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Asymptotic Efficiency in a Partly Autoregressive Model*

LIANG HUA
(Institute of System Science, Academia Sinica, Beijing, 100080)

Abstract

Consider the model $X_t = X_{t-1}\beta + g(U_t) + \varepsilon_t$ for $t \geq 1$. Here g is an unknown function, β is an unknown parameter to be estimated and ε_j are i.i.d. with mean 0 and variance σ^2 and U_t are i.i.d. random variables obeyed uniformly on [0, 1]. The order of convergence of consistent estimators and the bound of asymptotic efficiency in sense of Takeuchi are given. Meanwhile we give a necessary and sufficient condition that the least squares of β is asymptotically efficient, and we also show that the MLE is asymptotically efficient.

Keywords: asymptotic efficiency, partly autoregressive model, piecewise polynomial.

AMS Subject Classification: 62M10.

§1. Introduction

Consider the partly linear autoregressive model

$$X_t = \beta X_{t-1} + g(U_t) + \varepsilon_t \quad (t = 1, 2, \cdots), \tag{1.1}$$

where $X_0 = 0, g(\cdot)$ is an unknown function in R^1, $\{U_t . t = 1, 2, \cdots\}$ is i.i.d. sequence of random variables obeyed uniformly on [0, 1] and $\{\varepsilon_t : t = 1, 2, \cdots\}$ is a sequence of i.i.d. random variables having a density function $f(\cdot)$ (with respect to the Lebesgue measure) with mean 0, variance σ^2 and finite fourth moment. Assume that $|\beta| < 1$ and $f(u) > 0$ for all u.

The model defined (1.1) belongs to the class of partly linear regression models, which was first proposed by Robinson(1988). In recently years, a lot of literature discussed partly linear model $Y_t = X_i^T\beta + g(U_i) + \varepsilon_i$, $i = 1, 2, \cdots$ .Here (X_i, U_i) are i.i.d. random samples or known design points. See, for example, Heckmen (1986, 1988), Rice(1988), Speckmen(1988), Hidalgo(1992), Liang(1995), Liang and Cheng(1993).

More recently, Gao and Liang(1993) has investigated the asymptotic normality of LS estimator for the model $X_t = \beta X_{t-1} + g(X_{t-2}) + \varepsilon_t \quad (t = 3, \cdots)$, which based on g

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estimated by piecewise polynomial.

In this paper, we consider the asymptotic efficiency of some estimator of β for the model (1.1). In section 2, we shall give the order of consistent estimator. In section 3, we shall obtain the bound of the asymptotic normality in the sense of Takeuchi. In section 4, we shall give a necessary and sufficient condition that the least squares estimator of β is asymptotically efficient, finally, we shall also show that the MLE of β is asymptotically efficient.

Suppose that $\mathcal{B}$ is an open interval (-1. 1). Assume g is square integrable and $\int_0^1 g(u)du = 0$. Denote $J(g) = \int_0^1 g^2(u)du$,

$$\boldsymbol{X}_n = (X_1, \cdots, X_n)^\tau, \qquad \boldsymbol{X}_n^* = (X_0, \cdots, X_{n-1})^\tau,$$
$$\boldsymbol{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)^\tau, \qquad \boldsymbol{g}(U) = (g(U_1), \cdots, g(U_n))^\tau.$$

§2 The order of convergence of consistent estimators

In this section we consider the order of convergence of consistent estimators of β in model (1.1). Let

$$I_n(\beta_1, \beta_2) = \int \frac{\prod d\mathsf{P}_{\beta_1,t}}{\prod dx_t} \log\left(\frac{\prod d\mathsf{P}_{\beta_1,t}}{\prod d\mathsf{P}_{\beta_2,t}}\right) \prod dx_t.$$

Definition 2.1 For an increasing sequence of positive numbers $\{C_n\}$ (C_n tending to infinity), an estimator β_n is called consistent with order $\{C_n\}$if for any $\zeta > 0$ and every η of $\mathcal{B}$, there exist a sufficiently small positive number δ and sufficiently large positive number L satisfying the following:

$$\overline{\lim_{n\to\infty}} \sup_{\|\beta-\eta\|\le\delta} \mathsf{P}_{\beta,n}\{C_n\|\beta_n - \beta\| \ge L\} < \zeta. \tag{2.1}$$

Since the joint density of $X_1, \cdots, X_n$ is given by $\prod_{t=1}^{n} f(x_t - \beta x_{t-1} - g(u_t))$, it follows from that for every β_1, β_2 with $\beta_1 \ne \beta_2$

$$I_n(\beta_1, \beta_2) = \int \cdots \int \prod_{t=1}^{n} f(x_t - \beta_1 x_{t-1} - g(u_t)) \sum_{t=1}^{n} \log \frac{f(x_t - \beta_1 x_{t-1} - g(u_t))}{f(x_t - \beta_2 x_{t-1} - g(u_t))} \prod_{t=1}^{n} dx_t.$$

Putting $\beta_2 = \beta_1 + \Delta$, we have

$$\begin{aligned} I_n(\beta_1, \beta_2) &= \int \cdots \int \prod_{t=1}^{n} f(\varepsilon_t) \sum_{t=1}^{n} \log \frac{f(\varepsilon_t)}{f(u_t - \Delta x_{t-1})} \prod_{t=1}^{n} d\varepsilon_t \\ &= \sum_{t=1}^{n} \mathsf{E}\left[\int f(\varepsilon_t) \log \frac{f(\varepsilon_t)}{f(\varepsilon_t - \Delta x_{t-1})} d\varepsilon_t\right]. \end{aligned}$$

For all $|\Delta|$ we obtain $\int f(u)\log[f(u)/f(u-\Delta)] = O(\Delta^2)$. If $\log[f(u)/f(u-\Delta)] = O(\Delta^2)$ for large $|\Delta|$, then we have $\sup_{0<|\Delta|<\infty} \Delta^{-2} \int f(u)\log[f(u)/f(u-\Delta)] = O(1)$. Thus we have

$$I_n(\beta_1, \beta_2) = O\left(\Delta^2 \sum_{t=1}^{n} \mathsf{E}(X_{t-1}^2)\right). \tag{2.2}$$

Since for each t, $\mathsf{E}(X_t) = \beta\mathsf{E}(\mathsf{E}_{t-1}) = \cdots = \beta^t\mathsf{E}(X_0) = 0$,

$$\mathsf{E}(X_t^2) = \beta^2\mathsf{E}(X_{t-1}^2) + \mathsf{E}(\varepsilon_t^2) + J(g) = (\beta^{2(t-1)} + \cdots + \beta^2 + 1)(\sigma^2 + J(g))$$
$$= \frac{1-\beta^{2t}}{1-\beta^2}(\sigma^2 + J(g)),$$

we obtain

$$\sum_{t=1}^{n} \mathsf{E}(X_{t-1}^2) = \left(\frac{n}{1-\beta^2} - \frac{1-\beta^{2n}}{(1-\beta^2)^2}\right)(\sigma^2 + J(g)). \tag{2.3}$$

In order to show that the order of convergence of consistent estimators is $\sqrt{n}$, we need the following theorem.

Theorem 2.1 Suppose that for each $n, \{X_n : \prod_{t=1}^{n} f(X_t - \beta x_{t-1} - g(u_t)) > 0\}$ does not depend on β. If there exists a $\{C_n\}$-consistent estimator, then the following holds: for every $\beta \in \mathcal{B}$ and every $a \neq 0$

$$\lim_{L\to\infty} \underline{\lim}_{n\to\infty} I_n(\beta, \beta + aLC_n^{-1}) = \infty. \tag{2.4}$$

Proof See the proof of Theorem 2.2.2 of Akahira and Takeuchi(1981).

Letting $\Delta = LC_n^{-1}$, in order to get $\lim_{L\to\infty} \underline{\lim}_{n\to\infty} I_n(\beta, \beta + aLC_n^{-1}) = \infty$ from (2.2), (2.3) and Theorem 2.1 we have to obtain $C_n = O(\sqrt{n})$. Hence, it is seen that the order of convergence of consistent estimator is $\sqrt{n}$.

§3 The bound of asymptotic distribution of AMU estimators

In this section, we shall give the bound of asymptotic distribution of AMU estimator in the sense of Takeuchi. At first, we list some assumptions which are sufficient for our main results.

A.S.1 $f(\cdot)$ is differentiable and $f(u) > 0$ for all u and $\lim_{|u|\to\infty} f(u) = 0$.

A.S.2 $f(\cdot)$ is three times continuously differentiable on the real line and $\lim_{|u|\to\infty} f'(u) = 0$.

A.S.3 $\dfrac{\partial^2 f(u)}{\partial u^2}$ is a bounded function and $E|\varepsilon_i|^4 < \infty$.

A.S.4 For each $\beta_0 \in \mathcal{B}$ the following hold:

(a) $\displaystyle\lim_{n\to\infty} n^{-3/2} \sum_{t=1}^{n} \mathsf{E}_{\beta_j}\left\{|X_{t-1}|^3 \sup_{0<|\eta|<un^{-1/2}|X_{t-1}|} |k'(\varepsilon_t + \eta)|\right\} = 0, \quad (j = 0, 1);$

(b) $\displaystyle\lim_{n\to\infty} n^{-3} \sum_{t=1}^{n} \mathsf{E}_{\beta_j}\left\{|X_{t-1}|^3 \sup_{0<|\eta|<un^{-1/2}|X_{t-1}|} |k'(\varepsilon_t + \eta)|\right\}^2 = 0, \quad (j = 0, 1),$

where $k(x) = -\dfrac{\partial^2 \log f(x)}{\partial x^2}$.

A.S.5 $\mathsf{E}\left|\dfrac{f'(\varepsilon_t)}{f(\varepsilon_t)}\right|^4 < \infty.$

A.S.6 $\lim_{|u|\to\infty} uf(u) = 0.$

Definition 3.1 β_n is asymptotically median unbiased (AMU for short) if for any $\nu \in \mathcal{B}$, there exists a positive number δ such that

$$\lim_{n\to\infty} \sup_{\beta:|\beta-\nu|<\delta} C_n|\mathsf{P}_{\beta,n}\{\beta_n \le \beta\} - 1/2| = 0,$$

$$\lim_{n\to\infty} \sup_{\beta:|\beta-\nu|<\delta} C_n|\mathsf{P}_{\beta,n}\{\beta_n \ge \beta\} - 1/2| = 0.$$

Definition 3.2 For β_n asymptotically median unbiased, a distribution function $F(t,\beta)$ is called an asymptotically distribution of $C_n(\beta_n - \beta)$ if for each t, $F(t,\beta)$ is continuous in β and for every continuity point of $F(t,\beta)$

$$C_n|\mathsf{P}_{\beta,n}\{C_n(\beta_n - \beta) \le t\} - F(t,\beta)| \to 0.$$

Since β_n is a $\{C_n\}$-consistent estimator, it follows that $F(-\infty,\beta) = 0$ and $F(\infty,\beta) = 1$.Let β_n be an AMU estimator. Then it follows that $F(0,\beta^-) \le 1/2$ and $F(0,\beta^+) \ge 1/2$ for all $\beta \in \mathcal{B}$. For β_n AMU, that G_β^+ and G_β^- are defined as follows:

$$G_\beta^+ = \overline{\lim_{n\to\infty}}\, \mathsf{P}_{\beta,n}\{C_n(\beta_n - \beta) \le t\} \quad \forall t \ge 0, \tag{3.1}$$

$$G_\beta^- = \underline{\lim_{n\to\infty}}\, \mathsf{P}_{\beta,n}\{C_n(\beta_n - \beta) \le t\} \quad \forall t \le 0, \tag{3.2}$$

Let β_0 ($\in \mathcal{B}$) be arbitrary but fixed, we consider the problem of testing hypothesis

$$H^+: \quad \beta = \beta_1 = \beta_0 + \frac{u}{\sqrt{n}}(u > 0) \leftrightarrow K: \quad \beta = \beta_0.$$

We define

$$\Gamma_{\beta_0}^+ = \sup_{\phi\in\Phi_{1/2}} \overline{\lim_{n\to\infty}}\, E_{\beta_0,n}\phi_n, \tag{3.3}$$

where $\Phi_{1/2} = \{\phi_n : \mathsf{E}_{\beta_0+u/C_n,n}\phi_n = \frac{1}{2}, 0 \le \phi_n(X_n) \le 1$ for all X_n $(n = 1,\cdots)\}$. Putting $A_{\beta_n,\beta_0} = \{\sqrt{n}(\beta_n - \beta_0) \le u\}$, we have for all $u > 0$.

$$\mathsf{P}_{\beta_0+u/C_n,n}(A_{\beta_n,\beta_0}) = \mathsf{P}_{\beta_0+u/C_n,n}\{C_n(\beta_n - \beta_0 - u/C_n) \le 0\} \to \frac{1}{2}.$$

Since a sequence $\{\chi_{A_{\beta_n,\beta_0}}\}$ of the indicators of A_{β_n,β_0} $(n = 1,2,\cdots)$ belongs to $\Phi_{1/2}$, it follows from (3.1) and (3.3) that

$$G_{\beta_0}^+ \le \Gamma_{\beta_)}^+ \quad \forall u > 0. \tag{3.4}$$

Similarly to consider the next problem of testing the hypothesis

$$H^-: \quad \beta = \beta_0 + u/C_n\,(u < 0) \leftrightarrow K: \quad \beta = \beta_0.$$

We define

$$\Gamma_{\beta_0}^- = \inf_{\phi_n\in\Phi_{1/2}} \underline{\lim_{n\to\infty}}\, E_{\beta_0,n}\phi_n. \tag{3.5}$$

Note that

$$\Gamma^-_{\beta_0} = 1 - \sup_{\phi_n \in \Phi_{1/2}} \overline{\lim_{n\to\infty}} E_{\beta_0,n}\phi_n. \tag{3.6}$$

In the similarly way as the case $u > 0$, we have

$$G^-_{\beta_0} \geq \Gamma^-_{\beta_0} \quad \forall u < 0. \tag{3.7}$$

Since β_0 is arbitrary, the bound of the asymptotic distribution of AMU estimators are given by:

$$G^+_{\beta_0} \leq \Gamma^+_{\beta_0} \quad \forall u > 0, \quad G^-_{\beta_0} \geq \Gamma^-_{\beta_0} \quad \forall u < 0.$$

Definition 3.3 An AMU estimator β_n is asymptotically efficient if for each $\beta \in B$

$$F(u,\beta) = \begin{cases} \Gamma^+_{\beta_0}, & \forall\, u > 0, \\ \Gamma^-_{\beta_0}, & \forall\, u < 0. \end{cases}$$

It can easily see that for any AMU estimator β_n and any $\beta \in B$

$$\overline{\lim_{n\to\infty}} \mathrm{P}_{\beta,n}\{C_n(\beta_n - \beta) < u\} \leq \Gamma^+_{\beta_0}, \qquad \forall u > 0,$$
$$\underline{\lim_{n\to\infty}} \mathrm{P}_{\beta,n}\{C_n(\beta_n - \beta) < u\} \geq \Gamma^-_{\beta_0}, \quad \forall u < 0.$$

Hence if β^*_n satisfies the above condition of the asymptotic efficiency, we have for any AMU estimator β_n

$$\overline{\lim_{n\to\infty}} (\mathrm{P}_{\beta,n}\{-a < C_n(\beta^*_n - \beta) < b\} - \mathrm{P}_{\beta,n}\{-a < C_n(\beta_n - \beta) < b\}) \geq 0$$

for all positive number a and b and each $\beta \in B$.

In the following we shall obtain the bound of the asymptotic normality of the likelihood ratio statistics and the sufficient condition that an AMU estimator be asymptotic efficient. In fact, Theorem 3.2 shows that under some regular conditions the bound of asymptotic distribution of AMU estimators of β is a normal distribution with mean 0 and variance $(1-\beta^2)/((\sigma^2 + J(g))I)$, where I is the Fisher information of f. Thus it is easily seen that an AMU estimator is asymptotically efficient if it has an asymptotic normal distribution with variance equal to the above bound. In this section it is enough to consider the case $C_n = \sqrt{n}$. Here we first state several preliminary conclusions.

Lemma 3.1 Let $\{Z_n : n = 1, \cdots,\}$be a sequence of random variables satisfying the followings:

(1) $Z_n = Z_{n,N} + R_{n,N}\,(n > N)$;

(2) For each fixed N, the asymptotic distribution of $Z_{n,N}$ is normal with mean 0 and variance σ^2_N;

(3) $\lim_{N\to\infty} \sigma^2_N = \sigma^2$;

(4) $R_{n,N}$ converges in probability to 0 uniformly in n.

Then Z_n has a limiting normal distribution with mean 0 and variance σ^2.

Proof See the proof of Lemma 3.2.1 of Akahira and Takeuchi (1981).

Lemma 3.2 Let $Y_1, Y_2, \cdots$ be m-dependent sequence of random variables such that

(1) $\mathsf{E}(Y_i) = 0, \mathsf{E}(|Y_i|^3) \le R < \infty\ (i = 1, \cdots)$,

(2) $\lim_{p\to\infty} \frac{1}{p}\sum_{h=1}^{p} A_{i+h} = A$ exists, uniformly for all $i = 1, \cdots$.

Then $\sum_{i=1}^{n} Y_i$ is asymptotically normal with mean 0 and variance nA.

Proof See Hoeffding and Robbins (1948).

Lemma 3.3 Under the assumption (A.S.1), if $\mathsf{E}|\varepsilon_t|^3 < \infty$ and $\mathsf{E}|f'(\varepsilon_t)/f(\varepsilon_t)|^3 < \infty$, then the limiting distribution of $n^{-1/2}\sum\limits_{t=1}^{n} \dfrac{f'(\varepsilon_t)}{f(\varepsilon_t)} X_{t-1}$ is normal with mean 0 and variance $\dfrac{(\sigma^2 + J(g))I}{1-\beta^2}$, with $I = \displaystyle\int \frac{f'^2(u)}{f(u)} du$.

Proof See the proof of Lemma 3.2.3 of Akahira and Takeuchi (1981).

Next it will be shown that the bound of AMU estimators of β is obtained using the best test statistics and the least squares estimator of β is asymptotically efficient if and only if f is a normal distribution with mean 0 and variance σ^2.

Let β_0 arbitrary but fixed in R^1, considering the problem of testing the hypothesis

$$H^+ : \beta = \beta_1 = \beta_0 + \frac{u}{\sqrt{n}}\ (u > 0) \leftrightarrow K : \beta = \beta_0.$$

Putting $\beta_1 = \beta_0 + \Delta$ with $\Delta = u/\sqrt{n}$. We define

$$Z_{ni} = \log \frac{f(X_t - \beta_0 X_{t-1} - g(U_t))}{f(X_t - \beta_1 X_{t-1} - g(U_t))}.$$

We assume that f is twice continuously differentiable. If $\beta = \beta_0$, then

$$\begin{aligned}\sum_{t=1}^{n} Z_{nt} &= \sum_{t=1}^{n} \log \frac{f(\varepsilon_t)}{f(\varepsilon_t - un^{-1/2}X_{t-1})} = \sum_{t=1}^{n}\{\log f(\varepsilon_t) - \log f(\varepsilon_t - un^{-1/2}X_{t-1})\}\\ &= un^{-1/2}\sum_{t=1}^{n}\frac{f'(\varepsilon_t)}{f(\varepsilon_t)}X_{t-1} - \frac{u^2}{2n}\sum_{t=1}^{n}\frac{\partial^2 f(\varepsilon_t^*)}{\partial \varepsilon_t^2}X_{t-1}. \qquad (3.8)\end{aligned}$$

ε_t^* lies between ε_t and $\varepsilon_t - un^{-1/2}X_{t-1}$. If $\beta = \beta_1$, then

$$\begin{aligned}\sum_{t=1}^{n} Z_{nt} &= \sum_{t=1}^{n} \log \frac{f(\varepsilon_t + un^{-1/2}X_{t-1})}{f(\varepsilon_t)} = \sum_{t=1}^{n}\{\log f(\varepsilon_t + un^{-1/2}X_{t-1}) - \log f(\varepsilon_t)\}\\ &= un^{-1/2}\sum_{t=1}^{n}\frac{f'(\varepsilon_t)}{f(\varepsilon_t)}X_{t-1} + \frac{u^2}{2n}\sum_{t=1}^{n}\frac{\partial^2 f(\varepsilon_y^{**})}{\partial \varepsilon_t^2}X_{t-1}. \qquad (3.9)\end{aligned}$$

ε_t^{**} lies between ε_t and $\varepsilon_t + un^{-1/2}X_{t-1}$.

Putting $T_n = \sum_{t=1}^{n} X_{t-1}^2 k(\varepsilon_t), T_n^* = \sum_{t=1}^{n} X_{t-1}^2 k(\varepsilon_t^*)$, and $T_n^{**} = \sum_{t=1}^{n} X_{t-1}^2 k(\varepsilon_t^{**})$.

Lemma 3.4 Under assumption (A.S.1) – (A.S.4) the following hold

$$\begin{aligned}\mathsf{E}_{\beta_j}(T_n) &= \sum_{t=1}^{n} \mathsf{E}_{\beta_j}\left\{X_{t-1}^2\left\{\frac{f'(\varepsilon_t)}{f(\varepsilon_t)}\right\}^2\right\}\\ &= (\sigma^2 + J(g))I\left\{\frac{n-1}{1-\beta_j^2} - \frac{\beta_j^4(1-\beta_j^{2(n-1)})}{(1-\beta_j^2)^2}\right\}; \qquad (j = 0, 1)\end{aligned}$$

$$\lim_{n\to\infty} |\mathsf{E}_{\beta_0}(T_n^*/n) - \mathsf{E}_{\beta_0}(t_n/n)| = 0, \qquad \lim_{n\to\infty} |\mathsf{E}_{\beta_1}(T_n^{**}/n - \mathsf{E}_{\beta_1}(T_n/n)| = 0;$$
$$\lim_{n\to\infty} |\mathsf{E}_{\beta_0}(T_n^{*2}/n^2) - \mathsf{E}_{\beta_0}(T_n^2/n^2)| = 0, \quad \lim_{n\to\infty} |\mathsf{E}_{\beta_1}(T - n^{**2}/n^2) - \mathsf{E}_{\beta_1}(T_n^2/n^2)| = 0.$$

Proof See the proof of Lemma 3.2.4 of Akahira and Takeuchi (1981).

Lemma 3.5 Under the Assumption (A.S.1) – (A.S.5), both of the sequences $T_n^*/(E_{\beta_0}T_n^*)$ and $T_n^{**}/(E_{\beta_0}T_n^{**})$ converge with probability 1.

Proof See the proof of Lemma 3.2.5 of Akahira and Takeuchi (1981).

Theorem 3.1 Suppose that Assumption (A.S.1) – (A.S.5) hold. If $\beta = \beta_0$, then $\sum_{t=1}^{n} Z_{nt}$ has a limiting normal distribution with mean $\dfrac{u^2(\sigma^2 + J(g))I}{2(1-\beta_0^2)}$ and variance $\dfrac{u^2(\sigma^2 + J(g))I}{(1-\beta_0^2)}$. If $\beta = \beta_1$, then $\sum_{t=1}^{n} Z_{nt}$ has a limiting normal distribution with mean $-\dfrac{u^2(\sigma^2 + J(g))I}{2(1-\beta_0^2)}$ and variance $\dfrac{u^2(\sigma^2 + J(g))I}{1-\beta_0^2}$.

Proof If $\beta = \beta_0$, then it follows from (3.8) that

$$\begin{aligned}\sum_{t=1}^{n} Z_{nt} &= un^{-1/2}\sum_{t=1}^{n}\frac{f'(\varepsilon_t)}{f(\varepsilon_t)}X_{t-1} - \frac{u^2}{2n}\sum_{t=1}^{n} T_n^* \\ &= n^{-1}\mathsf{E}_{\beta_0}(T_n^*)\left\{u\frac{n^{-1/2}\sum_{t=1}^{n}\frac{f'(\varepsilon_t)}{f(\varepsilon_t)}X_{t-1}}{n^{-1}\mathsf{E}_{\beta_0}(T_n^*)} - \frac{u^2}{2}\frac{T_n^*}{\mathsf{E}_{\beta_0}(T_n^*)}\right\}. \end{aligned} \tag{3.10}$$

If $\beta = \beta_1$, then it follows from (3.9) that

$$\begin{aligned}\sum_{t=1}^{n} Z_{nt} &= un^{-1/2}\sum_{t=1}^{n}\frac{f'(\varepsilon_t)}{f(\varepsilon_t)}X_{t-1} - \frac{u^2}{2n}\sum_{t=1}^{n} T_n^{**} \\ &= n^{-1}\mathsf{E}_{\beta_1}(T_n^{**})\left\{u\frac{n^{-1/2}\sum_{t=1}^{n}\frac{f'(\varepsilon_t)}{f(\varepsilon_t)}X_{t-1}}{n^{-1}\mathsf{E}_{\beta_1}(T_n^{**})} - \frac{u^2}{2}\frac{T_n^{**}}{\mathsf{E}_{\beta_0}(T^{**}{}_n)}\right\}. \end{aligned} \tag{3.11}$$

It follows from the first two formulas of Lemma 3.4 that

$$\lim_{n\to\infty} n^{-1}\mathsf{E}_{\beta_j}(T_n^*) = \lim_{n\to\infty} n^{-1}\mathsf{E}_{\beta_1}(T_n^{**}) = \frac{(\sigma^2 + J(g))I}{1-\beta_0^2}\ (j = 0, 1).$$

Hence it follows from Lemma 3.3 that both of the sequences of

$$\frac{n^{-1/2}\sum_{t=1}^{n}\frac{f'(\varepsilon_t)}{f(\varepsilon_t)}X_{t-1}}{n^{-1}\mathsf{E}_{\beta_0}(T_n^*)} \quad \text{and} \quad \frac{n^{-1/2}\sum_{t=1}^{n}\frac{f'(\varepsilon_t)}{f(\varepsilon_t)}X_{t-1}}{n^{-1}\mathsf{E}_{\beta_1}(T_n^{**})}$$

have a limiting normal distribution with mean 0 and variance $(1-\beta_0^2)/[(\sigma^2 + J(g))I]$. Therefore it follows from (3.10), (3.11) and Lemma 3.5 that if $\beta = \beta_0$, then $\sum_{t=1}^{n} Z_{nt}$ has a limiting normal distribution with mean $\dfrac{u^2(\sigma^2 + J(g))I}{2(1-\beta_0^2)}$ and variance $\dfrac{u^2(\sigma^2 + J(g))I}{1-\beta_0^2}$. If $\beta = \beta_1$, then $\sum_{t=1}^{n} Z_{nt}$ has a limiting normal distribution with mean $-\dfrac{u^2(\sigma^2 + J(g))I}{2(1-\beta_0^2)}$ and variance $\dfrac{u^2(\sigma^2 + J(g))I}{1-\beta_0^2}$.

Theorem 3.2 Suppose that assumptions (A.S.1) – (A.S.5) hold. The bound of asymptotic distribution of AMU estimators β_n is given as follows: for each $\beta \in \mathcal{B}$

$$\overline{\lim_{n\to\infty}} \mathsf{P}_{\beta,n}\{\sqrt{n}(\beta_n - \beta) \le u\} \le \Phi\left(\frac{u\sqrt{(\sigma^2 + J(g))I}}{\sqrt{1-\beta^2}}\right) \qquad \forall u \ge 0, \qquad (3.12)$$

$$\underline{\lim_{n\to\infty}} \mathsf{P}_{\beta,n}\{\sqrt{n}(\beta_n - \beta) \ge u\} \ge \Phi\left(\frac{u\sqrt{(\sigma^2 + J(g))I}}{\sqrt{1-\beta^2}}\right) \qquad \forall u < 0, \qquad (3.13)$$

where $\Phi(\cdot)$ is the standard normal distribution.

Proof Let β_0 be arbitrary but fixed in $\mathcal{B}$. Let u be arbitrary positive number.Then we consider the problem of testing the hypothesis

$$H^+ : \beta = \beta_0 + un^{-1/2} \leftrightarrow K : \beta = \beta_0.$$

If we choose a sequence $\{k_n\}$ such that $\lim_{n\to\infty} \mathsf{P}_{\beta_0+un^{-1/2},n}\{\sum_{t=1}^n Z_{nt} > k_n\} = 1/2$, then it follows from Theorem 3.1 that $\lim_{n\to\infty} k_n = -u^2\sigma^2 I/[2(1-\beta_0^2)]$. Furthermore we have from Theorem 3.1

$$\lim_{n\to\infty} \mathsf{P}_{\beta_0,n}\left\{\sum_{t=1}^{n} Z_{nt} > k\right\} = \lim_{n\to\infty} \mathsf{P}_{\beta_0,n}\left\{\frac{\sum_{t=1}^n Z_{nt} - J/2}{\sqrt{I^*}} > \frac{k_n - J/2}{\sqrt{I^*}}\right\},$$

$$1 - \Phi(-\sqrt{I^*}) = \Phi(\sqrt{I^*}), \quad I^* = \frac{u^2(\sigma^2 + J(g))I}{1-\beta_0}.$$

Hence it follows from (3.3) and the fundamental lemma of Neyman and Pearson that for each $u > 0$ $\Gamma_{\beta_0}^+(u) = \Phi\big(u\sqrt{I(\sigma^2 + J(g))}/(\sqrt{1-\beta^2})\big)$. From (3.1) and (3.40) we obtain for every $u > 0$

$$\overline{\lim_{n\to\infty}} \mathsf{P}_{\beta_0,n}\{\sqrt{n}(\beta_n - \beta_0) \le u\} \le \Phi\left(\frac{u\sqrt{I(\sigma^2 + (g))}}{\sqrt{1-\beta^2}}\right).$$

Since β_n is AMU, $\Gamma_{\beta_0}^+(0) = \Phi(0) = 1/2$. It follows from the arbitrariness of β_0 that (3.12) holds.

Now let u be arbitrary negative number. Then we consider the problem of testing hypothesis

$$H^- : \beta = \beta_0 + un^{-1/2} \leftrightarrow K : \beta = \beta_0.$$

By a similar way as the case $u > 0$, we have from (3.6) $\Gamma_{\beta_0}^-(u) = \Phi\Big(\frac{u\sqrt{(\sigma^2 + J(g))I}}{\sqrt{1-\beta^2}}\Big)$ for all $u < 0$. Hence it follows from (3.2) and (3.70) that for every $u < 0$

$$\underline{\lim_{n\to\infty}} \mathsf{P}_{\beta_0,n}\{\sqrt{n}(\beta_n - \beta_0) \le u\} \ge \Phi\left(\frac{u\sqrt{(\sigma^2 + J(g))I}}{\sqrt{1-\beta^2}}\right).$$

It follows from the arbitrariness of β_0 that (3.13) holds. Thus we complete the proof.

From Theorem 3.3 and Definition 3.2 and 3.3 we get the following theorem.

Theorem 3.3 Under the assumptions (A.S.1) – (A.S.5), an AMU estimator is asymptotically efficient if and only if the limiting distribution of $\sqrt{n}(\beta_n - \beta_0)$ is normal with mean 0 and variance $(1-\beta^2)/\big((\sigma^2 + J(g))I\big)$.

§4. Asymptotic efficiency of LS estimator and MLE estimator

In this section, we shall give a necessary and sufficient condition that LS estimator of β is asymptotically efficient, then we also show that MLE of β is asymptotically efficient. We adopt piecewise polynomial to approximate g and to construct the LS estimator and MLE of β.The following condition is sufficient for the statement of the results.

Condition 1 Let $r\,(0 < r < 1), m\,(m = 1, \cdots)$ and $M(> 0)$ be nonnegative real constants such that

$$|g^{(m)}(y') - g^{(m)}(y)| \le M|y' - y|, \quad \text{for } 0 \le y, y' \le 1.$$

Think of $p = m + r$ as a measure of smoothness of the function g.

First we describe a piecewise polynomial estimator of g defined by Chen (1988), which has been investigated by some others. Given a positive M_n, the estimator has the form of a piecewise polynomial of degree m based on M_n intervals of length $1/M_n$, where the $(m+1)M_n$ coefficients are chosen by the method of least squares on the basis of the data $X_1, \cdots, X_n$. Let $I_{n\nu} = [(\nu-1)/M_n, \nu/M_n)$ for $1 \le \nu < M_n$ and $I_{nM_n} = [1 - 1/M_n, 1]$. Let $\chi_{n\nu}$ denote the indicator function for the interval $I_{n\nu}$, so that $\chi_{n\nu} = 1$ or 0 according to $x \in I_{n\nu}$ or $x \notin I_{n\nu}$. Consider the piecewise polynomial estimators of g of degree m given by $\widehat{g}(x) = \sum_{\nu=1}^{M_n} \chi_{n\nu}(x)\widehat{P}_{nm\nu}(x)$, where $\{\widehat{P}_{nm\nu}(x)\}$ are polynomial of degree m chosen to minimize the residual sum squares

$$\sum_{t=1}^{n}(X_t - X_{t-1}\beta_{LS} - \widehat{g}(U_t))^2 = \min. \tag{4.1}$$

For convenience and simplicity, some notations are introduced

$$P_{nm\nu}(x) = a_{0\nu} + a_{1\nu}x + \cdots + a_{m\nu}x^m,$$

$$Z = \begin{pmatrix} \chi_{n1}(U_1) & \cdots & \chi_{n1}(U_1)U_1^m & \cdots & \chi_{nM_n}(U_1) & \cdots & \chi_{nM_n}(U_1)U_1^m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \chi_{n1}(U_n) & \cdots & \chi_{n1}(U_n)U_n^m & \cdots & \chi_{nM_n}(U_n) & \cdots & \chi_{nM_n}(U_n)U_n^m \end{pmatrix},$$

$$\alpha = (a_{01}, \cdots, a_{m1}, a_{02}, \cdots, a_{m2}, a_{0M_n}, \cdots, a_{mM_n})^{\tau}_{(m+1)M_n},$$

$$\widehat{G} \widehat{=} \begin{pmatrix} \widehat{g}(U_1) \\ \cdots \\ \widehat{g}(U_n) \end{pmatrix} = \begin{pmatrix} \sum_{\nu=1}^{M_n} \chi_{n\nu}(U_1)\widehat{P}_{nm\nu}(U_1) \\ \cdots \\ \sum_{\nu=1}^{M_n} \chi_{n\nu}(U_n)\widehat{P}_{nm\nu}(U_n) \end{pmatrix} = Z_{\alpha}. \tag{4.2}$$

Then based on the model

$$X_n = \beta X_n^* + \widehat{G} + \epsilon,$$

the LS estimator $(\widehat{\alpha}, \beta_{LS})$ of (α, β) can be obtained as follows

$$\widehat{\alpha} = A(X_n - X_n^* \widehat{\beta}_n), \quad \beta_{LS} = (X_n^{*\tau}(\mathcal{F} - P)X_n^*)^{-1} X_n^{*\tau}((\mathcal{F} - P)X_n, \tag{4.3}$$

where $A = (Z^\tau Z)^{-1} Z^\tau$ and $P = ZA$ and $\mathcal{F}$ is the $(m+1)M_n$-order identity matrix.

Theorem 4.1 Assume that condition 1 hold, and that $\lim_{n\to\infty} n^{-q} M_n = 0$ for some $q \in (0,1)$ and $\lim_{n\to\infty} n M_n^{-2p} = 0$, then

$$\sqrt{n}(\beta_{LS} - \beta) \xrightarrow{L} N(0, \sigma^2 \Gamma^{-2}), \tag{4.4}$$

where β_{LS} is defined by (4.3) and $\Gamma^2 = \sigma^2 + J(g)$.

Proof Similarly prove as the proof of the Theorem 2.1 of Gao and Liang (1995), we omit the details.

It will be proved that the least square estimator of β is asymptotically efficient if and only if $f'(u)/f(u) = cu$, where c is some constant. Indeed, since

$$\sigma^2 I = \int u^2 f(u)du \cdot \int \frac{f^{*2}(u)}{f(u)} du \geq \left\{ \int u f'(u) du \right\}^2 = I,$$

"=" is obtained if and only if $f'(u)/f(u) = cu$. It follows from Theorem 3.2 that the limiting distribution of $\sqrt{n}(\beta_{LS} - \beta)$ attains the bound of the asymptotic distributions if and only if f is a normal density function with mean 0 and variance σ^2. Hence it is seen by Theorem 3.3 that the least square estimator is asymptotically efficient if and only if f is a normal density function with mean 0 and variance σ^2. Therefore we have now established.

Theorem 4.2 Under the assumptions (A.S.1) – (A.S.5), a necessary and sufficient condition that the least squares estimator of β is asymptotically efficient is that f is a normal density function with mean 0 and variance σ^2.

Remark As is immediately seen from above, assumptions (A.3.1)–(A.3.6) are not necessary for the proof of sufficiency.

We now consider the asymptotic efficiency of the maximum likelihood estimator (MLE), which is defined as the solution of the following

$$\sum_{t=1}^n \frac{\partial}{\partial \beta} \log f(X_t - \beta X_{t-1} - \widehat{g}(U_t)) = 0.$$

By Taylor expansion we have

$$\begin{aligned} 0 &= \sum_{t=1}^n \frac{\partial}{\partial \beta} \log f(X_t - \beta X_{t-1} - \widehat{g}(U_t)) \\ &= \sum_{t=1}^n \Big[\frac{\partial}{\partial \beta} \log f(X_t - \beta X_{t-1} - g(U_t)) \\ &\quad + \frac{\partial^2}{\partial \beta^2} \log f(X_t - \beta^* X_{t-1} - \widehat{\beta}^*(U_t)) (\beta_{ML} - \beta + \widehat{g}(U_t) - g(U_t)) \Big] \\ &\triangleq \sum_{t=1}^n \big[-X_{t-1} \psi'(X_t - \beta X_{t-1} - g(U_t)) \\ &\quad + X_{t-1}^2 \psi''(X_t - \beta^* X_{t-1} - \widehat{g}^*(U_t)) (\beta_{ML} - \beta + \widehat{g}(U_t) - g(U_t)) \big], \end{aligned}$$

where $|\beta^* - \beta| \le |\beta_{ML} - \beta|$, $|\widehat{g}^*(U_t) - g(U_t)| \le |\widehat{g}(U_t) - g(U_t)|$ and $\psi(u) = \log f(u)$.

We have from lemma 3.3 that the limiting distribution of

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_{t-1}\psi'(X_t - \beta X_{t-1} - g(U_t))$$

is normal with mean 0 and variance $((\sigma^2 + J(g))I)/(1-\beta^2)$. It follows from Lemma 3.4 that $\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2\psi''(X_t - \beta^* X_{t-1} - \widehat{g}^*(U_t))$ converges to $((\sigma^2 + J(g))I)/(1-\beta^2)$ in probability. On the other hand, it follows from condition 1 that

$$|\widehat{g}(U_t) - g(U_t)| = \left|\sum_{\nu=1}^{M_n} \chi_{n\nu}(U_t)|g(U_t) - P_{mn\nu}(U_t)|\right| \le CM_n^{-p}.$$

Assume $\lim_{n\to\infty} M_n^{-p} n^{1/2} = 0$. Hence we have

$$\sqrt{n}(\beta_{ML} - \beta) = \frac{\frac{1}{\sqrt{n}}\sum_{t=1}^{n} X_{t-1}\psi'(X_t - \beta X_{t-1} - g(U_t))}{\frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2\psi''(X_t - \beta^* X_{t-1} - \widehat{g}^*(U_t))} + o_p(1).$$

Using Theorem 3.3, we get the following conclusion:

Theorem 4.3 Suppose condition 1 and (4.5) hold. Under the assumptions (A.S.1) - (A.S.6), the maximum likelihood estimator is asymptotically efficient.

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一个部份自回归模型的渐近有效性

梁　华

(中国科学院系统科学研究所, 北京)

本文考虑部分自回归模型 $X_t = X_{t-1}\beta + g(U_t) + \varepsilon_t$, $t \geq 1$. 这里 g 是一未知函数, β是一待估参数, ε_j 是具有 0 均值和方差 σ^2 的 i.i.d. 误差, U_t i.i.d. 服从 [0, 1] 上均匀分布. 本文首先给出了相合估计的收敛阶和 Takeuchi 意义下渐近有效界. 同时给出了 β 最小二乘估计是有效的充要条件. 最后证明了 MLE 是渐近有效的.

关键词: 渐近有效性, 部分自回归模型, 分段多项式.

学科分类号: 212.1.

《应用概率统计》编委会致上海金宇建设联合发展公司的感谢信

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