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具有有限记忆的随机分形的重分形分解

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摘　要

Dryakhlov和Tempelman对具有有限记忆的随机分形集的Hausdorff维数进行了研究, 本文对具有有限记忆的随机分形集$K(\omega)$的重分形分解集$K_\alpha(\omega)$进行研究, 得到了在一定条件下, 这种随机分形集重分形分解集$K_\alpha(\omega)$的Hausdorff维数表达式.

关键词: 具有有限记忆的随机分形, 重分形分解, Perron-Frobenius定理, 鞅, 局部维数.

学科分类号: O153.1, O189.1.

§1. 引言与预备知识

重分形分解是研究分形的一个重要手段. Mauldin和Williams 1986年对一类随机递归结构进行了研究, 在这类随机递归结构中, 每个随机集生成下一代随机集的个数是未必有限的, 并且要求随机比率是独立同分布的; 他们研究了由这类随机递归结构产生的随机集的Hausdorff维数. 苏峰和赵兴球于1997年考虑了Mauldin和Williams引入的随机结构的特殊情形, 即限制每个随机集生成下一代随机集的个数是有限的; 他们研究了由这类随机递归结构产生的随机集的重分形分解问题. 然而应该指出的是, 在Mauldin和Williams, 苏峰和赵兴球所研究的随机结构中, 都要求随机比率是独立同分布的. 继而, Dryakhlov和Tempelman 2001年考虑了更一般的情形, 他们并不要求随机比率是同分布的, 而是将随机比率同分布的条件放宽为: 随机比率的分布由前$m(m>1)$步的分布来决定, 称之为有限记忆性. Dryakhlov和Tempelman对这种具有有限记忆的随机结构, 通过构造一个转移算子$V^{(\beta)}$, 证明了具有有限记忆的随机分形集的Hausdorff维数是方程$\rho(\beta)=1$的唯一解d, 其中$\rho(\beta)$是$V^{(\beta)}$的谱半径. 本文对具有有限记忆的随机分形集重分形分解进行研究, 获得了这种随机分形集重分形分解集的Hausdorff维数的结果.

先叙述符号空间中的一些概念. 设$N(\geq 2)$是正整数, 记

$$\triangle := \{1,2,\cdots,N\}, \qquad \triangle^n := \{\sigma:\sigma=(\sigma_1,\sigma_2,\cdots,\sigma_n),\sigma_i\in\triangle, i=1,2,\cdots,n\},$$
$$\triangle^* := \bigcup_{n=1}^{\infty}\triangle^n, \qquad \triangle^{\mathbf{N}} := \{\sigma:\sigma=(\sigma_1,\sigma_2,\cdots),\sigma_i\in\triangle, i=1,2,\cdots\},$$

规定$\triangle^0=\emptyset$; $\triangle^*$中的两个元素σ,τ的连接记为(σ,τ)或$\sigma*\tau=(\sigma_1,\sigma_2,\cdots,\sigma_n,\tau_1,\tau_2,\cdots,\tau_m)$, 对任意的$\sigma\in\triangle^*$, 再记$|\sigma|=$“符号序列的长度”, 例如若$\sigma=(\sigma_1,\sigma_2,\cdots,\sigma_k)$, 则$|\sigma|=k$. 对

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任意的自然数k, $\sigma\in\triangle^*$, 且$|\sigma|\geq k$, 则记$\sigma|k=(\sigma_1,\sigma_2,\cdots,\sigma_k)$为包含$\sigma$前$k$个符号的序列; $\triangle^*$上的偏序结构定义为$\sigma\prec\tau\Leftrightarrow\exists\,\eta\in\triangle^*$, s.t. $(\sigma,\eta)=\tau$.

设$\mathbf{R}^d$为欧氏空间, I为$\mathbf{R}^d$中的紧子集, 且$I=\overline{\mathrm{int}(I)}$. 设$(\Omega,\Sigma,\mathsf{P})$为概率空间. 若$\forall\,\omega\in\Omega$, $\mathbf{R}^d$中的非空闭子集族$I(\omega):=\{I_\sigma(\omega):\sigma\in\triangle^*\}$满足

$$\lim_{n\to\infty}\max_{\sigma\in\triangle^n}\mathrm{diam}(I_\sigma(\omega))=0,\qquad\forall\,\omega\in\Omega,\tag{1.1}$$

和

$$I_\sigma\subset I_\eta,\qquad\forall\,\eta\prec\sigma,\tag{1.2}$$

则称I为随机结构.

定义随机集

$$K(\omega):=\bigcap_{n=1}^{\infty}\bigcup_{\sigma\in\triangle^n}I_\sigma(\omega).\tag{1.3}$$

$\forall\,\sigma\in\triangle^{\mathbf{N}}$, 令$C_n(\sigma)$为由与$\sigma$的前$n$个符号相同的无限长符号序列所组成的的集合, 即

$$C_n(\sigma):=\{\tau:\tau\in\triangle^{\mathbf{N}},\tau|n=\sigma|n\},$$

称之为$\triangle^{\mathbf{N}}$中的柱集.

记$\mathrm{diam}(I_\sigma)=l_\sigma$, $\forall\,\sigma\in\triangle^*$.

设$b>0$, $N_0\in\mathbf{N}$, $r>0$: 若$bl_{\sigma|N_0}\geq r$, 则记$k(r,\sigma,b)\geq N_0$为满足$bl_{\sigma|k(r,\sigma,b)+1}<r\leq bl_{\sigma|k(r,\sigma,b)}$的正整数; 若$bl_{\sigma|N_0}<r$, 则令$k(r,\sigma,b)=N_0$.

设$n\geq N_0$, $\forall\,\sigma\in\triangle^{\mathbf{N}}$, 记$L_{n,\sigma}(b)$为满足条件"使得$B(\sigma,bl_{\sigma|n})\cap C_{k(bl_{\sigma|n},\tau^{(i)},b)}(\tau^{(i)})\neq\emptyset$的两两不交的柱集$C_{k(bl_{\sigma|n},\tau^{(i)},b)}(\tau^{(i)})$, $\tau^{(i)}\in\triangle^{\mathbf{N}}$的数目不超过$L_{n,\sigma}(b)$"的最小数目. 令$L_\omega(b):=\sup\{L_{n,\sigma}(b):\sigma\in\triangle^{\mathbf{N}},n\geq N_0\}$, 称之为对应于常数$b$关于$\rho$的Moran指数.

记$L_{\sigma*x}:=l_{\sigma*x}/l_\sigma$, $\forall\,\sigma\in\triangle^*$.

设下列条件满足:

(1) $\forall\,\sigma\in\triangle^*$, 随机向量族$(L_{\sigma*1},L_{\sigma*2},\cdots,L_{\sigma*N})$独立;

(2) 存在整数$m>1$, 使得$\forall\,\sigma\in\triangle^*$, $\forall\,\eta\in\triangle^{m-1}$, $(L_{\sigma*\eta*1},L_{\sigma*\eta*2},\cdots,L_{\sigma*\eta*N})$与$(L_{\eta*1},L_{\eta*2},\cdots,L_{\eta*N})$同分布.

有时, 我们还要求下列条件成立:

(3) 若$\sigma\not\prec\eta$且$\eta\not\prec\sigma$, 则对几乎处处的ω, 有

$$I_\sigma(\omega)\cap I_\eta(\omega)\cap K(\omega)=\emptyset;\tag{1.4}$$

(4) 对几乎处处的ω, 成立

$$\lim_{n\to\infty}\frac{\log l_{[\pi|n+1]}(\omega)}{\log l_{[\pi|n]}(\omega)}=1,\qquad\forall\,\pi\in\triangle^{\mathbf{N}};\tag{1.5}$$

(5) 对几乎处处的ω, 存在$b=b(\omega)>0$, 使得

$$L_\omega(b)<+\infty.\tag{1.6}$$

今记$g_\omega(\sigma) := \bigcap_{k=1}^{\infty} I_{\sigma|k}(\omega)$, $\sigma \in \triangle^{\mathbf{N}}$, 易证$g_\omega$具有如下性质:

(1) g_ω为$\triangle^{\mathbf{N}}$到$K(\omega)$上的连续映射;

(2) 若条件(3)成立, 则g_ω是一个一一映射.

下面通过定义码空间$\triangle^{\mathbf{N}}$上的无穷乘积测度来定义随机集$K(\omega)$上的随机测度.

设$\{(p_{\sigma*1}, p_{\sigma*2}, \cdots, p_{\sigma*N}) : \sigma \in \triangle^*\}$是一列独立的随机向量, 满足

(1) $\forall\sigma\in\triangle^*$, $\forall\eta\in\triangle^{m-1}$, 随机向量$(p_{\sigma*\eta*1}, p_{\sigma*\eta*2}, \cdots, p_{\sigma*\eta*N})$与$(p_{\eta*1}, p_{\eta*2}, \cdots, p_{\eta*N})$同分布;

(2) 当$\sigma \in \triangle^*$时, $(p_{\sigma*1}, p_{\sigma*2}, \cdots, p_{\sigma*N})$与$(L_{\sigma*1}, L_{\sigma*2}, \cdots, L_{\sigma*N})$独立;

(3) $\forall\,\sigma \in \triangle^*$, $\sum_{i=1}^{N} p_{\sigma*i} = 1$, a.s.;

(4) 存在$p_1, p_2, L_1, L_2 \in \mathbf{R}$, 使得$0 < p_1 \le p_\sigma \le p_2 < 1$, $0 < L_1 \le L_\sigma \le L_2 < 1$, $\forall\sigma\in\triangle^*$.

利用测度论的知识可以定义$\triangle^{\mathbf{N}}$上的无穷乘积测度$\overline{\mu}_\omega(C_n(\sigma)) := \prod_{i=1}^{n} p_{\sigma|i}$, $\forall\,\sigma \in \triangle^{\mathbf{N}}$.

以下为方便起见, 记$P_\sigma := \prod_{i=1}^{|\sigma|} p_{\sigma|i}$, $\forall\,\sigma \in \triangle^*$.

设μ_ω为由g_ω导出的随机集$K(\omega)$上的测度, 即$\mu_\omega(A) = \overline{\mu}_\omega(g_\omega^{-1}(A))$, $A \subset K(\omega) \subset \mathbf{R}^d$.

定义 1.1 对实数α及几乎处处的ω, 定义

$$\overline{K}_\alpha(\omega) := \Big\{\sigma \in \triangle^{\mathbf{N}} : \lim_{k\to+\infty} \frac{\log P_{\sigma|k}}{\log l_{\sigma|k}} = \alpha\Big\}, \qquad K_\alpha(\omega) := g_\omega(\overline{K}_\alpha(\omega)), \tag{1.7}$$

则称集簇为$\{K_\alpha(\omega)\}$对应于测度μ_ω构成$K(\omega)$的重分形分解集, 其中$K_\alpha(\omega)$叫做$K(\omega)$多重分形分解的分量.

§2. 辅助函数的性质

设$m > 1$, 考虑N^{m-1}维实向量空间$\Phi = \{u(x_1, x_2, \cdots, x_{m-1}) : 1 \le x_1, x_2, \cdots, x_{m-1} \le N\}$. 对任意的$q \in \mathbf{R}$, $\beta \in \mathbf{R}$, 在Φ中定义算子$V^{(q,\beta)} : u \longmapsto uV^{(q,\beta)} = \omega$, 满足

$$V^{(q,\beta)}(u) = \omega(x_2, \cdots, x_m) := \sum_{x_1\in\triangle} u(x_1, \cdots, x_{m-1})\mathsf{E}(p^q_{x_1,\cdots,x_m} L^\beta_{x_1,\cdots,x_m}). \tag{2.1}$$

设$\rho(q,\beta)$表示$V^{(q,\beta)}$的谱半径. 由于$V^{(q,\beta)}$是一个不可约的非负矩阵, 利用Perron-Frobenius定理可以得到$\rho(q,\beta)$的一些基本性质.

性质 2.1 (1) $\rho(q,\beta) : \mathbf{R} \times \mathbf{R} \longmapsto (0,\infty)$是解析的.

(2) $\rho(q,\beta)$关于q, β均是严格递减的.

(3) 固定q,

$$\lim_{\beta\to+\infty} \rho(q,\beta) = 0, \qquad \lim_{\beta\to-\infty} \rho(q,\beta) = +\infty;$$

固定β,

$$\lim_{q\to+\infty} \rho(q,\beta) = 0, \qquad \lim_{q\to-\infty} \rho(q,\beta) = +\infty.$$

注记 1 对于固定的q, 函数$\rho(q,\beta)$是关于β的连续函数, $\rho(q,\beta)$的取值范围从$0\ (\beta\to+\infty)$到$+\infty\ (\beta\to-\infty)$, 因此由介值定理知存在$\beta$, 满足$\rho(q,\beta)=1$, 又由性质2.1(2)知$\rho(q,\beta)$关于$\beta$是严格递减的, 从而$\beta$是唯一的. 如此, 可以定义关于$q$的函数$\beta(q)$, 满足$\rho(q,\beta(q))=1$.

性质 2.2 设$\beta=\beta(q)$满足$\rho(q,\beta(q))=1$, 则

(1) β是关于变量q的解析函数;

(2) β是严格递减的, 即若$q_1<q_2$, 则$\beta(q_1)>\beta(q_2)$;

(3) $\lim\limits_{q\to-\infty}\beta(q)=+\infty$, $\lim\limits_{q\to+\infty}\beta(q)=-\infty$.

以下考虑$\beta'(q)$. 记$\beta=\beta(q)$. 由于$V^{(q,\beta)}$是不可约矩阵, 由Perron-Frobenius定理知, 对应于$\rho(q,\beta)=1$, 存在一维正的右特征向量$r_1^{(q,\beta)}:=(r_1^{(q,\beta)}(x_2,\cdots,x_m):1\le x_2,\cdots,x_m\le N)$. 令

$$r^{(q,\beta)}(x_2,\cdots,x_m):=\frac{r_1^{(q,\beta)}(x_2,\cdots,x_m)}{\sum\limits_{x_2,\cdots,x_m\in\triangle}r_1^{(q,\beta)}(x_2,\cdots,x_m)},$$
$$r^{(q,\beta)}:=(r^{(q,\beta)}(x_2,\cdots,x_m):1\le x_2,\cdots,x_m\le N),$$

则$r^{(q,\beta)}$是唯一的; 对应于$\rho(q,\beta)=1$, 存在一维的正的左特征向量$r_2^{(q,\beta)}=(r_2^{(q,\beta)}(x_1,x_2,\cdots,x_{m-1}):1\le x_1,x_2,\cdots,x_{m-1}\le N)$, 令

$$\overline{r}^{(q,\beta)}(x_1,\cdots,x_{m-1}):=\frac{r_2^{(q,\beta)}(x_1,x_2,\cdots,x_{m-1})}{\sum\limits_{x_1,\cdots,x_{m-1}\in\triangle}r_2^{(q,\beta)}(x_1,x_2,\cdots,x_{m-1})r^{(q,\beta)}(x_1,\cdots,x_{m-1})},$$
$$\overline{r}^{(q,\beta)}:=(\overline{r}^{(q,\beta)}(x_1,\cdots,x_{m-1}):1\le x_1,\cdots,x_{m-1}\le N),$$

则$\overline{r}^{(q,\beta)}(x_1,\cdots,x_{m-1})$是唯一的, 利用与文献[5]中类似的方法得

$$\beta'(q)=-\frac{\sum\limits_{x_1,\cdots,x_m\in\triangle}\overline{r}^{(q,\beta)}(x_1,\cdots,x_{m-1})\mathsf{E}(p^q_{x_1,\cdots,x_m}L^\beta_{x_1,\cdots,x_m}\log p_{x_1,\cdots,x_m})r^{(q,\beta)}(x_2,\cdots,x_m)}{\sum\limits_{x_1,\cdots,x_m\in\triangle}\overline{r}^{(q,\beta)}(x_1,\cdots,x_{m-1})\mathsf{E}(p^q_{x_1,\cdots,x_m}L^\beta_{x_1,\cdots,x_m}\log L_{x_1,\cdots,x_m})r^{(q,\beta)}(x_2,\cdots,x_m)}. \tag{2.2}$$

于是$\beta'(q)<0$, 与性质2.2(2)相一致.

令$\alpha=-\mathrm{d}\beta/\mathrm{d}q$, 则$\alpha>0$, 且

$$\alpha=\frac{\sum\limits_{x_1,\cdots,x_m\in\triangle}\overline{r}^{(q,\beta)}(x_1,\cdots,x_{m-1})\mathsf{E}(p^q_{x_1,\cdots,x_m}L^\beta_{x_1,\cdots,x_m}\log p_{x_1,\cdots,x_m})r^{(q,\beta)}(x_2,\cdots,x_m)}{\sum\limits_{x_1,\cdots,x_m\in\triangle}\overline{r}^{(q,\beta)}(x_1,\cdots,x_{m-1})\mathsf{E}(p^q_{x_1,\cdots,x_m}L^\beta_{x_1,\cdots,x_m}\log L_{x_1,\cdots,x_m})r^{(q,\beta)}(x_2,\cdots,x_m)}. \tag{2.3}$$

在以下的论述中, 记$f(q)=q\alpha+\beta(q)$.

§3. $K_\alpha(\omega)$的Hausdorff维数的上界

定义随机变量

$$S_{q,\beta,n}^{(x_2,\cdots,x_m)} := \sum_{\substack{\sigma\in\triangle^n \\ \sigma_{n-m+2}=x_2,\cdots,\sigma_n=x_m}} P_\sigma^q l_\sigma^\beta, \qquad x_2,\cdots,x_m\in\triangle,\ n\geq m,$$

记$S_{q,\beta,n} := (S_{q,\beta,n}^{(x_2,\cdots,x_m)} : 1\leq x_2,\cdots,x_m\leq N)$.

考虑σ-代数: $\mathcal{F}_n := \sigma(\{L_\sigma, p_\sigma : \sigma\in\triangle^*, |\sigma|\leq n\})$, $\mathcal{F} := \bigvee_{n=1}^{\infty}\mathcal{F}_n = \sigma(\bigcup_{n=1}^{\infty}\mathcal{F}_n)$.

引理 3.1 $\forall\, q,\beta\in\mathbf{R}$, $n\geq m$, 成立$\mathsf{E}[S_{q,\beta,n}|\mathcal{F}_{n-1}] = S_{q,\beta,n-1}V^{(q,\beta)}$.

证明: 对于任意的$x_2,\cdots,x_m\in\triangle$, 有

$$\begin{aligned}
\mathsf{E}[S_{q,\beta,n}^{(x_2,\cdots,x_m)}|\mathcal{F}_{n-1}] &= \mathsf{E}\Big[\sum_{\substack{\sigma\in\triangle^n \\ \sigma_{n-m+2}=x_2,\cdots,\sigma_n=x_m}} P_\sigma^q l_\sigma^\beta|\mathcal{F}_{n-1}\Big] \\
&= \mathsf{E}\Big[\sum_{x_1\in\triangle}\sum_{\sigma\in\triangle^{n-m}} P^q_{\sigma,x_1,\cdots,x_m} l^\beta_{\sigma,x_1,\cdots,x_m}|\mathcal{F}_{n-1}\Big] \\
&= \sum_{x_1\in\triangle}\sum_{\sigma\in\triangle^{n-m}} P^q_{\sigma,x_1,\cdots,x_{m-1}} l^\beta_{\sigma,x_1,\cdots,x_{m-1}} \mathsf{E}[p^q_{\sigma,x_1,\cdots,x_m} L^\beta_{\sigma,x_1,\cdots,x_m}|\mathcal{F}_{n-1}] \\
&= \sum_{x_1\in\triangle}\sum_{\sigma\in\triangle^{n-m}} P^q_{\sigma,x_1,\cdots,x_{m-1}} l^\beta_{\sigma,x_1,\cdots,x_{m-1}} \mathsf{E}(p^q_{\sigma,x_1,\cdots,x_m} L^\beta_{\sigma,x_1,\cdots,x_m}) \\
&= \sum_{x_1\in\triangle}\sum_{\sigma\in\triangle^{n-m}} P^q_{\sigma,x_1,\cdots,x_{m-1}} l^\beta_{\sigma,x_1,\cdots,x_{m-1}} \mathsf{E}(p^q_{x_1,\cdots,x_m} L^\beta_{x_1,\cdots,x_m}) \\
&= \sum_{x_1\in\triangle} S^{(x_1,\cdots,x_{m-1})}_{q,\beta,n-1}\mathsf{E}(p^q_{x_1,\cdots,x_m} L^\beta_{x_1,\cdots,x_m}),
\end{aligned}$$

故命题获证. $\square$

由于$V^{(q,\beta)}$是一个不可约的非负矩阵, 利用Perron-Frobenius定理知$\rho(q,\beta)$是$V^{(q,\beta)}$的一个特征值, 且对应$\rho(q,\beta)$存在一个正的右特征向量$r^{(q,\beta)} = (r^{(q,\beta)}(x_2,\cdots,x_m) : 1\leq x_2,\cdots,x_m\leq N)$, 记$u^{(q,\beta)}(x_1,\cdots,x_m) = \mathsf{E}(p^q_{x_1,\cdots,x_m} L^\beta_{x_1,\cdots,x_m})$, 作如下的变换:

$$\overline{u}^{(q,\beta)}(x_1,\cdots,x_m) := \frac{u^{(q,\beta)}(x_1,\cdots,x_m)r^{(q,\beta)}(x_2,\cdots,x_m)}{\rho(q,\beta)r^{(q,\beta)}(x_1,\cdots,x_{m-1})},$$

于是$\overline{u}^{(q,\beta)} = (\overline{u}^{(q,\beta)}(x_1,\cdots,x_m) : 1\leq x_1,\cdots,x_m\leq N)$, 满足$\sum_{x_m\in\triangle}\overline{u}^{(q,\beta)}(x_1,\cdots,x_m) = 1$.

记

$$\begin{aligned}
\overline{S}^{(x_2,\cdots,x_m)}_{q,\beta,n} &:= \frac{r^{(q,\beta)}(x_2,\cdots,x_m)S^{(x_2,\cdots,x_m)}_{q,\beta,n}}{\rho^n(q,\beta)}, \qquad n\geq m, \\
\overline{S}_{q,\beta,n} &:= (\overline{S}^{(x_2,\cdots,x_m)}_{q,\beta,n} : 1\leq x_2,\cdots,x_m\leq N).
\end{aligned}$$

引理 3.2 $\forall\, q,\beta\in\mathbf{R}$, $n\geq m$, 成立$\mathsf{E}[\overline{S}_{q,\beta,n}|\mathcal{F}_{n-1}] = \overline{S}_{q,\beta,n-1}\overline{u}^{(q,\beta)}$.

证明: 对于任意的$x_2,\cdots,x_m\in\triangle$,有

$$\begin{aligned}&\mathsf{E}[\overline{S}^{(x_2,\cdots,x_m)}_{q,\beta,n}|\mathcal{F}_{n-1}]\\ =&\frac{r^{(q,\beta)}(x_2,\cdots,x_m)\sum\limits_{x_1\in\triangle}S^{(x_1,\cdots,x_{m-1})}_{q,\beta,n-1}\mathsf{E}(p^q_{x_1,\cdots,x_m}L^\beta_{x_1,\cdots,x_m})}{\rho^n(q,\beta)}\\ =&\sum_{x_1\in\triangle}\frac{r^{(q,\beta)}(x_1,\cdots,x_{m-1})S^{(x_1,\cdots,x_{m-1})}_{q,\beta,n-1}}{\rho^{n-1}(q,\beta)}\cdot\frac{\mathsf{E}(p^q_{x_1,\cdots,x_m}L^\beta_{x_1,\cdots,x_m})r^{(q,\beta)}(x_2,\cdots,x_m)}{\rho(q,\beta)r^{(q,\beta)}(x_1,\cdots,x_{m-1})}\\ =&\sum_{x_1\in\triangle}\overline{S}^{(x_1,\cdots,x_{m-1})}_{q,\beta,n-1}\overline{u}^{(q,\beta)}(x_1,\cdots,x_m),\end{aligned}$$

于是引理获证. □

记随机变量$\overline{Z}_{q,\beta,n}:=\sum\limits_{x_2,\cdots,x_m\in\triangle}\overline{S}^{(x_2,\cdots,x_m)}_{q,\beta,n}$, $n\ge m$.

引理 3.3 随机序列$(\overline{Z}_{q,\beta,n},\mathcal{F}_n)_{n\ge m}$是正鞅.

证明: 由于

$$\begin{aligned}\mathsf{E}[\overline{Z}_{q,\beta,n}|\mathcal{F}_{n-1}]&=\sum_{x_2,\cdots,x_m\in\triangle}\mathsf{E}[\overline{S}^{(x_2,\cdots,x_m)}_{q,\beta,n}|\mathcal{F}_{n-1}]\\ &=\sum_{x_2,\cdots,x_m\in\triangle}\sum_{x_1\in\triangle}\overline{S}^{(x_1,\cdots,x_{m-1})}_{q,\beta,n-1}\overline{u}^{(q,\beta)}(x_1,\cdots,x_m)\\ &=\sum_{x_1,\cdots,x_{m-1}\in\triangle}\overline{S}^{(x_1,\cdots,x_{m-1})}_{q,\beta,n-1}\sum_{x_m\in\triangle}\overline{u}^{(q,\beta)}(x_1,\cdots,x_m)\\ &=\sum_{x_1,\cdots,x_{m-1}\in\triangle}\overline{S}^{(x_1,\cdots,x_{m-1})}_{q,\beta,n-1}=Z_{q,\beta,n-1},\end{aligned}$$

所以$(\overline{Z}_{q,\beta,n},\mathcal{F}_n)_{n\ge m}$是正鞅. 利用鞅收敛定理知$\lim\limits_{n\to\infty}\overline{Z}_{q,\beta,n}$几乎处处$\omega$存在; 记

$$X_q:=\lim_{n\to\infty}\overline{Z}_{q,\beta(q),n},\qquad \text{a.s. }\omega.\qquad\square$$

定理 3.1 $\forall\,q\in\mathbf{R}$, $\mathsf{P}(\dim_H K_{\alpha(q)}\le f(q))=1$.

证明: 设$\overline{\beta}>\beta(q)$, 由$\rho(q,\beta)$关于$\beta$严格递减知$0<\rho(q,\overline{\beta})<1$, 记$Z_{q,\overline{\beta},n}:=\sum\limits_{\sigma\in\triangle^n}P^q_\sigma l^{\overline{\beta}}_\sigma$. 由Perron-Frobenius定理知, 对应$\rho(q,\overline{\beta})$存在正的右特征向量$r^{(q,\overline{\beta})}=(r^{(q,\overline{\beta})}(x_1,\cdots,x_{m-1}):1\le x_1,\cdots,x_{m-1}\le N)$. 记$1/r_0:=\min\limits_{x_1,\cdots,x_{m-1}\in\triangle}r^{(q,\overline{\beta})}(x_1,\cdots,x_{m-1})>0$, 再记$\Omega_0:=\{\omega:X_{q,\overline{\beta}}=\lim\limits_{n\to\infty}\overline{Z}_{q,\overline{\beta},n}\}\cap\{\omega:\lim\limits_{n\to\infty}\max\limits_{\sigma\in\triangle^n}\mathrm{diam}(I_\sigma(\omega))=0\}$, 则$\mathsf{P}(\Omega_0)=1$.

任取$\omega\in\Omega_0$, 对于给定的$n\ge m$, 有

$$\overline{Z}_{q,\overline{\beta},n}=\sum_{x_2,\cdots,x_m\in\triangle}\frac{r^{(q,\overline{\beta})}(x_2,\cdots,x_m)S^{(x_2,\cdots,x_m)}_{q,\overline{\beta},n}}{\rho^n(q,\overline{\beta})}\ge\sum_{x_2,\cdots,x_m\in\triangle}\frac{S^{(x_2,\cdots,x_m)}_{q,\overline{\beta},n}}{r_0\rho^n(q,\overline{\beta})}=\frac{Z_{q,\overline{\beta},n}}{r_0\rho^n(q,\overline{\beta})},$$

于是$0\le Z_{q,\overline{\beta},n}\le r_0\rho^n(q,\overline{\beta})\overline{Z}_{q,\overline{\beta},n}$, 令$n\to\infty$, 由$0<\rho(q,\overline{\beta})<1$得$\lim\limits_{n\to\infty}Z_{q,\overline{\beta},n}=0$.

(1) $q=0$时的情形

此时结论显然成立.

(2) $q>0$时的情形

令$\overline{A}_s(k):=\{\sigma\in\triangle^{\mathbf{N}}:P_{\sigma|k}\geq(l_{\sigma|k})^s\}$, $A_s(k)=g_\omega(\overline{A}_s(k))$, 则

$$K_{\alpha(q)}\subset\bigcap_{\alpha(q)<s}\bigcup_{j\geq1}\bigcap_{k\geq j}A_s(k).$$

因为$\lim\limits_{n\to\infty}\max\limits_{\sigma\in\triangle^n}\operatorname{diam}(I_\sigma(\omega))=0$, 所以$\forall\varepsilon>0$, $\exists N\in\mathbf{N}$, 当$k\geq N$时, $\max\limits_{\sigma\in\triangle^k}l_\sigma<\varepsilon$, 因此$\{l_\sigma:\sigma\in\triangle^k\cap\{\sigma\in\triangle^*:P_\sigma\geq(l_\sigma)^s\}\}$为$\bigcap\limits_{k\geq j}A_s(k)\,(j\geq1)$的一个$\varepsilon$-覆盖. 对这个覆盖经计算有

$$\begin{aligned}\sum_{\sigma\in\triangle^k\cap\{\sigma\in\triangle^*:P_\sigma\geq(l_\sigma)^s\}}l_\sigma^{qs+\overline{\beta}}&=\sum_{\sigma\in\triangle^k\cap\{\sigma\in\triangle^*:P_\sigma\geq(l_\sigma)^s\}}l_\sigma^{qs}l_\sigma^{\overline{\beta}}\\&\leq\sum_{\sigma\in\triangle^k\cap\{\sigma\in\triangle^*:P_\sigma\geq(l_\sigma)^s\}}P_\sigma^q l_\sigma^{\overline{\beta}}\leq\sum_{\sigma\in\triangle^k}P_\sigma^q l_\sigma^{\overline{\beta}}\\&=Z_{q,\overline{\beta},k}\longrightarrow0\qquad(k\to\infty),\end{aligned}$$

于是$\mathcal{H}^{qs+\overline{\beta}}\big(\bigcap\limits_{k\geq j}A_s(k)\big)=0$, 故据Hausdorff维数的定义知

$$\dim_H\Big(\bigcap_{k\geq j}A_s(k)\Big)\leq qs+\overline{\beta},\qquad j\geq1.$$

再利用Hausdorff维数的σ-稳定性知

$$\dim_H\Big(\bigcup_{j\geq1}\bigcap_{k\geq j}A_s(k)\Big)=\sup_{j\geq1}\dim_H\Big(\bigcap_{k\geq j}A_s(k)\Big)\leq qs+\overline{\beta},$$

进而有

$$\dim_H\Big(\bigcap_{\alpha(q)<s}\bigcup_{j\geq1}\bigcap_{k\geq j}A_s(k)\Big)\leq\inf_{\alpha(q)<s}(qs+\overline{\beta})=q\alpha(q)+\overline{\beta},\qquad\forall\overline{\beta}>\beta(q).$$

从而

$$\dim_H\Big(\bigcap_{\alpha(q)<s}\bigcup_{j\geq1}\bigcap_{k\geq j}A_s(k)\Big)\leq q\alpha(q)+\beta(q);$$

再利用Hausdorff维数的单调性知

$$\dim_H(K_{\alpha(q)}(\omega))\leq q\alpha(q)+\beta(q)=f(q),\qquad\text{a.s. }\omega.$$

(3) $q<0$时的情形类似可证.

综合(1)、(2)、(3)知定理获证. $\square$

§4. $K_\alpha(\omega)$的Hausdorff维数的下界

任取$\sigma\in\triangle^*$, 记

$$S^{(x_2,\cdots,x_m)}_{\sigma,q,\beta,n}:=\frac{\sum\limits_{\eta\in\triangle^{n-m+1}}P^q_{\sigma,\eta,x_2,\cdots,x_m}l^\beta_{\sigma,\eta,x_2,\cdots,x_m}}{P^q_\sigma l^\beta_\sigma},$$

$$S_{\sigma,q,\beta,n}:=(S^{(x_2,\cdots,x_m)}_{\sigma,q,\beta,n}:1\le x_1,\cdots,x_m\le N).$$

引理 4.1 $\forall\, q,\beta\in\mathbf{R}$, 有$\mathsf{E}[S_{\sigma,q,\beta,n}|\mathcal{F}_{n+|\sigma|-1}]=S_{\sigma,q,\beta,n-1}V^{(q,\beta)}$, $n\ge m$.

证明: 任取$x_2,\cdots,x_m\in\triangle$, 有

$$\begin{aligned}
&\mathsf{E}[S^{(x_2,\cdots,x_m)}_{\sigma,q,\beta,n}|\mathcal{F}_{n+|\sigma|-1}]\\
=&\ \mathsf{E}\Big[\sum_{x_1\in\triangle}\sum_{\eta\in\triangle^{n-m}}\frac{p^q_{\sigma,\eta,x_1,\cdots,x_m}L^\beta_{\sigma,\eta,x_1,\cdots,x_m}P^q_{\sigma,\eta,x_1,\cdots,x_{m-1}}l^\beta_{\sigma,\eta,x_1,\cdots,x_{m-1}}}{P^q_\sigma l^\beta_\sigma}\Big|\mathcal{F}_{n+|\sigma|-1}\Big]\\
=&\ \sum_{x_1\in\triangle}\sum_{\eta\in\triangle^{n-m}}\frac{P^q_{\sigma,\eta,x_1,\cdots,x_{m-1}}l^\beta_{\sigma,\eta,x_1,\cdots,x_{m-1}}}{P^q_\sigma l^\beta_\sigma}\cdot\mathsf{E}(p^q_{\sigma,\eta,x_1,\cdots,x_m}L^\beta_{\sigma,\eta,x_1,\cdots,x_m})\\
=&\ \sum_{x_1\in\triangle}S^{(x_1,\cdots,x_{m-1})}_{\sigma,q,\beta,n-1}\mathsf{E}(p^q_{\sigma,\eta,x_1,\cdots,x_m}L^\beta_{\sigma,\eta,x_1,\cdots,x_m}),
\end{aligned}$$

于是引理获证. □

记

$$Z_{\sigma,q,\beta,n}:=\frac{\sum\limits_{x_2,\cdots,x_m\in\triangle}r^{(q,\beta)}(x_2,\cdots,x_m)S^{(x_2,\cdots,x_m)}_{\sigma,q,\beta,n}}{\rho^n(q,\beta)}.$$

引理 4.2 $(Z_{\sigma,q,\beta,n},\mathcal{F}_{n+|\sigma|})_{n\ge m}$是鞅.

证明: 利用引理4.1的证明过程, 有

$$\begin{aligned}
&\mathsf{E}[Z_{\sigma,q,\beta,n}|\mathcal{F}_{n+|\sigma|-1}]\\
=&\sum_{x_2,\cdots,x_m\in\triangle}\frac{r^{(q,\beta)}(x_2,\cdots,x_m)}{\rho^n(q,\beta)}\mathsf{E}[S^{(x_2,\cdots,x_m)}_{\sigma,q,\beta,n}|\mathcal{F}_{n+|\sigma|-1}]\\
=&\sum_{x_2,\cdots,x_m\in\triangle}\frac{r^{(q,\beta)}(x_2,\cdots,x_m)}{\rho^n(q,\beta)}\sum_{x_1\in\triangle}S^{(x_1,\cdots,x_{m-1})}_{\sigma,q,\beta,n-1}\mathsf{E}(p^q_{x_1,\cdots,x_m}L^\beta_{x_1,\cdots,x_m})\\
=&\sum_{x_1,\cdots,x_{m-1}\in\triangle}\frac{r^{(q,\beta)}(x_1,\cdots,x_{m-1})S^{(x_1,\cdots,x_{m-1})}_{\sigma,q,\beta,n-1}}{\rho^{n-1}(q,\beta)}\\
&\cdot\sum_{x_m\in\triangle}\frac{r^{(q,\beta)}(x_2,\cdots,x_m)\mathsf{E}(p^q_{x_1,\cdots,x_m}L^\beta_{x_1,\cdots,x_m})}{\rho(q,\beta)r^{(q,\beta)}(x_1,\cdots,x_{m-1})}\\
=&\ Z_{\sigma,q,\beta,n-1}\sum_{x_m\in\triangle}\overline{u}^{(q,\beta)}(x_2,\cdots,x_m)=Z_{\sigma,q,\beta,n-1},
\end{aligned}$$

故$(Z_{\sigma,q,\beta,n},\mathcal{F}_{n+|\sigma|})_{n\geq m}$是鞅. $\square$

利用鞅收敛定理知$\lim\limits_{n\to\infty} Z_{\sigma,q,\beta,n}$几乎处处$\omega$存在; 记$X_{\sigma,q}:=\lim\limits_{n\to\infty} Z_{\sigma,q,\beta(q),n}$, a.s.$\,\omega$.

引理 4.3 (1) $X_{\emptyset,q}=X_q/(P_\emptyset^q l_\emptyset^\beta)$;

(2) $P_\sigma^q l_\sigma^\beta X_{\sigma,q}=\sum\limits_{\eta\in\triangle^k} P_{\sigma,\eta}^q l_{\sigma,\eta}^\beta X_{\sigma,\eta,q}$;

(3) $X_q=\sum\limits_{\eta\in\triangle^k} P_\eta^q l_\eta^\beta X_{\eta,q}$.

证明: (1) 对于任意的$n\geq m$, $x_2,\cdots,x_m\in\triangle$, 有

$$S_{\emptyset,q,\beta,n}^{(x_2,\cdots,x_m)}=\sum_{\eta\in\triangle^{n-m+1}}\frac{P_{\eta,x_2,\cdots,x_m}^q l_{\eta,x_2,\cdots,x_m}^\beta}{P_\emptyset^q l_\emptyset^\beta}=\frac{S_{q,\beta,n}^{(x_2,\cdots,x_m)}}{P_\emptyset^q l_\emptyset^\beta},$$

从而$Z_{\emptyset,q,\beta,n}=Z_{q,\beta,n}/(P_\emptyset^q l_\emptyset^\beta)$. 再令$n\to\infty$, 即得$X_{\emptyset,q}=X_q/(P_\emptyset^q l_\emptyset^\beta)$.

(2) 固定$x_2,\cdots,x_m\in\triangle$, 记$\overline{\triangle}_{n-m+1}:=\{\gamma\in\triangle^n:\gamma_{n-m+2}=x_2,\cdots,\gamma_n=x_m\}$, 则

$$S_{\sigma,q,\beta,n}^{(x_2,\cdots,x_m)}=\sum_{\gamma\in\overline{\triangle}_{n-m+1}}\frac{P_{\sigma,\gamma}^q l_{\sigma,\gamma}^\beta}{P_\sigma^q l_\sigma^\beta}=\sum_{\gamma\in\overline{\triangle}_{n-m+1}}\prod_{t=1}^n p_{\sigma,\gamma|t}^q L_{\sigma,\gamma|t}^\beta. \tag{4.1}$$

设$\eta\in\triangle^k$, 则

$$P_{\sigma,\eta}l_{\sigma,\eta}=P_\sigma l_\sigma\prod_{t=1}^k p_{\sigma,\eta|t}L_{\sigma,\eta|t}, \tag{4.2}$$

又如果$\gamma\in\overline{\triangle}_{n-m+1}$, 则$\forall\,\sigma\in\triangle^*$显然有

$$(\sigma,\eta|t)=(\sigma,(\eta,\gamma)|t),\qquad 1\leq t\leq k, \tag{4.3}$$
$$(\sigma,\eta,\gamma|t)=(\sigma,(\eta,\gamma)|t+k),\qquad 1\leq t\leq n, \tag{4.4}$$

于是由(4.1)、(4.2)、(4.3)、(4.4), 有

$$\begin{aligned}
&\sum_{\eta\in\triangle^k}P_{\sigma,\eta}^q l_{\sigma,\eta}^\beta S_{\sigma,\eta,q,\beta,n}^{(x_2,\cdots,x_m)}\\
=&\sum_{\eta\in\triangle^k}P_{\sigma,\eta}^q l_{\sigma,\eta}^\beta\Big(\sum_{\gamma\in\overline{\triangle}_{n-m+1}}\prod_{t=1}^n p_{\sigma,\eta,\gamma|t}^q L_{\sigma,\eta,\gamma|t}^\beta\Big)\\
=&P_\sigma^q l_\sigma^\beta\sum_{\eta\in\triangle^k}\Big(\prod_{t=1}^k p_{\sigma,(\eta\gamma)|t}^q L_{\sigma,(\eta\gamma)t}^\beta\sum_{\gamma\in\overline{\triangle}_{n-m+1}}\prod_{t=k+1}^{k+n}p_{\sigma,(\eta\gamma)|t}^q L_{\sigma,(\eta\gamma)|t}^\beta\Big)\\
=&P_\sigma^q l_\sigma^\beta\Big(\sum_{\eta\in\triangle^k}\prod_{t=1}^{k+n}p_{\sigma,(\eta\gamma)|t}^q L_{\sigma,(\eta\gamma)|t}^\beta\Big)\\
=&P_\sigma^q l_\sigma^\beta\sum_{\eta\in\overline{\triangle}_{n+k-m+1}}\prod_{t=1}^{k+n}p_{\sigma,\eta|t}^q L_{\sigma,\eta|t}^\beta=P_\sigma^q l_\sigma^\beta S_{\sigma,q,\beta,n+k}^{(x_2,\cdots,x_m)}.
\end{aligned}$$

故对于任意的$\sigma\in\triangle^*$, $n\geq m$和k, 有

$$P_\sigma^q l_\sigma^\beta Z_{\sigma,q,\beta,n+k}=\sum_{\eta\in\triangle^k}P_{\sigma,\eta}^q l_{\sigma,\eta}^\beta Z_{\sigma,\eta,q,\beta,n}.$$

再对上式令$n\to\infty$取极限, 便得

$$P_\sigma^q l_\sigma^\beta X_{\sigma,q}=\sum_{\eta\in\triangle^k}P_{\sigma,\eta}^q l_{\sigma,\eta}^\beta X_{\sigma,\eta,q}.$$

(3) 利用已证的结论(1)和(2)有

$$X_q=X_{\emptyset,q}P_\emptyset^q l_\emptyset^\beta=\sum_{\eta\in\triangle^k}P_{\emptyset,\eta}^q l_{\emptyset,\eta}^\beta X_{\emptyset,\eta,q}=\sum_{\eta\in\triangle^k}P_\eta^q l_\eta^\beta X_{\eta,q}. \qquad \square$$

定理 4.1 对每一$\sigma\in\triangle^*$, 随机序列$\{Z_{\sigma,q,\beta(q),n},n\geq m\}$是$L^k$-有界的, 这里$k\in\mathbf{N}$.

证明: 为简便计, 设$\sigma=\emptyset$, 对$\sigma\neq\emptyset$的情形类似可证. 此时有

$$Z_{q,\beta,n}=\frac{\sum\limits_{\sigma\in\triangle^{n-m+1}}\sum\limits_{x_2,\cdots,x_m\in\triangle}P_{\sigma,x_2,\cdots,x_m}^q l_{\sigma,x_2,\cdots,x_m}^\beta r^{(q,\beta)}(x_2,\cdots,x_m)}{\rho^n(q,\beta)}.$$

以下记$\widehat{Z}_{q,\beta,n}:=\rho^n(q,\beta)Z_{q,\beta,n}$.

下面用数学归纳法证明$\forall\, q\in\mathbf{R}$, $\beta\geq\beta(q)$, $t\in\mathbf{N}$, 有

$$\mathsf{E}(\widehat{Z}_{q,\beta,n}^t)\leq ab^n,\qquad n\geq m, \tag{4.5}$$

其中$a=a(q,\beta,t)>0$, $b=b(q,\beta,t)\in(0,1]$.

当$t=1$时, $(Z_{q,\beta,n},\mathcal{F}_n)$是一个鞅, 此时

$$\mathsf{E}(\widehat{Z}_{q,\beta,n})=\mathsf{E}(Z_{q,\beta,n})\rho^n(q,\beta),$$

其中$b=\rho^n(q,\beta)$, $a=\mathsf{E}(Z_{q,\beta,n})$, 即当$t=1$时(4.5)成立.

设$k>1$, 假定(4.5)对任意$t\in\{1,2,\cdots,k-1\}$成立, 则对$t=k$, 由于

$$\widehat{Z}_{q,\beta,n+1}=\sum_{\eta=(\sigma,x_1,\cdots,x_{m-1})\in\triangle^n}P_\eta^q l_\eta^\beta\Big(\sum_{x_m\in\triangle}p_{\eta,x_m}^q L_{\eta,x_m}^\beta r^{(q,\beta)}(x_2,\cdots,x_m)\Big),$$

从而

$$\begin{aligned}\widehat{Z}_{q,\beta,n+1}^k &= \sum_{h=1}^k\sum_{\substack{j_1\geq\cdots\geq j_h\geq1\\ j_1+\cdots+j_h=k}}\sum_{\substack{\eta^{(1)},\cdots,\eta^{(h)}\in\triangle^n\\ \eta^{(s)}\neq\eta^{(t)},s\neq t}}\prod_{i=1}^h P_{\eta^{(i)}}^{j_iq}l_{\eta^{(i)}}^{j_i\beta}\\ &\quad\cdot\Big(\sum_{x_m\in\triangle}p_{\eta^{(i)},x_m}^q L_{\eta^{(i)},x_m}^\beta r^{(q,\beta)}(x_2^{(i)},\cdots,x_{m-1}^{(i)},x_m)\Big)^{j_i},\\ &\qquad\text{其中 }\eta^{(i)}=(\sigma^{(i)},x_1^{(i)},\cdots,x_{m-1}^{(i)})\in\triangle^n.\end{aligned}$$

因为$\{\sum\limits_{x_m\in\triangle}P_{\eta,x_m}^q l_{\eta,x_m}^\beta r^{(q,\beta)}(x_2,\cdots,x_m):\eta\in\triangle^n\}$独立, 且它与$\mathcal{F}_n$独立, 故得

$$\begin{aligned}\mathsf{E}(\widehat{Z}_{q,\beta,n+1}^k|\mathcal{F}_n) &= \sum_{h=1}^k\sum_{\substack{j_1\geq\cdots\geq j_h\geq1\\ j_1+\cdots+j_h=k}}\sum_{\substack{\eta^{(1)},\cdots,\eta^{(h)}\in\triangle^n\\ \eta^{(s)}\neq\eta^{(t)},s\neq t}}\prod_{i=1}^h P_{\eta^{(i)}}^{j_iq}l_{\eta^{(i)}}^{j_i\beta}\\ &\quad\cdot\mathsf{E}\Big[\Big(\sum_{x_m\in\triangle}P_{\eta^{(i)},x_m}^q l_{\eta^{(i)},x_m}^\beta r^{(q,\beta)}(x_2^{(i)},\cdots,x_{m-1}^{(i)},x_m)\Big)^{j_i}\Big].\end{aligned} \tag{4.6}$$

在(4.6)中, 若$h=k$, 则有$j_1=\cdots=j_k=1$. 又

$$\sum_{x_m\in\triangle}\mathsf{E}(p^q_{x_1,\cdots,x_m}L^\beta_{x_1,\cdots,x_m})r^{(q,\beta)}(x_2,\cdots,x_m)=\rho(q,\beta)r^{(q,\beta)}(x_1,\cdots,x_{m-1}),$$

从而在(4.6)式的右端中, 当$h=k$时的那一项为

$$\begin{aligned}&\sum_{\substack{\eta^{(1)},\cdots,\eta^{(h)}\in\triangle^n\\ \eta^{(s)}\neq\eta^{(t)},s\neq t}}\prod_{i=1}^h P^q_{\eta^{(i)}}l^\beta_{\eta^{(i)}}\mathsf{E}\Big(\sum_{x_m\in\triangle}P^q_{\eta^{(i)},x_m}l^\beta_{\eta^{(i)},x_m}\Big)r^{(q,\beta)}(x_2^{(i)},\cdots,x_{m-1}^{(i)},x_m)\\ =\;&\rho^k(q,\beta)\sum_{\substack{\eta^{(1)},\cdots,\eta^{(h)}\in\triangle^n\\ \eta^{(s)}\neq\eta^{(t)},s\neq t}}\prod_{i=1}^h P^q_{\eta^{(i)}}l^\beta_{\eta^{(i)}}r^{(q,\beta)}(x_1^{(i)},\cdots,x_{m-1}^{(i)})\le\rho^k(q,\beta)\widehat{Z}^k_{q,\beta,n},\end{aligned}\tag{4.7}$$

在(4.6)中, 若$h<k$, 则$j_1\ge2$. 因为

$$\max_{1\le j\le k}\max_{x_1,\cdots,x_{m-1}\in\triangle}\mathsf{E}[(p^q_{x_1,\cdots,x_m}L^\beta_{x_1,\cdots,x_m}r^{(q,\beta)}(x_2,\cdots,x_m))^j]<+\infty,$$

且

$$\min_{1\le j\le k}\min_{x_1,\cdots,x_{m-1}\in\triangle}r^{(jq,j\beta)}(x_1,\cdots,x_{m-1})>0,$$

故存在常数$c_1=c_1(q,\beta,k)$, 使得

$$\begin{aligned}&\sum_{\substack{\eta^{(1)},\cdots,\eta^{(h)}\in\triangle^n\\ \eta^{(s)}\neq\eta^{(t)},s\neq t}}\prod_{i=1}^h P^{j_iq}_{\eta^{(i)}}l^{j_i\beta}_{\eta^{(i)}}\mathsf{E}\Big[\Big(\sum_{x_m}P^q_{\eta^{(i)},x_m}l^\beta_{\eta^{(i)},x_m}r^{(q,\beta)}(x_2^{(i)},\cdots,x_m^{(i)})\Big)^{j_i}\Big]\\ \le\;&c_1\sum_{\substack{\eta^{(1)},\cdots,\eta^{(h)}\in\triangle^n\\ \eta^{(s)}\neq\eta^{(t)},s\neq t}}\prod_{i=1}^h P^{j_iq}_{\eta^{(i)}}l^{j_i\beta}_{\eta^{(i)}}r^{(j_iq,j_i\beta)}(x_1^{(i)},\cdots,x_{m-1}^{(i)})\le c_1\prod_{i=1}^h Z_{j_iq,j_i\beta,n},\end{aligned}\tag{4.8}$$

从而由(4.6)、(4.7)、(4.8)得

$$\mathsf{E}(\widehat{Z}^k_{q,\beta,n+1}|\mathcal{F}_n)\le\rho^k(q,\beta)\widehat{Z}^k_{q,\beta,n}+c_1\sum_{h=1}^{k-1}\sum_{\substack{j_1\ge\cdots\ge j_h\ge1\\ j_1+\cdots+j_h=k}}\prod_{i=1}^h Z_{j_iq,j_i\beta,n},\tag{4.9}$$

对(4.9)两边取数学期望, 并利用Hölder不等式的推广形式

$$\mathsf{E}\Big(\prod_{i=1}^h\widehat{Z}_{j_iq,j_i\beta,n}\Big)\le\prod_{i=1}^h\|\widehat{Z}_{j_iq,j_i\beta,n}\|_h,$$

便有

$$\mathsf{E}(\widehat{Z}^k_{q,\beta,n+1})\le\rho^k(q,\beta)\mathsf{E}(\widehat{Z}^k_{q,\beta,n})+c_1\sum_{h=1}^{k-1}\sum_{\substack{j_1\ge\cdots\ge j_h\ge1\\ j_1+\cdots+j_h=k}}\mathsf{E}\Big(\prod_{i=1}^h\widehat{Z}_{j_iq,j_i\beta,n}\Big),\tag{4.10}$$

记$M_n := \sum_{h=1}^{k-1} \sum_{\substack{j_1 \ge \cdots \ge j_h \ge 1 \\ j_1+\cdots+j_h=k}} \prod_{i=1}^{h} \|\widehat{Z}_{j_i q, j_i \beta, n}\|_h$, 则得

$$\mathsf{E}(\widehat{Z}^k_{q,\beta,n+1}) \le \rho^k(q,\beta)\mathsf{E}(\widehat{Z}^k_{q,\beta,n}) + c_1 M_n. \tag{4.11}$$

对(4.11)式进行迭代得

$$\mathsf{E}(\widehat{Z}^k_{q,\beta,n+1}) \le \rho^{(n-m+1)k}(q,\beta)\mathsf{E}(\widehat{Z}^k_{q,\beta,m}) + c_1 \sum_{t=m}^{n} (\rho(q,\beta))^{(n-t)k} M_t. \tag{4.12}$$

现在$h \le k-1$, 由$j_i\beta \ge \beta(j_i q)$, 利用归纳假设, 如果$j_i \ge 1$, 有$\sup\limits_n \|\widehat{Z}_{j_i q, j_i \beta, n}\|_h < +\infty$. 因为$j_1 \ge 2$, $j_1\beta \ge \beta(j_1 q)$, 再次利用归纳假设得

$$\mathsf{E}(\widehat{Z}^h_{j_1 q, j_1 \beta, n}) \le a(j_1 q, j_1 \beta, h) b(j_1 q, j_1 \beta, h)^n,$$

其中$a(j_1 q, j_1\beta, h) > 0$, $b(j_1 q, j_1 \beta, h) \in (0,1)$, 从而由(4.12)知(4.5)当$t=k$时成立. $\square$

推论 4.1 $\forall \sigma \in \triangle^*$, $k \in \mathbf{N}$, 有$\mathsf{E}(X^k_{\sigma,q}) < +\infty$.

推论 4.2 $\mathsf{P}(X_{\sigma,q} > 0) = 1$, $\forall \sigma \in \triangle^*$.

证明: 因为$\forall \sigma \in \triangle^*$, $X_{\sigma,q}$是鞅$(Z_{\sigma,q,n}, \mathcal{F}_{|\sigma|+n})_{n\ge m}$的右闭元, 所以有$\mathsf{E}(X_{\sigma,q}) = \mathsf{E}(Z_{\sigma,q,\beta,m}) > 0$, 从而有$\mathsf{P}(X_{\sigma,q} > 0) > 0$. 考虑$\sigma$-代数$\mathcal{F}^n_n = \sigma\{p_\eta, L_\eta : \eta \in \triangle^n\}$, $\mathcal{F}^n = \sigma\{p_\eta, L_\eta : \eta \in \triangle^*, |\eta| \ge n\}$, $n \in \mathbf{N}$. 因为$\{\mathcal{F}^n_n\}$是独立的, 利用随机变量$X_{\sigma,q}$的结构知, $\forall \sigma \in \triangle^*$, 事件$\{X_{\sigma,q} > 0\}$属于尾$\sigma$-代数$\mathcal{F}^n$, $\forall n > 0$, 从而由Kolmogorov's 0-1律知$\mathsf{P}(X_{\sigma,q} > 0) = 1$. $\square$

以下记$\triangle^{(n)} := \bigcup\limits_{t=1}^{n} \triangle^t$, $\triangle^{(n)}_0 := \bigcup\limits_{t=0}^{n} \triangle^t$.

引理 4.4 若$|\sigma| = k \ge m-1$, 且$\sigma_{k-m+2} = x_1, \cdots, \sigma_k = x_{m-1}$, 则$\forall n \in \mathbf{N}$, 随机向量$(p_{\sigma,\eta} L_{\sigma,\eta}; \eta \in \triangle^{(n)})$与$(p_{x_1,\cdots,x_{m-1},\eta} L_{x_1,\cdots,x_{m-1},\eta}; \eta \in \triangle^{(n)})$同分布.

证明: 欲证结论成立, 只需证明对任何的Borel集族$(B_\eta : \eta \in \triangle^{(n)})$, 有

$$\mathsf{P}(p_{\sigma,\eta} L_{\sigma,\eta} \in B_\eta : \eta \in \triangle^{(n)}) = \mathsf{P}(p_{x_1,\cdots,x_{m-1},\eta} L_{x_1,\cdots,x_{m-1},\eta} \in B_\eta : \eta \in \triangle^{(n)}).$$

事实上

$$\begin{aligned}
&\mathsf{P}(p_{\sigma,\eta} L_{\sigma,\eta} \in B_\eta : \eta \in \triangle^{(n)}) \\
=\ & \mathsf{P}(p_{\sigma,\eta,x} L_{\sigma,\eta,x} \in B_{\eta,x} : \eta \in \triangle^{(n-1)}_0, x \in \triangle) \\
=\ & \prod_{\eta \in \triangle^{(n-1)}_0} \mathsf{P}(p_{\sigma,\eta,x} L_{\sigma,\eta,x} \in B_{\eta,x} : x \in \triangle) \\
=\ & \prod_{\eta \in \triangle^{(n-1)}_0} \mathsf{P}(p_{x_1,\cdots,x_{m-1},\eta,x} L_{x_1,\cdots,x_{m-1},\eta,x} \in B_{\eta,x} : x \in \triangle) \\
=\ & \mathsf{P}(p_{x_1,\cdots,x_{m-1},\eta,x} L_{x_1,\cdots,x_{m-1},\eta,x} \in B_{\eta,x} : \eta \in \triangle^{(n-1)}_0, x \in \triangle) \\
=\ & \mathsf{P}(p_{x_1,\cdots,x_{m-1},\eta} L_{x_1,\cdots,x_{m-1},\eta} \in B_\eta : \eta \in \triangle^{(n)}). \qquad \square
\end{aligned}$$

引理 4.5 若$|\sigma| = k \geq m-1$, 且$\sigma_{k-m+2} = x_1, \cdots, \sigma_k = x_{m-1}$, 则$X_{\sigma,q}$与$X_{x_1,\cdots,x_{m-1}}$同分布.

证明: 易知$S^{(x_1,\cdots,x_{m-1})}_{\sigma,q,\beta,n} = \sum\limits_{\eta\in\triangle^{n-m+1}} \prod\limits_{t=1}^{n} p^q_{\sigma,(\eta,x_1,\cdots,x_{m-1})|t} L^{\beta}_{\sigma,(\eta,x_1,\cdots,x_{m-1})|t}$, $n \geq m-1$. 由引理4.4知$S^{(x_1,\cdots,x_{m-1})}_{\sigma,q,\beta,n}$与$S^{(x_1,\cdots,x_{m-1})}_{x_1,\cdots,x_{m-1},q,\beta,n}$同分布, 从而$Z_{\sigma,q,\beta,n}$与$Z_{x_1,\cdots,x_{m-1},q,\beta,n}$同分布, 又由$X_{\sigma,q} = \lim\limits_{n\to\infty} Z_{\sigma,q,\beta,n}$即知$X_{\sigma,q}$与$X_{x_1,\cdots,x_{m-1}}$同分布. □

4.1 $K_\alpha(\omega)$的Hausdorff维数的下界及主要结果

考虑集类

$$\mathcal{C} := \{C_n(\sigma) : n \in \mathbf{N}, \sigma \in \triangle^{\mathbf{N}}\} \cup \{\emptyset\},$$

则易见$\mathcal{C}$是一个半环, 令$\mathcal{A} = \sigma(\mathcal{C})$.

定理 4.2 设μ_ω是$\mathcal{C}$上的集函数, 满足

(1) $\mu_\omega(\emptyset) = 0$;

(2) $\mu_\omega(C_n(\sigma)) = P^q_\sigma l^\beta_\sigma X_{\sigma,q}$, $\forall C_n(\sigma) \in \mathcal{C}$, 则$\mu_\omega$是$\mathcal{A}$上的有限测度, 且$\mu_\omega(\triangle^{\mathbf{N}}) = X_q$, a.s. ω.

证明: 设$\{A_i\} \subset \mathcal{C}$, $A_i \cap A_j = \emptyset$, $i \neq j$, $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{C}$. 由于$\forall C_n(\sigma) \in \mathcal{C}$, 有$C_n(\sigma) = \bigcup\limits_{\eta\in\triangle^k} C_{n+k}((\sigma,\eta))$, $\forall k \in \mathbf{N}$. 表明可把低阶柱集表示成高阶的柱集之并, 故不妨设诸A_i同阶的. 因为$\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{C}$, 故存在$n_0$及$\sigma \in \triangle^{n_0}$, 使得$C_{n_0}(\sigma) = \bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{C}$, 于是可设$A_i$是$n_0 + m_0$阶的, 又由于对给定的$k$表达式$C_n(\sigma) = \bigcup\limits_{\eta\in\triangle^k} C_{n+k}((\sigma,\eta))$是唯一的, 因此

$$\begin{aligned}
\mu_\omega\Big(\bigcup_{i=1}^{\infty} A_i\Big) &= \mu_\omega(C_{n_0}(\sigma)) = P^q_\sigma l^\beta_\sigma X_{\sigma,q} = \sum_{\eta\in\triangle^{m_0}} P^q_{\sigma,\eta} l^\beta_{\sigma,\eta} X_{\sigma,\eta,q} \\
&= \sum_{\eta\in\triangle^{m_0}} \mu_\omega(C_{n_0+m_0}((\sigma,\eta))) = \sum_{i=1}^{\infty} \mu_\omega(A_i),
\end{aligned}$$

从而μ_ω是$\mathcal{C}$上的测度; 再利用Carathéodory 测度扩张定理知μ_ω可扩张成$\mathcal{A}$的测度且唯一, 仍记作μ_ω, 且$\mu_\omega(\triangle^{\mathbf{N}}) = \sum\limits_{\sigma\in\triangle^n} \mu_\omega(C_n(\sigma)) = \sum\limits_{\sigma\in\triangle^n} P^q_\sigma l^\beta_\sigma X_{\sigma,q} = X_q < +\infty$, a.s. ω. □

设μ为$\mathcal{A}$的有限测度, 称

$$\underline{d}_{\mathcal{B},\mu}(\xi) = \varliminf_{r\to 0} \frac{\log \mu(B(\xi,r))}{\log r}, \qquad \underline{d}_{\mathcal{C},\mu}(\xi) = \varliminf_{n\to\infty} \frac{\log \mu(C_n(\xi))}{\log(l_{[\xi|n]})} \tag{4.13}$$

分别为测度μ在点$\xi \in \triangle^{\mathbf{N}}$的球态下局部维数与柱态下局部维数. 显然上述定义对刚引入的μ_ω也适用.

引理 4.6[1] 如果条件(4)、(5)成立, 则有$\underline{d}_{\mathcal{B},\mu}(\xi) \geq \underline{d}_{\mathcal{C},\mu}(\xi)$, $\xi \in \triangle^{\mathbf{N}}$.

引理 4.7[1] 设μ为$\mathcal{A}$上的有限测度, 如果$\mu(F)>0$, 且在F上$\underline{d}_{\mathcal{B},\mu}(\xi)\geq s$, 则$\dim_H(F)\geq s$.

定理 4.3 $\mathsf{P}(\underline{d}_{\mathcal{C},\mu_\omega}(\pi)\geq f(q))=1, \forall\,\pi\in\overline{K}_\alpha(\omega)$.

证明: 利用推论4.1知$\forall\,t>0$, $\mathsf{E}(X^t_{\sigma,q})<+\infty$. 现固定$c>0$, 因为$P_\sigma l_\sigma$与$X_\sigma$是独立的, 所以$\forall\,\overline{\beta}<\beta$, 有

$$\mathsf{P}(P^q_\sigma l^\beta_\sigma X_{\sigma,q}>cP^q_\sigma l^{\overline{\beta}}_\sigma)\leq\frac{\mathsf{E}(l_\sigma^{(\beta-\overline{\beta})t})\mathsf{E}(X^t_{\sigma,q})}{c^t}\leq\frac{M}{c^t}\mathsf{E}(l_\sigma^{(\beta-\overline{\beta})t}),$$

其中$M=\max\limits_{x_1,\cdots,x_{m-1}\in\triangle}\mathsf{E}(X^t_{x_1,\cdots,x_{m-1},q})<+\infty$. 注意到$\overline{Z}_{q,\overline{\beta},n}=\sum\limits_{\sigma\in\triangle^n}P^q_\sigma l^{\overline{\beta}}_\sigma$, 于是对某常数$b>0$, 有$\overline{Z}_{q,\overline{\beta},n}\leq b\rho^n(q,\overline{\beta})Z_{q,\overline{\beta},n}$, 从而

$$\begin{aligned}\sum_{\sigma\in\triangle^n}\mathsf{P}(P^q_\sigma l^\beta_\sigma X_{\sigma,q}>cP^q_\sigma l^{\overline{\beta}}_\sigma)&\leq\sum_{\sigma\in\triangle^n}\frac{M}{c^t}\mathsf{E}(l_\sigma^{(\beta-\overline{\beta})t})=\frac{M}{c^t}\mathsf{E}(\overline{Z}_{0,(\beta-\overline{\beta})t,n})\\&\leq\frac{Mb}{c^t}\rho^n(0,(\beta-\overline{\beta})t)\mathsf{E}(Z_{0,(\beta-\overline{\beta})t,n}).\end{aligned}$$

今选择t_0使得$\rho(0,(\beta-\overline{\beta})t_0)<1$, 则有

$$\begin{aligned}\sum_{n=0}^\infty\mathsf{P}(\exists\,\sigma\in\triangle^n:P^q_\sigma l^\beta_\sigma X_{\sigma,q}>cP^q_\sigma l^{\overline{\beta}}_\sigma)&=\sum_{n=0}^\infty\sum_{\sigma\in\triangle^n}\mathsf{P}(P^q_\sigma l^\beta_\sigma X_{\sigma,q}>cP^q_\sigma l^{\overline{\beta}}_\sigma)\\&\leq\frac{Mb}{c^t}\mathsf{E}(Z_{0,(\beta-\overline{\beta})t,n})\sum_{n=0}^\infty\rho^n(0,(\beta-\overline{\beta})t_0)\\&<+\infty.\end{aligned}$$

利用Borel-Cantelli引理, 知

$$\mathsf{P}\Big(\bigcup_{k=1}^\infty\bigcap_{n=k}^\infty\{\forall\,\sigma\in\triangle^n:P^q_\sigma l^\beta_\sigma X_{\sigma,q}\leq cP^q_\sigma l^{\overline{\beta}}_\sigma\}\Big)=1.$$

注意到$\forall\,\pi\in\overline{K}_\alpha(\omega)$, $\pi|n\in\triangle^n$, 于是有

$$\mathsf{P}(\{\omega:\exists\,n_0,\forall\,n\geq n_0,\forall\,\pi\in\overline{K}_\alpha(\omega),P^q_{\pi|n}l^\beta_{\pi|n}X_{\pi|n,q}\leq kP^q_{\pi|n}l^{\overline{\beta}}_{\pi|n}\})=1.$$

故可以取定ω, 使得

$$P^q_{\pi|n}l^\beta_{\pi|n}X_{\pi|n,q}\leq kP^q_{\pi|n}l^{\overline{\beta}}_{\pi|n},$$

再注意到由于$0<l_{\pi|n}(\omega)<1$知$\log l_{\pi|n}(\omega)<0$, 于是

$$\frac{\log(P^q_{\pi|n}l^\beta_{\pi|n}X_{\pi|n,q})}{\log l_{\pi|n}}\geq\frac{\log(kP^q_{\pi|n}l^{\overline{\beta}}_{\pi|n})}{\log l_{\pi|n}}.$$

进而据定理4.2 (2), 有

$$\frac{\log(\mu_\omega(c_n(\pi)))}{\log l_{\pi|n}}\geq\frac{\log k}{\log l_{\pi|n}}+q\frac{\log P_{\pi|n}}{\log l_{\pi|n}}+\overline{\beta},\qquad\forall\,\overline{\beta}<\beta,$$

令$n \to \infty$, 由(4.13), 得

$$\underline{d}_{\mathcal{C},\mu}(\pi) \geq q\alpha + \beta, \qquad \forall \overline{\beta} < \beta,$$

从而

$$\underline{d}_{\mathcal{C},\mu}(\pi) \geq q\alpha(q) + \beta(q) = f(q). \qquad \square$$

下面给出本文的主要结果.

定理 4.4 设第一章假设中条件(3)、(4)、(5)成立, 则

$$\mathsf{P}(\dim_H K_{\alpha(q)} = f(q)) = 1.$$

证明: 记$\lambda_k = \rho \circ g_\omega^{-1}$, 则$\lambda_k$是$K(\omega)$上的一个度量. 当条件(3)成立时, $g_\omega : \triangle^N \longmapsto K(\omega)$是一个一一等距映射, 于是有$\dim_H(K_\alpha(\omega)) = \dim_H(\overline{K}_\alpha(\omega))$. 由定理4.3知

$$\underline{d}_{\mathcal{C},\mu_\omega}(\pi) \geq f(q), \ \ \text{a.s.}\,\omega, \qquad \forall\,\pi \in \overline{K}_\alpha(\omega),$$

运用引理4.6有$\underline{d}_{\mathcal{B},\mu_\omega}(\pi) \geq \underline{d}_{\mathcal{C},\mu_\omega}(\pi)$, 于是

$$d_{\mathcal{B},\mu}(\pi) \geq f(q), \ \ \text{a.s.}\,\omega, \qquad \forall\,\pi \in \overline{K}_\alpha(\omega),$$

再运用引理4.7有

$$\dim_H(K_\alpha(\omega)) \geq f(q), \qquad \text{a.s.}\,\omega,$$

结合第三章定理3.1 $\mathsf{P}(\dim_H K_{\alpha(q)} \leq f(q)) = 1$, 立得$\mathsf{P}(\dim_H K_{\alpha(q)} = f(q)) = 1$. 定理证毕. $\square$

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Mutlifractal Decomposition of Random Fractal with Finite Memory

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The Hausdorff dimension about random fractal sets with finite memory has been studied by Dryakhlov and Tempelman. In this paper, the multifractal decomposition of random fractal sets with finite memory is researched and under some certain conditions the Hausdorff dimension expression of the multifractal decomposition set $K_\alpha(\omega)$ for random fractal set with finite memory is given.

Keywords: Random fractal with finite memory, multifractal decomposition, Perron-Frobenius theorem, martingale, local dimension.

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