance company.

In order to give our main results for the model (1), we introduce the first passage time of our interested process U. Define the first up-crossing and down-crossing time of U by

$$\tau_b^+ := \inf\{t \geqslant 0; U_t > b\} \quad \text{and} \quad \tau_c^- := \inf\{t \geqslant 0; U_t < c\}.$$

For $0 \leqslant x \leqslant a$ and $q \geqslant 0$, it's well known that

$$\mathsf{E}_x\big(\mathrm{e}^{-q\tau_a^+}; \tau_a^+ < \tau_0^-\big) = \frac{W_{c_1}^{(q)}(x)}{W_{c_1}^{(q)}(a)} \tag{2}$$

and

$$\mathsf{E}_x\big(\mathrm{e}^{-q\tau_0^-}; \tau_0^- < \tau_a^+\big) = Z_{c_1}^{(q)}(x) - W_{c_1}^{(q)}(x)\frac{Z_{c_1}^{(q)}(a)}{W_{c_1}^{(q)}(a)}, \tag{3}$$

where $Z_c^{(q)}(x)$ and $W_c^{(q)}(x)$ (see (22) and (23) in the Appendix) respectively represent the first and second scale function of the compound Poisson process with Poisson arrival rate $\lambda > 0$, drift coefficients c and an exponential jump size with distribution function $F(x) = 1 - \mathrm{e}^{-\mu x}$, $\mu > 0$.

In this paper, we are interested in the joint Laplace transform of the occupation times of the disjoint sets $(0, a)$ and (a, b) for the process given in (1) prior to it's exit from the set $[0, b]$, which are the two primary objects in this paper. Here, 0 is chosen as the lower boundary because τ_0^- (i.e., the ruin time) is one of the important quantities of an insurance company, and at that time the surplus of the company is negative. On the other hand, the constant $b > 0$ is chosen as the upper boundary because b can be viewed as the threshold level of surplus process of an insurance company. When the surplus is above the threshold level b, insurance companies can be regarded as in a better operating environment because they have sufficient cash flow; on the contrary, when the surplus is lower than b, the insurance company can be regarded as in an early-warning environment. Thus, it's very useful to monitor the occupation time of the surplus process in $[0, b]$ for an insurance company. Now, define

$$f_1(x) := \mathsf{E}_x\Big\{\exp\Big[-q_1\int_0^{\tau_0^-}\mathbf{1}_{(0,a)}(U_s)\mathrm{d}s - q_2\int_0^{\tau_0^-}\mathbf{1}_{(a,b)}(U_s)\mathrm{d}s\Big]; \tau_0^- < \tau_b^+\Big\}$$

and

$$f_2(x) := \mathsf{E}_x\Big\{\exp\Big[-q_1\int_0^{\tau_b^+}\mathbf{1}_{(0,a)}(U_s)\mathrm{d}s - q_2\int_0^{\tau_b^+}\mathbf{1}_{(a,b)}(U_s)\mathrm{d}s\Big]; \tau_0^- > \tau_b^+\Big\}$$

for any $0 < a < b$, $0 \leqslant x \leqslant b$ and $q_1, q_2 \geqslant 0$. Then our objective is to derive the explicit expressions of $f_1(x)$ and $f_2(x)$. Before this, we introduce the following result and it can be seen as a special case in [25] and [26].

Lemma 1 When $c_1 = c_2 = c$, for the compound Poisson risk model in (1), we have

$$\mathsf{E}_x\big(\mathrm{e}^{-q\tau_a^- + \theta U_{\tau_a^-}}; \tau_a^- < \tau_b^+\big) = \mathrm{e}^{\theta a}\Big[Z_c^{(q)}(x-a,\theta) - \frac{Z_c^{(q)}(b-a,\theta)}{W_c^{(q)}(b-a)} W_c^{(q)}(x-a)\Big].$$

§3. Main Results

In this section, we will derive the explicit expressions of $f_1(x)$ and $f_2(x)$ through the q-scale functions.

Theorem 2 For $x \in [0, a]$, we have

$$\begin{aligned} f_1(x) = {} & \frac{A_1(q_1;c_1)}{c_1}\Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)} + \frac{q_1}{\theta_1(q_1;c_1)}\Big]\mathrm{e}^{\theta_1(q_1;c_1)x} \\ & - \frac{A_2(q_1;c_1)}{c_1}\Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)} + \frac{q_1}{\theta_2(q_1;c_1)}\Big]\mathrm{e}^{\theta_2(q_1;c_1)x}, \end{aligned} \tag{4}$$

$$f_2(x) = \frac{W_{c_1}^{(q_1)}(x)}{W_{c_1}^{(q_1)}(a)} f_2(a). \tag{5}$$

For $x \in [a, b]$, we have

$$\begin{aligned} f_1(x) = {} & \frac{A_1(q_1;c_1)}{c_1}\Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)} + \frac{q_1}{\theta_1(q_1;c_1)}\Big] \\ & \times \mathrm{e}^{\theta_1(q_1;c_1)a}\Big[Z_{c_2}^{(q_2)}(x-a,\theta_1(q_1;c_1)) - \frac{Z_{c_2}^{(q_2)}(b-a,\theta_1(q_1;c_1))}{W_{c_2}^{(q_2)}(b-a)} W_{c_2}^{(q_2)}(x-a)\Big] \\ & - \frac{A_2(q_1;c_1)}{c_1}\Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)} + \frac{q_1}{\theta_2(q_1;c_1)}\Big] \\ & \times \mathrm{e}^{\theta_2(q_1;c_1)a}\Big[Z_{c_2}^{(q_2)}(x-a,\theta_2(q_1;c_1)) - \frac{Z_{c_2}^{(q_2)}(b-a,\theta_2(q_1;c_1))}{W_{c_2}^{(q_2)}(b-a)} W_{c_2}^{(q_2)}(x-a)\Big], \end{aligned} \tag{6}$$

$$\begin{aligned} f_2(x) = {} & \frac{A_1(q_1;c_1)}{c_1}\frac{f_2(a)\,\mathrm{e}^{\theta_1(q_1;c_1)a}}{W_{c_1}^{(q_1)}(a)} \\ & \times \Big[Z_{c_2}^{(q_2)}(x-a,\theta_1(q_1;c_1)) - \frac{Z_{c_2}^{(q_2)}(b-a,\theta_1(q_1;c_1))}{W_{c_2}^{(q_2)}(b-a)} W_{c_2}^{(q_2)}(x-a)\Big] \\ & - \frac{A_2(q_1;c_1)}{c_1}\frac{f_2(a)\,\mathrm{e}^{\theta_2(q_1;c_1)a}}{W_{c_1}^{(q_1)}(a)} \\ & \times \Big[Z_{c_2}^{(q_2)}(x-a,\theta_2(q_1;c_1)) - \frac{Z_{c_2}^{(q_2)}(b-a,\theta_2(q_1;c_1))}{W_{c_2}^{(q_2)}(b-a)} W_{c_2}^{(q_2)}(x-a)\Big] \\ & + \frac{W_{c_2}^{(q_2)}(x-a)}{W_{c_2}^{(q_2)}(b-a)}. \end{aligned} \tag{7}$$

Here, the expressions of $f_1(a)$ and $f_2(a)$ are given by the following (12) and (17), respectively.

Proof We start with the proof for (4) and (6). Actually, using the strong Markov property and by (2) and (3), we have for $x \in [0, a]$

$$
\begin{aligned}
f_1(x) &= \mathsf{E}_x\big(\mathrm{e}^{-q_1\int_0^{\tau_0^-}\mathbf{1}_{(0,a)}(U_s)\mathrm{d}s - q_2\int_0^{\tau_0^-}\mathbf{1}_{(a,b)}(U_s)\mathrm{d}s}; \tau_a^+ < \tau_0^- < \tau_b^+\big) \\
&\quad + \mathsf{E}_x\big(\mathrm{e}^{-q_1\int_0^{\tau_0^-}\mathbf{1}_{(0,a)}(U_s)\mathrm{d}s - q_2\int_0^{\tau_0^-}\mathbf{1}_{(a,b)}(U_s)\mathrm{d}s}; \tau_0^- < \tau_a^+\big) \\
&= \mathsf{E}_x\big(\mathrm{e}^{-q_1\tau_a^+}; \tau_a^+ < \tau_0^-\big) f_1(a) + \mathsf{E}_x\big(\mathrm{e}^{-q_1\tau_0^-}; \tau_0^- < \tau_a^+\big) \\
&= \frac{W_{c_1}^{(q_1)}(x)}{W_{c_1}^{(q_1)}(a)} f_1(a) + Z_{c_1}^{(q_1)}(x) - Z_{c_1}^{(q_1)}(a)\frac{W_{c_1}^{(q_1)}(x)}{W_{c_1}^{(q_1)}(a)} \\
&= \Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}\Big] W_{c_1}^{(q_1)}(x) + Z_{c_1}^{(q_1)}(x) \\
&= \frac{A_1(q_1;c_1)}{c_1}\Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)} + \frac{q_1}{\theta_1(q_1;c_1)}\Big]\mathrm{e}^{\theta_1(q_1;c_1)x} \\
&\quad - \frac{A_2(q_1;c_1)}{c_1}\Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)} + \frac{q_1}{\theta_2(q_1;c_1)}\Big]\mathrm{e}^{\theta_2(q_1;c_1)x}.
\end{aligned} \tag{8}
$$

Meanwhile, note that (8) holds by definition for $x < 0$, then for any $x \in [a, b]$, we have

$$
\begin{aligned}
f_1(x) &= \mathsf{E}_x\big(\mathrm{e}^{-q_1\int_0^{\tau_0^-}\mathbf{1}_{(0,a)}(U_s)\mathrm{d}s - q_2\int_0^{\tau_0^-}\mathbf{1}_{(a,b)}(U_s)\mathrm{d}s}; \tau_a^- \leqslant \tau_0^- < \tau_b^+\big) \\
&= \mathsf{E}_x\big[\mathrm{e}^{-q_2\tau_a^-} f_1(U_{\tau_a^-}); \tau_a^- < \tau_b^+\big] \\
&= \Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}\Big]\mathsf{E}_x\big[\mathrm{e}^{-q_2\tau_a^-} W_{c_1}^{(q_1)}(U_{\tau_a^-}); \tau_a^- < \tau_b^+\big] \\
&\quad + \mathsf{E}_x\big[\mathrm{e}^{-q_2\tau_a^-} Z_{c_1}^{(q_1)}(U_{\tau_a^-}); \tau_a^- < \tau_b^+\big].
\end{aligned}
$$

Hence for $x \in [a, b]$, by (22), (23) and Lemma 1, we have

$$
\begin{aligned}
f_1(x) &= \Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}\Big]\mathsf{E}_x\big[\mathrm{e}^{-q_2\tau_a^-} W_{c_1}^{(q_1)}(U_{\tau_a^-}); \tau_a^- < \tau_b^+\big] \\
&\quad + \mathsf{E}_x\big[\mathrm{e}^{-q_2\tau_a^-} Z_{c_1}^{(q_1)}(U_{\tau_a^-}); \tau_a^- < \tau_b^+\big] \\
&= \Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}\Big]\Big[\frac{A_1(q_1;c_1)}{c_1}\mathsf{E}_x\big(\mathrm{e}^{-q_2\tau_a^-}\mathrm{e}^{\theta_1(q_1;c_1)U_{\tau_a^-}}; \tau_a^- < \tau_b^+\big) \\
&\quad - \frac{A_2(q_1;c_1)}{c_1}\mathsf{E}_x\big(\mathrm{e}^{-q_2\tau_a^-}\mathrm{e}^{\theta_2(q_1;c_1)U_{\tau_a^-}}; \tau_a^- < \tau_b^+\big)\Big] \\
&\quad + \frac{q_1 A_1(q_1;c_1)}{c_1\theta_1(q_1;c_1)}\mathsf{E}_x\big(\mathrm{e}^{-q_2\tau_a^-}\mathrm{e}^{\theta_1(q_1;c_1)U_{\tau_a^-}}; \tau_a^- < \tau_b^+\big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{q_1 A_2(q_1;c_1)}{c_1\theta_2(q_1;c_1)}\mathsf{E}_x\big(\mathrm{e}^{-q_2\tau_a^-}\mathrm{e}^{\theta_2(q_1;c_1)U_{\tau_a^-}};\tau_a^-<\tau_b^+\big)\\
=&\frac{A_1(q_1;c_1)}{c_1}\Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)}-\frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}+\frac{q_1}{\theta_1(q_1;c_1)}\Big]\mathsf{E}_x\big(\mathrm{e}^{-q_2\tau_a^-}\mathrm{e}^{\theta_1(q_1;c_1)U_{\tau_a^-}};\tau_a^-<\tau_b^+\big)\\
&-\frac{A_2(q_1;c_1)}{c_1}\Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)}-\frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}+\frac{q_1}{\theta_2(q_1;c_1)}\Big]\mathsf{E}_x\big(\mathrm{e}^{-q_2\tau_a^-}\mathrm{e}^{\theta_2(q_1;c_1)U_{\tau_a^-}};\tau_a^-<\tau_b^+\big)\\
=&\frac{A_1(q_1;c_1)}{c_1}\Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)}-\frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}+\frac{q_1}{\theta_1(q_1;c_1)}\Big]\\
&\times \mathrm{e}^{\theta_1(q_1;c_1)a}\Big[Z_{c_2}^{(q_2)}(x-a,\theta_1(q_1;c_1))-\frac{Z_{c_2}^{(q_2)}(b-a,\theta_1(q_1;c_1))}{W_{c_2}^{(q_2)}(b-a)}W_{c_2}^{(q_2)}(x-a)\Big]\\
&-\frac{A_2(q_1;c_1)}{c_1}\Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)}-\frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}+\frac{q_1}{\theta_2(q_1;c_1)}\Big]\\
&\times \mathrm{e}^{\theta_2(q_1;c_1)a}\Big[Z_{c_2}^{(q_2)}(x-a,\theta_2(q_1;c_1))-\frac{Z_{c_2}^{(q_2)}(b-a,\theta_2(q_1;c_1))}{W_{c_2}^{(q_2)}(b-a)}W_{c_2}^{(q_2)}(x-a)\Big]. \qquad (9)
\end{aligned}
$$

Equations (24) and (9) give iterative formulas of $f_1(x)$. In order to obtain an explicit formula for $f_1(x)$, we should get the explicit expression for $f_1(a)$. By conditioning on the first Poisson arrival time and the first claim size, we get

$$
\begin{aligned}
f_1(a)=&\int_0^{(b-a)/c_2}\lambda\,\mathrm{e}^{-\lambda t}\mathrm{d}t\Big[\int_0^{c_2t}\mu\,\mathrm{e}^{-\mu y}\mathrm{e}^{-q_2t}f_1(a+c_2t-y)\mathrm{d}y\\
&+\int_{c_2t}^{a+c_2t}\mu\,\mathrm{e}^{-\mu y}\mathrm{e}^{-q_2t}f_1(a+c_2t-y)\mathrm{d}y+\int_{a+c_2t}^{\infty}\mu\,\mathrm{e}^{-\mu y}\mathrm{e}^{-q_2t}\mathrm{d}y\Big]\\
=&\int_0^{(b-a)/c_2}\lambda\,\mathrm{e}^{-(\lambda+q_2)t}\mathrm{d}t\Big[\int_0^{c_2t}\mu\,\mathrm{e}^{-\mu y}f_1(a+c_2t-y)\mathrm{d}y\\
&+\int_{c_2t}^{a+c_2t}\mu\,\mathrm{e}^{-\mu y}f_1(a+c_2t-y)\mathrm{d}y+\mathrm{e}^{-\mu(a+c_2t)}\Big]. \qquad (10)
\end{aligned}
$$

Define

$$
\begin{aligned}
&B_i(q_1;c_1)\\
=&\,\mathrm{e}^{\theta_i(q_1;c_1)a}\int_0^{(b-a)/c_2}\mathrm{e}^{-(\lambda+q_2)t}\mathrm{d}t\\
&\times\int_0^{c_2t}\mathrm{e}^{-\mu y}\Big[Z_{c_2}^{(q_2)}(c_2t-y,\theta_i(q_1;c_1))-\frac{Z_{c_2}^{(q_2)}(b-a,\theta_i(q_1;c_1))}{W_{c_2}^{(q_2)}(b-a)}W_{c_2}^{(q_2)}(c_2t-y)\Big]\mathrm{d}y\\
&+\int_0^{(b-a)/c_2}\mathrm{e}^{-(\lambda+q_2)t}\mathrm{d}t\int_{c_2t}^{a+c_2t}\mathrm{e}^{-\mu y}\mathrm{e}^{\theta_i(q_1;c_1)(a+c_2t-y)}\mathrm{d}y\\
=&\,\mathrm{e}^{\theta_i(q_1;c_1)a}\Big\{\Big\{1-\frac{[q_2-\psi(\theta_i(q_1;c_1))]A_1(q_2;c_2)}{c_2[\theta_1(q_2;c_2)-\theta_i(q_1;c_1)]}+\frac{[q_2-\psi(\theta_i(q_1;c_1))]A_2(q_2;c_2)}{c_2[\theta_2(q_2;c_2)-\theta_i(q_1;c_1)]}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big\{\frac{\mathrm{e}^{-(\lambda+q_2+\mu c_2)(b-a)/c_2}-1}{(\lambda+q_2+\mu c_2)[\mu+\theta_i(q_1;c_1)]}-\frac{\mathrm{e}^{-[\lambda+q_2-\theta_i(q_1;c_1)c_2](b-a)/c_2}-1}{[\lambda+q_2-\theta_i(q_1;c_1)c_2][\mu+\theta_i(q_1;c_1)]}\Big\}\\
&+\frac{[q_2-\psi(\theta_i(q_1;c_1))]A_1(q_2;c_2)}{c_2[\theta_1(q_2;c_2)-\theta_i(q_1;c_1)]}\\
&\times\Big\{\frac{\mathrm{e}^{-(\lambda+q_2+\mu c_2)(b-a)/c_2}-1}{(\lambda+q_2+\mu c_2)[\mu+\theta_1(q_2;c_2)]}-\frac{\mathrm{e}^{-[\lambda+q_2-\theta_1(q_2;c_2)c_2](b-a)/c_2}-1}{[\lambda+q_2-\theta_1(q_2;c_2)c_2][\mu+\theta_1(q_2;c_2)]}\Big\}\\
&-\frac{[q_2-\psi(\theta_i(q_1;c_1))]A_2(q_2;c_2)}{c_2[\theta_2(q_2;c_2)-\theta_i(q_1;c_1)]}\\
&\times\Big\{\frac{\mathrm{e}^{-(\lambda+q_2+\mu c_2)(b-a)/c_2}-1}{(\lambda+q_2+\mu c_2)[\mu+\theta_2(q_2;c_2)]}-\frac{\mathrm{e}^{-[\lambda+q_2-\theta_2(q_2;c_2)c_2](b-a)/c_2}-1}{[\lambda+q_2-\theta_2(q_2;c_2)c_2][\mu+\theta_2(q_2;c_2)]}\Big\}\Big\}\\
&-\mathrm{e}^{\theta_i(q_1;c_1)a}\frac{Z_{c_2}^{(q_2)}(b-a,\theta_i(q_1;c_1))}{W_{c_2}^{(q_2)}(b-a)}\\
&\times\Big\{\frac{A_1(q_2;c_2)}{c_2[-\mu-\theta_1(q_2;c_2)]}\Big\{\frac{\mathrm{e}^{-(\lambda+q_2+\mu c_2)(b-a)/c_2}-1}{-(\lambda+q_2+\mu c_2)}-\frac{\mathrm{e}^{-[\lambda+q_2-\theta_1(q_2;c_2)c_2](b-a)/c_2}-1}{-[\lambda+q_2-\theta_1(q_2;c_2)c_2]}\Big\}\\
&-\frac{A_2(q_2;c_2)}{c_2[-\mu-\theta_2(q_2;c_2)]}\Big\{\frac{\mathrm{e}^{-(\lambda+q_2+\mu c_2)(b-a)/c_2}-1}{-(\lambda+q_2+\mu c_2)}-\frac{\mathrm{e}^{-[\lambda+q_2-\theta_2(q_2;c_2)c_2](b-a)/c_2}-1}{-[\lambda+q_2-\theta_2(q_2;c_2)c_2]}\Big\}\Big\}\\
&+\frac{(\mathrm{e}^{-\mu a}-\mathrm{e}^{\theta_i(q_1;c_1)a})(\mathrm{e}^{-(\lambda+q_2+\mu c_2)(b-a)/c_2}-1)}{[\mu+\theta_i(q_1;c_1)](\lambda+q_2+\mu c_2)},\qquad i=1,2.
\end{aligned}
$$

Then, equation (10) can be rewritten as

$$
\begin{aligned}
&f_1(a)\Big\{1-\Big[\lambda\mu\frac{A_1(q_1;c_1)}{c_1W_{c_1}^{(q_1)}(a)}B_1(q_1;c_1)-\lambda\mu\frac{A_2(q_1;c_1)}{c_1W_{c_1}^{(q_1)}(a)}B_2(q_1;c_1)\Big]\Big\}\\
&=\lambda\mu\frac{A_1(q_1;c_1)}{c_1}\Big[-\frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}+\frac{q_1}{\theta_1(q_1;c_1)}\Big]B_1(q_1;c_1)\\
&\quad-\lambda\mu\frac{A_2(q_1;c_1)}{c_1}\Big[-\frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}+\frac{q_1}{\theta_2(q_1;c_1)}\Big]B_2(q_1;c_1)\\
&\quad-\lambda\mathrm{e}^{\mu a}\frac{\mathrm{e}^{-(\lambda+q_2+\mu c_2)(b-a)/c_2}-1}{\lambda+q_2+\mu c_2}.
\end{aligned}\tag{11}
$$

Solving (11), we obtain

$$
f_1(a)=\frac{\ell_1(c_1,c_2,q_1,q_2)}{\ell_2(c_1,c_2,q_1,q_2)},\tag{12}
$$

where

$$
\ell_2(c_1,c_2,q_1,q_2)=1-\Big[\lambda\mu\frac{A_1(q_1;c_1)}{c_1W_{c_1}^{(q_1)}(a)}B_1(q_1;c_1)-\lambda\mu\frac{A_2(q_1;c_1)}{c_1W_{c_1}^{(q_1)}(a)}B_2(q_1;c_1)\Big],
$$

and

$$
\ell_1(c_1,c_2,q_1,q_2)=\lambda\mu\frac{A_1(q_1;c_1)}{c_1}\Big[-\frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}+\frac{q_1}{\theta_1(q_1;c_1)}\Big]B_1(q_1;c_1)
$$

$$
\begin{aligned}
&- \lambda\mu \frac{A_2(q_1;c_1)}{c_1}\Big[- \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)} + \frac{q_1}{\theta_2(q_1;c_1)}\Big] B_2(q_1;c_1) \\
&- \lambda \mathrm{e}^{\mu a} \frac{\mathrm{e}^{-(\lambda+q_2+\mu c_2)(b-a)/c_2} - 1}{\lambda + q_2 + \mu c_2}.
\end{aligned}
$$

Combining (24), (9) and (12), we can finally get the solution of $f_1(x)$ given by (4) and (6).

We now proceed to show (5) and (7). Again using the strong Markov property and by (2), we have for $x \in [0, a]$,

$$
\begin{aligned}
f_2(x) &= \mathsf{E}_x\Big\{ \exp\Big[- q_1 \int_0^{\tau_b^+} \mathbf{1}_{(0,a)}(U_s)\mathrm{d}s - q_2 \int_0^{\tau_b^+} \mathbf{1}_{(a,b)}(U_s)\mathrm{d}s \Big]; \tau_0^- > \tau_b^+ \Big\} \\
&= \mathsf{E}_x\Big\{ \exp\Big[- q_1 \int_0^{\tau_b^+} \mathbf{1}_{(0,a)}(U_s)\mathrm{d}s - q_2 \int_0^{\tau_b^+} \mathbf{1}_{(a,b)}(U_s)\mathrm{d}s \Big]; \tau_0^- > \tau_b^+ > \tau_a^+ \Big\} \\
&= \mathsf{E}_x\big(\mathrm{e}^{-q_1 \tau_a^+}; \tau_a^+ < \tau_0^- \big) f_2(a) \\
&= \frac{W_{c_1}^{(q_1)}(x)}{W_{c_1}^{(q_1)}(a)} f_2(a). \qquad (13)
\end{aligned}
$$

For $x \in [a, b]$, by (22), (23) and Lemma 1, we have

$$
\begin{aligned}
f_2(x) &= \mathsf{E}_x\Big\{ \exp\Big[- q_1 \int_0^{\tau_b^+} \mathbf{1}_{(0,a)}(U_s)\mathrm{d}s - q_2 \int_0^{\tau_b^+} \mathbf{1}_{(a,b)}(U_s)\mathrm{d}s \Big]; \tau_0^- > \tau_b^+ \Big\} \\
&= \mathsf{E}_x\Big\{ \exp\Big[- q_1 \int_0^{\tau_b^+} \mathbf{1}_{(0,a)}(U_s)\mathrm{d}s - q_2 \int_0^{\tau_b^+} \mathbf{1}_{(a,b)}(U_s)\mathrm{d}s \Big]; \tau_0^- > \tau_b^+ > \tau_a^- \Big\} \\
&\quad + \mathsf{E}_x\Big\{ \exp\Big[- q_1 \int_0^{\tau_b^+} \mathbf{1}_{(0,a)}(U_s)\mathrm{d}s - q_2 \int_0^{\tau_b^+} \mathbf{1}_{(a,b)}(U_s)\mathrm{d}s \Big]; \tau_0^- \geqslant \tau_a^- > \tau_b^+ \Big\} \\
&= \frac{f_2(a)}{W_{c_1}^{(q_1)}(a)} \mathsf{E}_x\big(\mathrm{e}^{-q_2 \tau_a^-} W_{c_1}^{(q_1)}(U_{\tau_a^-}); \tau_a^- < \tau_b^+ \big) + \frac{W_{c_2}^{(q_2)}(x-a)}{W_{c_2}^{(q_2)}(b-a)} \\
&= \frac{f_2(a)}{W_{c_1}^{(q_1)}(a)} \mathsf{E}_x\Big\{ \mathrm{e}^{-q_2 \tau_a^-} \Big[\frac{A_1(q_1;c_1)}{c_1} \mathrm{e}^{\theta_1(q_1;c_1) U_{\tau_a^-}} - \frac{A_2(q_1;c_1)}{c_1} \mathrm{e}^{\theta_2(q_1;c_1) U_{\tau_a^-}} \Big]; \tau_a^- < \tau_b^+ \Big\} \\
&\quad + \frac{W_{c_2}^{(q_2)}(x-a)}{W_{c_2}^{(q_2)}(b-a)} \\
&= \frac{A_1(q_1;c_1)}{c_1} \frac{f_2(a)}{W_{c_1}^{(q_1)}(a)} \mathsf{E}_x\big(\mathrm{e}^{-q_2 \tau_a^-} \mathrm{e}^{\theta_1(q_1;c_1) U_{\tau_a^-}}; \tau_a^- < \tau_b^+ \big) \\
&\quad - \frac{A_2(q_1;c_1)}{c_1} \frac{f_2(a)}{W_{c_1}^{(q_1)}(a)} \mathsf{E}_x\big(\mathrm{e}^{-q_2 \tau_a^-} \mathrm{e}^{\theta_2(q_1;c_1) U_{\tau_a^-}}; \tau_a^- < \tau_b^+ \big) + \frac{W_{c_2}^{(q_2)}(x-a)}{W_{c_2}^{(q_2)}(b-a)} \\
&= \frac{A_1(q_1;c_1)}{c_1} \frac{f_2(a)\, \mathrm{e}^{\theta_1(q_1;c_1) a}}{W_{c_1}^{(q_1)}(a)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big[Z_{c_2}^{(q_2)}(x-a,\theta_1(q_1;c_1)) - \frac{Z_{c_2}^{(q_2)}(b-a,\theta_1(q_1;c_1))}{W_{c_2}^{(q_2)}(b-a)} W_{c_2}^{(q_2)}(x-a)\Big] \\
&- \frac{A_2(q_1;c_1)}{c_1} \frac{f_2(a)\,\mathrm{e}^{\theta_2(q_1;c_1)a}}{W_{c_1}^{(q_1)}(a)} \\
&\times \Big[Z_{c_2}^{(q_2)}(x-a,\theta_2(q_1;c_1)) - \frac{Z_{c_2}^{(q_2)}(b-a,\theta_2(q_1;c_1))}{W_{c_2}^{(q_2)}(b-a)} W_{c_2}^{(q_2)}(x-a)\Big] \\
&+ \frac{W_{c_2}^{(q_2)}(x-a)}{W_{c_2}^{(q_2)}(b-a)}. \qquad (14)
\end{aligned}
$$

Equations (13) and (14) give iterative formulas of $f_2(x)$. In order to obtain an explicit formula for $f_2(x)$, we should get the explicit expression for $f_2(a)$. By conditioning on the first Poisson arrival time and the first claim size again we get

$$
\begin{aligned}
f_2(a) &= \int_0^{(b-a)/c_2} \lambda \mathrm{e}^{-\lambda t}\mathrm{d}t \Big[\int_0^{c_2 t} \mu \mathrm{e}^{-\mu y}\mathrm{e}^{-q_2 t} f_2(a+c_2 t-y)\mathrm{d}y \\
&\quad + \int_{c_2 t}^{a+c_2 t} \mu \mathrm{e}^{-\mu y}\mathrm{e}^{-q_2 t} f_2(a+c_2 t-y)\mathrm{d}y\Big] + \mathrm{e}^{-q_2(b-a)/c_2} \int_{(b-a)/c_2}^{\infty} \lambda \mathrm{e}^{-\lambda t}\mathrm{d}t \\
&= \int_0^{(b-a)/c_2} \lambda \mathrm{e}^{-(\lambda+q_2)t}\mathrm{d}t \Big[\int_0^{c_2 t} \mu \mathrm{e}^{-\mu y} f_2(a+c_2 t-y)\mathrm{d}y \\
&\quad + \int_{c_2 t}^{a+c_2 t} \mu \mathrm{e}^{-\mu y} f_2(a+c_2 t-y)\mathrm{d}y\Big] + \mathrm{e}^{-(q_2+\lambda)(b-a)/c_2}. \qquad (15)
\end{aligned}
$$

It can be checked that

$$
\begin{aligned}
D_i(q_2;c_2) &:= \int_0^{(b-a)/c_2} \mathrm{e}^{-(\lambda+q_2)t}\mathrm{d}t \int_0^{c_2 t} \mathrm{e}^{-\mu y}\mathrm{e}^{\theta_i(q_2;c_2)(c_2 t-y)}\mathrm{d}y \\
&= \frac{1}{\mu+\theta_i(q_2;c_2)}\Big[\frac{\mathrm{e}^{-[\lambda+q_2-\theta_i(q_2;c_2)c_2](b-a)/c_2}-1}{-(\lambda+q_2)+\theta_i(q_2;c_2)c_2} - \frac{\mathrm{e}^{-(\lambda+q_2+\mu c_2)(b-a)/c_2}-1}{-(\lambda+q_2)-\mu c_2}\Big], \\
&\qquad i=1,2.
\end{aligned}
$$

Then, recalling the definition of $B_1(q_1;c_1)$ and $B_2(q_1;c_1)$ given right below (10), one can rewrite equation (15) as

$$
\begin{aligned}
& f_2(a)\Big\{1 - \frac{\lambda\mu}{c_1 W_{c_1}^{(q_1)}(a)}[A_1(q_1;c_1)B_1(q_1;c_1) - A_2(q_1;c_1)B_2(q_1;c_1)]\Big\} \\
&= \frac{\lambda\mu}{c_2 W_{c_2}^{(q_2)}(b-a)}[A_1(q_2;c_2)D_1(q_2;c_2) - A_2(q_2;c_2)D_2(q_2;c_2)] + \mathrm{e}^{-(\lambda+q_2)(b-a)/c_2}. \qquad (16)
\end{aligned}
$$

Solving (16), we arrive at

$$
f_2(a) = \frac{\ell_3(c_1,c_2,q_1,q_2)}{\ell_4(c_1,c_2,q_1,q_2)}, \qquad (17)
$$

where

$$\ell_4(c_1,c_2,q_1,q_2)=1-\frac{\lambda\mu}{c_1W_{c_1}^{(q_1)}(a)}[A_1(q_1;c_1)B_1(q_1;c_1)-A_2(q_1;c_1)B_2(q_1;c_1)],$$

and

$$\begin{aligned}\ell_3(c_1,c_2,q_1,q_2)=&\frac{\lambda\mu}{c_2W_{c_2}^{(q_2)}(b-a)}[A_1(q_2;c_2)D_1(q_2;c_2)-A_2(q_2;c_2)D_2(q_2;c_2)]\\&+\mathrm{e}^{-(\lambda+q_2)(b-a)/c_2}.\end{aligned}$$

Finally, combining (13), (14) and (16) yields the desired expressions for $f_2(x)$ given by (5) and (7). $\square$

Remark 3 When $c_1=c_2$, the surplus process U in (1) agrees with the classical compound Poisson risk model with a constant premium rate, and so Theorem 2 provides some new expressions of the Laplace transforms of the joint occupation times, which can be seen as an extension of a special case of spectrally negative Lévy processes discussed in [24], or an extension of a special case (the tax rate $\gamma(x)\equiv 0$) of classical compound Poisson risk model with tax discussed in [10].

We conclude this paper with the following corollary. Define

$$f_1^{\infty}(x):=\mathsf{E}_x\Big\{\exp\Big[-q_1\int_0^{\tau_0^-}\mathbf{1}_{(0,a)}(U_s)\mathrm{d}s-q_2\int_0^{\tau_0^-}\mathbf{1}_{(a,\infty)}(U_s)\mathrm{d}s\Big];\tau_0^-<\infty\Big\},$$

then the following result gives the expression of joint Laplace transform of the occupation times of the disjoint sets $(0,a)$ and (a,∞) for the process given in (1).

Corollary 4 For any $x,a>0$ and $0<x\leqslant a$, we have

$$\begin{aligned}f_1^{\infty}(x)=&\frac{A_1(q_1;c_1)}{c_1}\Big[\frac{f_1^{\infty}(a)}{W_{c_1}^{(q_1)}(a)}-\frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}+\frac{q_1}{\theta_1(q_1;c_1)}\Big]\mathrm{e}^{\theta_1(q_1;c_1)x}\\&-\frac{A_2(q_1;c_1)}{c_1}\Big[\frac{f_1^{\infty}(a)}{W_{c_1}^{(q_1)}(a)}-\frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}+\frac{q_1}{\theta_2(q_1;c_1)}\Big]\mathrm{e}^{\theta_2(q_1;c_1)x},\end{aligned}\tag{18}$$

and for $x>a$, we have

$$\begin{aligned}f_1^{\infty}(x)=&\frac{A_1(q_1;c_1)}{c_1}\Big[\frac{f_1^{\infty}(a)}{W_{c_1}^{(q_1)}(a)}-\frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}+\frac{q_1}{\theta_1(q_1;c_1)}\Big]\\&\times\mathrm{e}^{\theta_1(q_1;c_1)a}\Big[Z_{c_2}^{(q_2)}(x-a,\theta_1(q_1;c_1))-\frac{q_2-\psi(\theta_1(q_1;c_1))}{\theta_2(q_2;c_2)-\theta_1(q_1;c_1)}W_{c_2}^{(q_2)}(x-a)\Big]\\&-\frac{A_2(q_1;c_1)}{c_1}\Big[\frac{f_1^{\infty}(a)}{W_{c_1}^{(q_1)}(a)}-\frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}+\frac{q_1}{\theta_2(q_1;c_1)}\Big]\end{aligned}$$

$$\times\, \mathrm{e}^{\theta_2(q_1;c_1)a}\Big[Z_{c_2}^{(q_2)}(x-a,\theta_2(q_1;c_1)) - \frac{q_2-\psi(\theta_2(q_1;c_1))}{\theta_2(q_2;c_2)-\theta_2(q_1;c_1)}W_{c_2}^{(q_2)}(x-a)\Big]. \quad (19)$$

Here, the expression of $f_1^\infty(a)$ is given by the following (21).

Proof First, by (4) and letting $b\to\infty$, we have

$$\begin{aligned} f_1^\infty(x) &= \lim_{b\to\infty}\Big\{\frac{A_1(q_1;c_1)}{c_1}\Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)} + \frac{q_1}{\theta_1(q_1;c_1)}\Big]\mathrm{e}^{\theta_1(q_1;c_1)x} \\ &\quad - \frac{A_2(q_1;c_1)}{c_1}\Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)} + \frac{q_1}{\theta_2(q_1;c_1)}\Big]\mathrm{e}^{\theta_2(q_1;c_1)x}\Big\} \\ &= \frac{A_1(q_1;c_1)}{c_1}\Big[\frac{f_1^\infty(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)} + \frac{q_1}{\theta_1(q_1;c_1)}\Big]\mathrm{e}^{\theta_1(q_1;c_1)x} \\ &\quad - \frac{A_2(q_1;c_1)}{c_1}\Big[\frac{f_1^\infty(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)} + \frac{q_1}{\theta_2(q_1;c_1)}\Big]\mathrm{e}^{\theta_2(q_1;c_1)x}. \end{aligned} \quad (20)$$

Next, by letting $b\to\infty$ in (6) and by (22), (23) together with some algebras, we have

$$\begin{aligned} f_1^\infty(x) = \lim_{b\to\infty}&\Big\{\frac{A_1(q_1;c_1)}{c_1}\Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)} + \frac{q_1}{\theta_1(q_1;c_1)}\Big] \\ &\times \mathrm{e}^{\theta_1(q_1;c_1)a}\Big[Z_{c_2}^{(q_2)}(x-a,\theta_1(q_1;c_1)) - \frac{Z_{c_2}^{(q_2)}(b-a,\theta_1(q_1;c_1))}{W_{c_2}^{(q_2)}(b-a)}W_{c_2}^{(q_2)}(x-a)\Big] \\ &- \frac{A_2(q_1;c_1)}{c_1}\Big[\frac{f_1(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)} + \frac{q_1}{\theta_2(q_1;c_1)}\Big] \\ &\times \mathrm{e}^{\theta_2(q_1;c_1)a}\Big[Z_{c_2}^{(q_2)}(x-a,\theta_2(q_1;c_1)) - \frac{Z_{c_2}^{(q_2)}(b-a,\theta_2(q_1;c_1))}{W_{c_2}^{(q_2)}(b-a)}W_{c_2}^{(q_2)}(x-a)\Big]\Big\} \\ = \frac{A_1(q_1;c_1)}{c_1}&\Big[\frac{f_1^\infty(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)} + \frac{q_1}{\theta_1(q_1;c_1)}\Big] \\ &\times \mathrm{e}^{\theta_1(q_1;c_1)a}\Big[Z_{c_2}^{(q_2)}(x-a,\theta_1(q_1;c_1)) - \frac{q_2-\psi(\theta_1(q_1;c_1))}{\theta_2(q_2;c_2)-\theta_1(q_1;c_1)}W_{c_2}^{(q_2)}(x-a)\Big] \\ &- \frac{A_2(q_1;c_1)}{c_1}\Big[\frac{f_1^\infty(a)}{W_{c_1}^{(q_1)}(a)} - \frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)} + \frac{q_1}{\theta_2(q_1;c_1)}\Big] \\ &\times \mathrm{e}^{\theta_2(q_1;c_1)a}\Big[Z_{c_2}^{(q_2)}(x-a,\theta_2(q_1;c_1)) - \frac{q_2-\psi(\theta_2(q_1;c_1))}{\theta_2(q_2;c_2)-\theta_2(q_1;c_1)}W_{c_2}^{(q_2)}(x-a)\Big], \end{aligned}$$

where $f_1^\infty(a)$ is given by

$$\begin{aligned} f_1^\infty(a) &= \lim_{b\to\infty}\frac{\ell_1(c_1,c_2,q_1,q_2)}{\ell_2(c_1,c_2,q_1,q_2)} \\ &= \Big\{1 - \Big[\lambda\mu\frac{A_1(q_1;c_1)}{c_1W_{c_1}^{(q_1)}(a)}B_1^\infty(q_1;c_1) - \lambda\mu\frac{A_2(q_1;c_1)}{c_1W_{c_1}^{(q_1)}(a)}B_2^\infty(q_1;c_1)\Big]\Big\}^{-1} \end{aligned}$$

$$\times\Big\{\lambda\mu\frac{A_1(q_1;c_1)}{c_1}\Big[-\frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}+\frac{q_1}{\theta_1(q_1;c_1)}\Big]B_1^{\infty}(q_1;c_1)$$
$$-\lambda\mu\frac{A_2(q_1;c_1)}{c_1}\Big[-\frac{Z_{c_1}^{(q_1)}(a)}{W_{c_1}^{(q_1)}(a)}+\frac{q_1}{\theta_2(q_1;c_1)}\Big]B_2^{\infty}(q_1;c_1)\Big\} \tag{21}$$

and

$$\begin{aligned}B_i^{\infty}(q_1;c_1)={}&\mathrm{e}^{\theta_i(q_1;c_1)a}\Big\{\Big\{1-\frac{[q_2-\psi(\theta_i(q_1;c_1))]A_1(q_2;c_2)}{c_2[\theta_1(q_2;c_2)-\theta_i(q_1;c_1)]}\\&+\frac{[q_2-\psi(\theta_i(q_1;c_1))]A_2(q_2;c_2)}{c_2[\theta_2(q_2;c_2)-\theta_i(q_1;c_1)]}\Big\}\frac{c_2}{(\lambda+q_2+\mu c_2)[\lambda+q_2-\theta_i(q_1;c_1)c_2]}\\&+\frac{[q_2-\psi(\theta_i(q_1;c_1))]A_1(q_2;c_2)}{[\theta_1(q_2;c_2)-\theta_i(q_1;c_1)](\lambda+q_2+\mu c_2)[\lambda+q_2-\theta_1(q_2;c_2)c_2]}\\&-\frac{[q_2-\psi(\theta_i(q_1;c_1))]A_2(q_2;c_2)}{[\theta_2(q_2;c_2)-\theta_i(q_1;c_1)](\lambda+q_2+\mu c_2)[\lambda+q_2-\theta_2(q_2;c_2)c_2]}\Big\}\\&-\mathrm{e}^{\theta_i(q_1;c_1)a}\frac{q_2-\psi(\theta_i(q_1;c_1))}{\theta_2(q_2;c_2)-\theta_i(q_1;c_1)}\\&\times\frac{A_1(q_2;c_2)[\lambda+q_2-\theta_2(q_2;c_2)c_2]-A_2(q_2;c_2)[\lambda+q_2-\theta_1(q_2;c_2)c_2]}{(\lambda+q_2+\mu c_2)[\lambda+q_2-\theta_1(q_2;c_2)c_2][\lambda+q_2-\theta_2(q_2;c_2)c_2]}\\&-\frac{\mathrm{e}^{-\mu a}-\mathrm{e}^{\theta_i(q_1;c_1)a}}{[\mu+\theta_i(q_1;c_1)](\lambda+q_2+\mu c_2)},\qquad i=1,2.\end{aligned}$$

Then we complete the proof. □

§4. Appendix

In this Appendix, we shall provide some preliminaries for the q-scale function related to the Lévy process and some identities for the classical compound Poisson risk model.

As in [27], the scale functions $\{W^{(q)};\ q\geqslant 0\}$ corresponding to a Lévy process X are defined as follows. For each $q\geqslant 0$, $W^{(q)}:[0,\infty)\to[0,\infty)$ is the unique strictly increasing and continuous function with Laplace transform

$$\int_0^{\infty}\mathrm{e}^{-\theta x}W^{(q)}(x)\mathrm{d}x=\frac{1}{\psi(\theta)-q},\qquad \theta>\Phi_q,$$

where Φ_q is the largest solution of the equation $\psi(\theta)=q$ (there are at most two). For convenience, the domain of $W^{(q)}$ can be extended to the whole real line by setting $W^{(q)}(x)=0$ for all $x<0$. In particular, write $W=W^{(0)}$ when $q=0$. Further, define the function $Z^{(q)}(x)$, closely related to $W^{(q)}(x)$, by

$$Z^{(q)}(x)=1+q\int_0^x W^{(q)}(z)\mathrm{d}z,\qquad q\geqslant 0,\ x\geqslant 0,$$

and the so-called second scale function $Z^{(q)}(x,\theta)$ by

$$Z^{(q)}(x,\theta)=\mathrm{e}^{\theta x}\Big\{1+[q-\psi(\theta)]\int_0^x \mathrm{e}^{-\theta z}W^{(q)}(z)\mathrm{d}z\Big\},\qquad q,\theta\geqslant 0,\ x\geqslant 0,$$

with $Z^{(q)}(x)=1$ and $Z^{(q)}(x,\theta)=\mathrm{e}^{\theta x}$ for $x<0$.

In the following parts, we are going to list the explicit expressions of $W^{(q)}(x),Z^{(q)}(x)$ and $Z^{(q)}(x,\theta)$ for the compound Poisson process with Poisson arrival rate $\lambda>0$, drift coefficients c and an exponential jump size with distribution function $F(x)=1-\mathrm{e}^{-\mu x}$, $\mu>0$. Its scale function $W^{(q)}(x)$ is given by

$$W^{(q)}(x):=W_c^{(q)}(x)=\frac{A_1(q;c)}{c}\mathrm{e}^{\theta_1(q;c)x}-\frac{A_2(q;c)}{c}\mathrm{e}^{\theta_2(q;c)x},\qquad x\geqslant 0,\tag{22}$$

with $A_1(q;c)=[\mu+\theta_1(q;c)]/[\theta_1(q)-\theta_2(q;c)]$, $A_2(q;c)=[\mu+\theta_2(q;c)]/[\theta_1(q;c)-\theta_2(q;c)]$ and

$$\begin{aligned}\theta_1(q;c)&=\frac{\lambda+q-c\mu+\sqrt{(c\mu-\lambda-q)^2+4cq\mu}}{2c},\\ \theta_2(q;c)&=\frac{\lambda+q-c\mu-\sqrt{(c\mu-\lambda-q)^2+4cq\mu}}{2c}.\end{aligned}$$

Further, the expressions of $Z^{(q)}(x)$ and $Z^{(q)}(x,\theta)$ can be calculated as follows.

$$\begin{aligned}Z^{(q)}(x):=Z_c^{(q)}(x)&=1+q\int_0^x\Big[\frac{A_1(q;c)}{c}\mathrm{e}^{\theta_1(q;c)y}-\frac{A_2(q;c)}{c}\mathrm{e}^{\theta_2(q;c)y}\Big]\mathrm{d}y\\ &=1+\frac{qA_1(q;c)}{c\theta_1(q;c)}\big(\mathrm{e}^{\theta_1(q;c)x}-1\big)-\frac{qA_2(q;c)}{c\theta_2(q;c)}\big(\mathrm{e}^{\theta_2(q;c)x}-1\big)\\ &=\frac{qA_1(q;c)}{c\theta_1(q;c)}\mathrm{e}^{\theta_1(q;c)x}-\frac{qA_2(q;c)}{c\theta_2(q;c)}\mathrm{e}^{\theta_2(q;c)x},\qquad x\geqslant 0,\end{aligned}\tag{23}$$

and

$$\begin{aligned}Z^{(q)}(x,\theta):=Z_c^{(q)}(x,\theta)&=\mathrm{e}^{\theta x}\Big\{1+[q-\psi(\theta)]\int_0^x\mathrm{e}^{-\theta z}W_c^{(q)}(z)\mathrm{d}z\Big\}\\ &=\mathrm{e}^{\theta x}\Big\{1+[q-\psi(\theta)]\Big\{\frac{A_1(q;c)}{c[\theta_1(q;c)-\theta]}\big(\mathrm{e}^{(\theta_1(q;c)-\theta)x}-1\big)\\ &\quad-\frac{A_2(q;c)}{c[\theta_2(q;c)-\theta]}\big(\mathrm{e}^{(\theta_2(q;c)-\theta)x}-1\big)\Big\}\Big\}\\ &=\Big\{1-\frac{[q-\psi(\theta)]A_1(q;c)}{c[\theta_1(q;c)-\theta]}+\frac{[q-\psi(\theta)]A_2(q;c)}{c[\theta_2(q;c)-\theta]}\Big\}\mathrm{e}^{\theta x}\\ &\quad+\frac{[q-\psi(\theta)]A_1(q;c)}{c[\theta_1(q;c)-\theta]}\mathrm{e}^{\theta_1(q;c)x}-\frac{[q-\psi(\theta)]A_2(q;c)}{c[\theta_2(q;c)-\theta]}\mathrm{e}^{\theta_2(q;c)x},\\ &\qquad\theta\geqslant 0,\ x\geqslant 0.\end{aligned}\tag{24}$$

References

[1] PITMAN J, YOR M. Hitting, occupation and inverse local times of one-dimensional diffusions: martingale and excursion approaches [J]. *Bernoulli*, 2003, **9(1)**: 1–24.

[2] LI B, ZHOU X W. The joint Laplace transforms for diffusion occupation times [J]. *Adv Appl Probab*, 2013, **45(4)**: 1049–1067.

[3] DOS REIS A E. How long is the surplus below zero? [J]. *Insurance Math Econom*, 1993, **12(1)**: 23–38.

[4] ZHANG C S, WU R. Total duration of negative surplus for the compound Poisson process that is perturbed by diffusion [J]. *J Appl Probab*, 2002, **39(3)**: 517–532.

[5] BIARD R, LOISEL S, MACCI C, et al. Asymptotic behavior of the finite-time expected time-integrated negative part of some risk processes and optimal reserve allocation [J]. *J Math Anal Appl*, 2010, **367(2)**: 535–549.

[6] LANDRIAULT D, RENAUD J F, ZHOU X W. Occupation times of spectrally negative Lévy processes with applications [J]. *Stochastic Process Appl*, 2011, **121(11)**: 2629–2641.

[7] KYPRIANOU A E, PARDO J C, PÉREZ J L. Occupation times of refracted Lévy processes [J]. *J Theoret Probab*, 2014, **27(4)**: 1292–1315.

[8] LOEFFEN R L, RENAUD J F, ZHOU X W. Occupation times of intervals until first passage times for spectrally negative Lévy processes [J]. *Stochastic Process Appl*, 2014, **124(3)**: 1408–1435.

[9] YIN C C, YUEN K C. Exact joint laws associated with spectrally negative Lévy processes and applications to insurance risk theory [J]. *Front Math China*, 2014, **9(6)**: 1453–1471.

[10] WANG W Y, WU X Y, PENG X C, et al. A note on joint occupation times of spectrally negative Lévy risk processes with tax [J]. *Statist Probab Lett*, 2018, **140**: 13–22.

[11] CAI N, CHEN N, WAN X W. Occupation times of jump-diffusion processes with double exponential jumps and the pricing of options [J]. *Math Oper Res*, 2010, **35(2)**: 412–437.

[12] PÉREZ J L, YAMAZAKI K. On the refracted-reflected spectrally negative Lévy processes [J]. *S-tochastic Process Appl*, 2018, **128(1)**: 306–331.

[13] ASMUSSEN S. *Ruin Probabilities* [M]. Singapore: World Scientific Press, 2000.

[14] ZHANG H Y, ZHOU M, GUO J Y. The Gerber-Shiu discounted penalty function for classical risk model with a two-step premium rate J]. *Statist Probab Lett*, 2006, **76(12)**: 1211–1218.

[15] KARNAUKH I. Risk process with stochastic income and two-step premium rate [J]. *Appl Math Comput*, 2010, **217(2)**: 775–781.

[16] LIN X S, PAVLOVA K P. The compound Poisson risk model with a threshold dividend strategy [J]. *Insurance Math Econom*, 2006, **38(1)**: 57–80.

[17] XU L, YANG H L, WANG R M. Cox risk model with variable premium rate and stochastic return on investment [J]. *J Comput Appl Math*, 2014, **256**: 52–64.

[18] EPISCOPOS A. Bank capital regulation in a barrier option framework [J]. *J Bank Financ*, 2008, **32(8)**: 1677–1686.

[19] HAN Z, HU Y Z, LEE C. Optimal pricing barriers in a regulated market using reflected diffusion processes [J]. *Quant Finance*, 2016, **16(4)**: 639–647.

[20] HAN Z, HU Y Z, LEE C. On pricing barrier control in a regime-switching regulated market [J]. *Quant Finance*, 2019, **19(3)**: 491–499.

[21] KRUGMAN P R. Target zones and exchange rate dynamics [J]. *Quart J Econom*, 1991, **106(3)**: 669–682.

[22] BO L J, WANG Y J, YANG X W. Some integral functionals of reflected SDEs and their applications in finance [J]. *Quant Finance*, 2011, **11(3)**: 343–348.

[23] BO L J, WANG Y J, YANG X W. On the default probability in a regime-switching regulated market [J]. *Methodol Comput Appl Probab*, 2014, **16(1)**: 101–113.

[24] LI Y Q, ZHOU X W. On pre-exit joint occupation times for spectrally negative Lévy processes [J]. *Statist Probab Lett*, 2014, **94**: 48–55.

[25] KUZNETSOV A, KYPRIANOU A E, RIVERO V. The theory of scale functions for spectrally negative Lévy processes [M] // COHEN S, KUZNETSOV A, KYPRIANOU A E, et al. *Lévy Matters II*. Berlin: Springer, 2012: 97–186.

[26] KYPRIANOU A E. *Fluctuations of Lévy Processes with Applications: Introductory Lectures* [M]. 2nd ed. Heidelberg: Springer, 2014.

[27] BERTOIN J. *Lévy Processes* [M]. Cambridge: Cambridge University Press, 1996.

带两步保费率的复合 Poisson 风险模型的占位时

张爱丽 刘 章

(南京审计大学统计与数学学院, 南京, 211815) (江西农业大学计算机与信息工程学院, 南昌, 330045)

摘 要: 本文考虑了带两步保费率的经典复合 Poisson 风险模型. 使用一种替代方法, 找到了两个不相交时间间隔的联合占位时对应 Laplace 变换的显式表达式. 其中, Laplace 变换用 Lévy 过程的尺度函数来表示.

关键词: 复合 Poisson 风险模型; 联合占位时; 两步保费率; 尺度函数

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