

# Computing the Stationary Distribution of Absorbing Markov Chains with One Eigenvector of Diagonalizable Transition Matrices \*

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**Abstract:** An absorbing Markov chain is an important statistic model and widely used in algorithm modeling for many disciplines, such as digital image processing, network analysis and so on. In order to get the stationary distribution for such model, the inverse of the transition matrix usually needs to be calculated. However, it is still difficult and costly for large matrices. In this paper, for absorbing Markov chains with two absorbing states, we propose a simple method to compute the stationary distribution for models with diagonalizable transition matrices. With this approach, only an eigenvector with eigenvalue 1 needs to be calculated. We also use this method to derive probabilities of the gambler's ruin problem from a matrix perspective. And, it is able to handle expansions of this problem. In fact, this approach is a variant of the general method for absorbing Markov chains. Similar techniques can be used to avoid calculating the inverse matrix in the general method.

**Keywords:** random walk; absorbing Markov chain; stationary distribution; gambler's ruin

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## §1. Introduction

An absorbing Markov chain is a classical stochastic process, widely applied in many fields, such as cyber security analytics<sup>[1]</sup> and digital image processing<sup>[2]</sup>. In these applications, some events or features are always set to the absorbing states and Markov chains are used to construct the model. It is necessary to compute the stationary distribution which

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is an important parameter to estimate the stability or efficiency of these systems. Conventional calculations generally involve computing the inverse of the transition matrix<sup>[3]</sup>, which is still not easy to calculate for large matrices. However, in this paper, for Markov chains with two absorbing states and diagonalizable transition matrices, we propose a simple method to compute the stationary distribution. Instead of computing the inverse matrix, this approach solves the stationary distribution by calculating an eigenvector with eigenvalue 1, which is only equivalent to solving a system of linear equations. This method can be used to derive analytic results for some problems, e.g. the gambler's ruin problem.

We discuss a kind of one-dimensional random walk problem, which can also be described by an absorbing Markov chain. The transition matrix is the same after each step, showing a typical feature of a time-homogeneous Markov chain.

Here an absorbing Markov chain that satisfies the following model is defined.

In an ordered and finite state space  $S = \{s_1, s_2, s_3, \dots, s_N\}$ , where  $N$  is the number of states, a random walk can be performed. The observed sequence of random variables is  $X_1, X_2, X_3, \dots$ , while the agent moves  $h$  steps forward or backward to the next state with a certain probability and the following restrictions.

1. The probability of moving forward or backward is non-zero.
2.  $h \in \mathbb{Z}$  and  $0 \leq h \leq N - 2$  for each  $h$ .
3.  $s_1$  and  $s_N$  are two absorbing states, also called boundaries. When the agent reaches  $s_1$  or  $s_N$ , the walk ends.
4. If it would go out of the boundaries after moving, set the next state to the nearest boundary.
5. If the next state is not  $s_1$  and  $s_N$ , the probability of moving  $h$  steps forward or backward is the same. That is

$$P(X_{t+1} = s_{i \pm h} | X_t = s_i) = P(X_{t+n+1} = s_{j \pm h} | X_t = s_j),$$

where  $t$  and  $n$  are any time,  $s_{i \pm h}$ ,  $s_{j \pm h}$ ,  $s_i$  and  $s_j$  are not  $s_1$  and  $s_N$ .

This is a time-homogeneous Markov chain. In each transition, let the probability of not moving be  $t_0$  and probabilities of moving  $h$  steps forward and backward be  $t_h$  and  $t_{-h}$  respectively. According to the Restriction 4, the probability of reaching each boundary is the summation of probabilities of crossing this boundary. Therefore, the transition matrix

$\mathbf{P} = (p_{ij})_{N \times N}$  can be written as

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \sum_{i=1}^{N-2} t_{-i} & t_0 & t_1 & t_2 & \cdots & t_{N-4} & t_{N-3} & t_{N-2} \\ \sum_{i=2}^{N-2} t_{-i} & t_{-1} & t_0 & t_1 & \cdots & t_{N-5} & t_{N-4} & \sum_{i=N-3}^{N-2} t_i \\ \sum_{i=3}^{N-2} t_{-i} & t_{-2} & t_{-1} & t_0 & \cdots & t_{N-6} & t_{N-5} & \sum_{i=N-4}^{N-2} t_i \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \sum_{i=N-3}^{N-2} t_{-i} & t_{-(N-4)} & t_{-(N-5)} & t_{-(N-6)} & \cdots & t_0 & t_1 & \sum_{i=2}^{N-2} t_i \\ t_{-(N-2)} & t_{-(N-3)} & t_{-(N-4)} & t_{-(N-5)} & \cdots & t_{-1} & t_0 & \sum_{i=1}^{N-2} t_i \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, \quad (1)$$

where  $p_{ij}$  is the probability of moving from state  $i$  to state  $j$ .

Note that the sum of each row is 1:

$$\sum_{i=1}^{N-2} t_{-i} + t_0 + \sum_{i=1}^{N-2} t_i = 1. \quad (2)$$

According to the Restriction 1,

$$\sum_{i=1}^{N-2} t_{-i} > 0 \quad \text{and} \quad \sum_{i=1}^{N-2} t_i > 0. \quad (3)$$

By adjusting the row and column positions of the absorption states,  $x_1$  and  $x_N$ ,  $\mathbf{P}$  can be rewritten as

$$\mathbf{P}_s = \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{R} & \mathbf{T} \end{pmatrix}, \quad (4)$$

where  $\mathbf{I}_2$  is a  $2 \times 2$  identity matrix,  $\mathbf{R}$  and  $\mathbf{T} = (a_{ij})_{(N-2) \times (N-2)}$  are

$$\mathbf{R} = \begin{pmatrix} \sum_{i=1}^{N-2} t_{-i} & \sum_{i=2}^{N-2} t_{-i} & \sum_{i=3}^{N-2} t_{-i} & \cdots & \sum_{i=N-3}^{N-2} t_{-i} & t_{-(N-2)} \\ t_{N-2} & \sum_{i=N-3}^{N-2} t_i & \sum_{i=N-4}^{N-2} t_i & \cdots & \sum_{i=2}^{N-2} t_i & \sum_{i=1}^{N-2} t_i \end{pmatrix}^T, \quad (5)$$

and

$$\mathbf{T} = \begin{pmatrix} t_0 & t_1 & t_2 & \cdots & t_{N-4} & t_{N-3} \\ t_{-1} & t_0 & t_1 & \cdots & t_{N-5} & t_{N-4} \\ t_{-2} & t_{-1} & t_0 & \cdots & t_{N-6} & t_{N-5} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{-(N-4)} & t_{-(N-5)} & t_{-(N-6)} & \cdots & t_0 & t_1 \\ t_{-(N-3)} & t_{-(N-4)} & t_{-(N-5)} & \cdots & t_{-1} & t_0 \end{pmatrix}. \quad (6)$$

In the following pages, all identity matrices are presented by  $\mathbf{I}$ ; and  $\mathbf{T}$  in Equation (4) is called the transient matrix, not only for our model, but also for all absorbing Markov chains.

Equation (4) is the canonical form of transition matrices of absorbing Markov chains. According to [4],

$$\mathbf{P}^k = \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{R} & \mathbf{T} \end{pmatrix}^k = \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{R}_k & \mathbf{T}^k \end{pmatrix},$$

where

$$\mathbf{R}_k = (\mathbf{I} + \mathbf{T} + \mathbf{T}^2 + \cdots + \mathbf{T}^{k-1})\mathbf{R} = \sum_{m=0}^{k-1} \mathbf{T}^m \mathbf{R};$$

since  $\lim_{k \rightarrow \infty} \mathbf{T}^k = \mathbf{0}$  and  $(\mathbf{I} - \mathbf{T})^{-1} = \sum_{m=0}^{\infty} \mathbf{T}^m$ ,

$$\lim_{k \rightarrow \infty} \mathbf{P}_s^k = \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{R}_{\infty} & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \mathbf{R}_{\infty} = (\mathbf{I} - \mathbf{T})^{-1} \mathbf{R}. \quad (7)$$

If the limit  $\lim_{k \rightarrow \infty} \mathbf{P}_s^k$  is known, it is easy to calculate the stationary distribution by multiplying the initial distribution. In addition, by adjusting the row and column positions of  $\lim_{k \rightarrow \infty} \mathbf{P}_s^k$ ,  $\lim_{k \rightarrow \infty} \mathbf{P}^k$  can also be obtained. Equation (7) is a general method to solve for absorbing Markov chain models. However, an inverse of a matrix needs to be calculated, which is difficult for large matrices.

It is obvious that the matrix power is the key to calculating the stationary distribution. If a matrix is diagonalizable, it is easy to get its matrix power. In addition, there is a close relationship among diagonalization, eigenvalues and eigenvectors. Fortunately, considering the particularity of this problem, eigenvalues and eigenvectors of the transition matrix have some special properties, which is elaborated in Sections 2–3. Then a better approach to this problem could be derived from these properties.

The paper is organized as follows. In Section 2, we present a proposition of eigenvalues of the transition matrix. Then, in Section 3, the condition of diagonalization of this

transition matrix is provided. Based on this condition, we propose a simple method to compute the stationary distribution. In Section 4, we use this method to derive probabilities of the gambler's ruin problem from a matrix perspective, and compute the stationary distribution of a numeric example of the complex expansion of the gambler's ruin problem. The relationship between the proposed approach and the conventional method is also discussed. Finally, we end with conclusions in Section 5.

## §2. Eigenvalues of the Transition Matrix

The power of a matrix is closely linked with eigenvalues and eigenvectors. Now a proposition of eigenvalues of this transition matrix will be given and forms the base of this new method.

**Proposition 1** If a transition matrix satisfies Equations (1), (2) and (3), the norm of its eigenvalues are not greater than 1 and the algebraic multiplicity of eigenvalue 1 is 2.

**Proof** Suppose that  $\lambda$  is an eigenvalue of the transition matrix  $\mathbf{P}$ , then  $|\lambda\mathbf{I} - \mathbf{P}| = 0$ , so that

$$(\lambda - 1)^2 \begin{vmatrix} \lambda - t_0 & -t_1 & -t_2 & \cdots & -t_{N-4} & -t_{N-3} \\ -t_{-1} & \lambda - t_0 & -t_1 & \cdots & -t_{N-5} & -t_{N-4} \\ -t_{-2} & t_{-1} & \lambda - t_0 & \cdots & -t_{N-6} & -t_{N-5} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -t_{-(N-4)} & -t_{-(N-5)} & -t_{-(N-6)} & \cdots & \lambda - t_0 & -t_1 \\ -t_{-(N-3)} & -t_{-(N-4)} & -t_{-(N-5)} & \cdots & -t_{-1} & \lambda - t_0 \end{vmatrix} = 0,$$

and hence

$$\lambda = 1 \text{ or } \begin{vmatrix} \lambda - t_0 & -t_1 & -t_2 & \cdots & -t_{N-4} & -t_{N-3} \\ -t_{-1} & \lambda - t_0 & -t_1 & \cdots & -t_{N-5} & -t_{N-4} \\ -t_{-2} & -t_{-1} & \lambda - t_0 & \cdots & -t_{N-6} & -t_{N-5} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -t_{-(N-4)} & -t_{-(N-5)} & -t_{-(N-6)} & \cdots & \lambda - t_0 & -t_1 \\ -t_{-(N-3)} & -t_{-(N-4)} & -t_{-(N-5)} & \cdots & -t_{-1} & \lambda - t_0 \end{vmatrix} = 0. \quad (8)$$

Obviously, 1 is an eigenvalue whose algebraic multiplicity is not less than 2. The second solution of  $\lambda$  in Equation (8) is the same as eigenvalues of the Toeplitz matrix  $\mathbf{T}$  in Equation (6). Suppose that  $\beta$  is an eigenvalue of  $\mathbf{T}$  whose eigenvector is  $\mathbf{x} =$

$(x_1, x_2, x_3, \dots, x_{N-2})^\top$ , then  $\mathbf{T}\mathbf{x} = \beta\mathbf{x}$ . Let  $|x_k| = \max_{1 \leq i \leq N-2} |x_i| = \|\mathbf{x}\|_\infty$ . Consider the  $k$ -th row of the above equation,

$$\sum_{j=1}^{N-2} a_{kj}x_j = \beta x_k \quad \text{or} \quad (\beta - a_{kk})x_k = \sum_{j \neq k} a_{kj}x_j,$$

and hence

$$|\beta - a_{kk}||x_k| \leq \sum_{j \neq k} |a_{kj}||x_j| \leq |x_k| \sum_{j \neq k} |a_{kj}|.$$

The eigenvector must be non-zero, so  $|x_k| > 0$ . Then

$$|\beta - a_{kk}| \leq \sum_{j \neq k} |a_{kj}|. \quad (9)$$

Two sides of Equation (9) are equal if and only if  $|x_j| = |x_k|$  for all  $j \neq k$ . From Equation (2) and Equation (9),

$$|\beta| \leq |a_{kk}| + \sum_{j \neq k} |a_{kj}| \leq 1. \quad (10)$$

What interests us is whether the range of  $\beta$  satisfies

$$|\beta| < 1. \quad (11)$$

Assume  $|\beta| = 1$  and its eigenvector  $\mathbf{x} = (x_1, x_2, x_3, \dots, x_{N-2})^\top$ . If  $|\beta| = 1$ , then  $|a_{kk}| + \sum_{j \neq k} |a_{kj}| = 1$  and  $|x_j| = |x_k|$  are valid for all  $j \neq k$  in Equation (9) and Equation (10). And since  $|x_k|$  is the infinity norm,  $k$  could be any integer of  $1, 2, 3, \dots, N-2$ . Because elements of  $\mathbf{T}$  are non-negative,

$$\sum_{j=1}^{N-2} a_{kj} = 1, \quad k = 1, 2, 3, \dots, N-2. \quad (12)$$

Let  $k = 1$ . According to Equation (12) and Equation (2)

$$\begin{cases} \sum_{j=1}^{N-2} |a_{1j}| = t_0 + \sum_{i=1}^{N-3} t_i = 1, \\ \sum_{i=1}^{N-2} t_{-i} = 0. \end{cases}$$

Obviously, it contradicts Equation (3). Hence the assumption  $|\beta| = 1$  is false, i.e.  $|\beta| \neq 1$ . Note that we have  $|\beta| \leq 1$  from Equation (10). Therefore, Equation (11) is true and the norm of the eigenvalues of  $\mathbf{T}$  is less than 1. And according to Equation (8), Proposition 1 is true.  $\square$

The range of eigenvalues of the transition matrix has been figured out. And the algebraic multiplicity of eigenvalue 1 is 2, which is the key to the simple approach to compute the stationary distribution elaborated in the next section.

### §3. The Stationary Distribution

The stationary distribution depends on the initial distribution and the transition matrix. Let  $\pi_0$  be the initial distribution and  $\pi_k$  be the distribution after  $k$ -step transition. For a time-homogeneous Markov chain,

$$\pi_k = \pi_0 P^k. \quad (13)$$

If the limit  $\lim_{k \rightarrow \infty} P^k$  exists and is found,  $\pi_\infty$  is the stationary distribution and could be easily determined for any starting distribution. Furthermore if  $P$  is diagonalizable, it will be easy to find this limit. Now a proposition about diagonalization will be proposed.

**Proposition 2** If a transition matrix  $P$  satisfies Proposition 1, a necessary and sufficient condition for  $P$  to be diagonalizable is that its transient matrix  $T$  is diagonalizable.

**Proof** Necessity is dealt first. Suppose that there exists an invertible matrix  $H$  and a diagonal matrix  $\Lambda$ , satisfying  $H\Lambda H^{-1} = P$ , i.e.  $P$  is diagonalizable.

$H$ ,  $\Lambda$  and  $H^{-1}$  could be partitioned as follows

$$H = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Lambda_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad H^{-1} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}. \quad (14)$$

Sizes of blocks at the same positions of  $H$ ,  $\Lambda$  and  $H^{-1}$  are the same. Let  $n = N - 2$ . Then the sizes of the first row are  $1 \times 1$ ,  $1 \times n$  and  $1 \times 1$ , respectively. The sizes of the first column are  $1 \times 1$ ,  $n \times 1$  and  $1 \times 1$ , respectively. The second column of  $H\Lambda H^{-1}$  satisfies

$$\begin{cases} L_{11}R_{12} + L_{12}\Lambda_1 R_{22} + L_{13}R_{32} = 0, \\ L_{21}R_{12} + L_{22}\Lambda_1 R_{22} + L_{23}R_{32} = T, \\ L_{31}R_{12} + L_{32}\Lambda_1 R_{22} + L_{33}R_{32} = 0. \end{cases} \quad (15)$$

Similarly, since  $HH^{-1} = I$ ,

$$\begin{cases} L_{11}R_{12} + L_{12}R_{22} + L_{13}R_{32} = 0, \\ L_{21}R_{12} + L_{22}R_{22} + L_{23}R_{32} = I, \\ L_{31}R_{12} + L_{32}R_{22} + L_{33}R_{32} = 0. \end{cases} \quad (16)$$

From Equation (15) and Equation (16),

$$\begin{cases} L_{12}(\Lambda_1 - I)R_{22} = 0, \\ L_{22}(\Lambda_1 - I)R_{22} = T - I, \\ L_{32}(\Lambda_1 - I)R_{22} = 0. \end{cases} \quad (17)$$

Obviously, diagonal elements of  $\Lambda_1 - I$  are the eigenvalues of  $T - I$ . From Equation (11), eigenvalues of  $T - I$  are not equal to 0. So that the determinant of  $\Lambda_1 - I$  is non-zero. From the second sub-equation of Equation (17),  $|L_{22}||R_{22}| = 1$ . Thus  $L_{22}$  and  $R_{22}$  are invertible. And since  $\Lambda_1 - I$  is invertible,

$$L_{12} = \mathbf{0} \quad \text{and} \quad L_{32} = \mathbf{0}. \quad (18)$$

Note that, the second block in the second row of  $H^{-1}H$  is actually  $I$ ,

$$R_{21}L_{12} + R_{22}L_{22} + R_{23}L_{32} = I. \quad (19)$$

Form Equation (18) and Equation (19),

$$L_{22}R_{22} = I. \quad (20)$$

Then form Equation (17) and Equation (20),

$$L_{22}\Lambda_1R_{22} = T.$$

So that  $T$  is diagonalizable. The necessity is proved.

Then sufficiency. Suppose that there exist such two invertible matrices  $L_{22}$  and  $R_{22}$ , satisfying  $L_{22}R_{22} = I$ , and a diagonal matrix  $\Lambda_1$  that satisfies  $L_{22}\Lambda_1R_{22} = T$ , i.e.  $T$  is diagonalizable.

Let three block matrices be

$$H = \begin{pmatrix} 1 & \mathbf{0} & 0 \\ L_{21} & L_{22} & L_{23} \\ 0 & \mathbf{0} & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & \mathbf{0} & 0 \\ 0 & \Lambda_1 & 0 \\ 0 & \mathbf{0} & 1 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 1 & \mathbf{0} & 0 \\ R_{21} & R_{22} & R_{23} \\ 0 & \mathbf{0} & 1 \end{pmatrix}. \quad (21)$$

Sizes of each blocks are the same as Equation (14). The condition for  $HG = I$  is

$$\begin{cases} L_{21} + L_{22}R_{21} = \mathbf{0}, \\ L_{22}R_{22} = I, \\ L_{22}R_{23} + L_{23} = \mathbf{0}. \end{cases} \quad (22)$$

Since  $L_{22}$  and  $R_{22}$  are known, we only need to find  $L_{21}$ ,  $L_{23}$ ,  $R_{21}$  and  $R_{23}$  to meet Equation (22).

The condition for  $H\Lambda G = P$  is

$$\begin{cases} L_{21} + L_{22}\Lambda_1R_{21} = A, \\ L_{22}\Lambda_1R_{22} = I, \\ L_{22}\Lambda_1R_{23} + L_{23} = B, \end{cases} \quad (23)$$



where  $\mathbf{A} = (p_{j,1})_{(N-2) \times 1}$  and  $\mathbf{B} = (p_{j,N})_{(N-2) \times 1}$  for  $j = 2, 3, 4, \dots, N-1$ .  $p$  is the same as the definition of the transition matrix  $\mathbf{P}$ . According to our assumption, the second sub-equations of Equation (22) and Equation (23) are true. A unique solution to  $\mathbf{L}_{21}$ ,  $\mathbf{L}_{23}$ ,  $\mathbf{R}_{21}$ , and  $\mathbf{R}_{23}$  can be sought by combining Equation (22) and Equation (23). So that there exist  $\mathbf{H}$ ,  $\mathbf{\Lambda}$  and  $\mathbf{G}$  such that  $\mathbf{H}\mathbf{G} = \mathbf{I}$  and  $\mathbf{H}\mathbf{\Lambda}\mathbf{G} = \mathbf{P}$ . Hence  $\mathbf{P}$  is diagonalizable. The sufficiency is proved. Thus, Proposition 2 is true.  $\square$

From Proposition 2, the Toeplitz matrix has the same diagonalizable property with transition matrix. Because of the properties of Toeplitz matrices, the workload of calculating its eigenvalues and determining whether it is diagonalizable is acceptable, especially for some special Toeplitz matrices which have been fully studied, e.g. circulant matrices (see, e.g. [5]). This greatly simplifies the problem of determining whether the transition matrix can be diagonalized. So we mainly focus on this situation.

Suppose that there exists an invertible matrix  $\mathbf{C}$  such that  $\mathbf{P} = \mathbf{C}\mathbf{D}\mathbf{C}^{-1}$ , where  $\mathbf{D} = \text{diag}(\lambda_1, \lambda_2, \lambda_2, \dots, \lambda_{N-2}, 1, 1)$ . From Proposition 1, we have

$$|\lambda_i| < 1, \quad i = 1, 2, 3, \dots, N-2.$$

$\mathbf{C}$  is the matrix composed of eigenvectors. Let  $\mathbf{P}_\infty = \lim_{k \rightarrow \infty} \mathbf{P}^k$ , then

$$\mathbf{P}_\infty = \mathbf{C} \lim_{k \rightarrow \infty} \mathbf{D}^k \mathbf{C}^{-1},$$

where  $\lim_{k \rightarrow \infty} \mathbf{D}^k = \text{diag}(0, 0, 0, \dots, 0, 1, 1)$ .

Let  $\mathbf{x}_1$  and  $\mathbf{x}_2$  be the eigenvectors with eigenvalue 1 of  $\mathbf{P}$ , where  $\mathbf{x}_1 = [x_{11}, x_{12}, x_{13}, \dots, x_{1N}]^T$  and  $\mathbf{x}_2 = [x_{21}, x_{22}, x_{23}, \dots, x_{2N}]^T$ . Because of the form of the first and last row of  $\mathbf{P}$ ,  $x_{11}$  and  $x_{1N}$  are arbitrary. However,  $x_{11}$  and  $x_{1N}$  cannot be 0 at the same time. Otherwise the algebraic multiplicity of eigenvalue 1 would be no more than 1, which is against Proposition 1. Similarly,  $x_{21}$  and  $x_{2N}$  can be arbitrary, while  $\mathbf{x}_1 \neq \mathbf{x}_2$  must be ensured. For simplicity, let

$$x_{11} = 0, \quad x_{1N} = 1, \quad x_{21} = 1 \quad \text{and} \quad x_{2N} = 0. \quad (24)$$

Then

$$\mathbf{C} \lim_{k \rightarrow \infty} \mathbf{D}^k = (\mathbf{0}, \mathbf{x}_1, \mathbf{x}_2).$$

$\mathbf{C}^{-1}$  can be written as  $\mathbf{C}^{-1} = (f_{ij})_{N \times N}$ . If, by whatever means, the last two rows of  $\mathbf{C}^{-1}$  are found, then  $\mathbf{P}_\infty$  can be easily determined. Since the stationary states are two

absorbing states,  $s_1$  and  $s_N$ , we have

$$\begin{cases} P_{\infty}(1, 1) = 1, & P_{\infty}(1, N) = 0; \\ P_{\infty}(N, 1) = 0, & P_{\infty}(N, N) = 1; \\ P_{\infty}(i, j) = 0, \end{cases} \quad (25)$$

where  $i, j = 1, 2, 3, \dots, N$ , but they can't be 1 or  $N$  at the same time.

From Equation (24) and Equation (25), the last two rows of  $\mathbf{C}^{-1}$  are found to be

$$\begin{cases} f_{N-1,N} = 1/x_{1N} = 1, & f_{N-1,i} = 0, & i = 1, 2, 3, \dots, N-1; \\ f_{N,1} = 1/x_{21} = 1, & f_{N,j} = 0, & j = 2, 3, 4, \dots, N. \end{cases} \quad (26)$$

Thus, the unknowns of  $\mathbf{P}_{\infty}$  are

$$\begin{cases} P_{\infty}(k, 1) = x_{2k}f_{N,1} = x_{2k}/x_{21} = x_{2k}; \\ P_{\infty}(k, N) = x_{1k}f_{N-1,N} = x_{1k}/x_{1N} = x_{1k}, \end{cases} \quad (27)$$

where  $k = 2, 3, 4, \dots, N-1$ . Note that the sum of these two probabilities is 1, we get

$$P_{\infty}(k, 1) + P_{\infty}(k, N) = 1. \quad (28)$$

Therefore, it only needs to calculate one of them. If the transition matrix is diagonalizable with 0 and 1 at the proposed position of the eigenvector, then  $\mathbf{C}$  is determined; its inverse exists, and Equation (26) is satisfied. Hence Equation (27) is reasonable, and the logic of our method is clear. We summarize it as Proposition 3.

**Proposition 3** For our model, if its transition matrix is diagonalizable, only an eigenvector with eigenvalue 1 needs to be computed, whose first element and last element are 0 and 1 respectively. Then,  $\lim_{k \rightarrow \infty} \mathbf{P}^k$  can be determined from Equation (25) and Equation (27), and the stationary distribution can be calculated with Equation (13).

Note that before using this method, it must ensure that the transition matrix is diagonalizable. Some judging techniques can be used, for example, the minimal polynomial [6].

In addition, the derivations in this section have nothing to do with the form of the transient matrix. We only used Proposition 1. Hence, even for an absorbing Markov chain which does not completely satisfy our model, Proposition 2 is still true if it satisfies Proposition 1. When the transition matrix is diagonalizable, we could still use Proposition 3 to compute the stationary distribution quickly.

## §4. Examples

### 4.1 Gambler's Ruin

Now we take this method to derive probabilities of the gambler's ruin problem. It is a classic problem in probability theory. Here is a quick review.

Suppose a gambler starts with  $m$  units of money, whose rival has  $n$  units of money. The bet of each play is one unit of money and he has a probability  $a$  of losing (and  $c = 1 - a$  of winning) in each play. The game ends when either one of them goes broken. We are interested in probabilities of losing and winning the game.

Since the total money is  $m + n$ , there are  $M = m + n + 1$  states. Let the state space be  $\{0, 1, 2, \dots, m + n\}$ , where the number represents the money that this gambler may have. The initial distribution can be written as  $\pi_0 = (0, 0, \dots, 0, 1, 0, 0, \dots, 0)$  whose size is  $1 \times M$ . Only the  $(m + 1)$ -th element is 1. Let  $\mathbf{P}_1$  be the  $M \times M$  transition matrix, then

$$\mathbf{P}_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a & 0 & c & \cdots & 0 & 0 & 0 \\ 0 & a & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 & c \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Obviously, this problem fits our model. Let  $N = m + n - 1$  and an  $N \times N$  Toeplitz matrix  $\mathbf{Q}_1$  be

$$\mathbf{Q}_1 = \begin{pmatrix} 0 & c & 0 & \cdots & 0 & 0 & 0 \\ a & 0 & c & \cdots & 0 & 0 & 0 \\ 0 & a & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 & c \\ 0 & 0 & 0 & \cdots & 0 & a & 0 \end{pmatrix}.$$

To apply Proposition 2, we must determine whether the Toeplitz matrix  $\mathbf{Q}_1$  is diagonalizable. As  $\mathbf{Q}_1$  is a tridiagonal matrix, according to [7], its eigenvalues are

$$\lambda_t = -2\sqrt{ac} \cos \frac{t\pi}{N+1}, \quad t = 1, 2, 3, \dots, N.$$

Hence,  $\mathbf{Q}_1$  has  $N$  different eigenvalues and is diagonalizable. And from Proposition 2, the transition matrix  $\mathbf{P}_1$  is diagonalizable. Our method is applicable for this problem.

Let an eigenvector with eigenvalue 1 of  $\mathbf{P}_1$  be  $\mathbf{x}_1 = (x_1, x_2, x_3, \dots, x_M)^T$ . It must satisfy

$$(\mathbf{I} - \mathbf{P}_1)\mathbf{x}_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -a & 1 & -c & \cdots & 0 & 0 & 0 \\ 0 & -a & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -a & 1 & -c \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{M-1} \\ x_M \end{pmatrix} = 0. \quad (29)$$

Equation (29) can be written as

$$cx_{k+1} - x_k + ax_{k+1} = 0, \quad k = 2, 3, \dots, M-1.$$

It is an order-2 homogeneous linear difference equation. According to [8], its characteristic equation has this form

$$cr^2 - r + a = 0, \quad (30)$$

where  $r$  are characteristic roots of this linear difference equation. Since  $a + c = 1$ , it is easy to find  $1 - 4ac \geq 0$ . For the situations  $1 - 4ac = 0$  and  $1 - 4ac > 0$ , the general solutions are different. Hence these two cases must be discussed respectively.

If  $1 - 4ac = 0$ , then  $a = c = 0.5$  and characteristic roots are 1. The general solution would be

$$x_j = (c_1 + c_2j)r^j = c_1 + c_2j.$$

From Equation (24),  $c_1 = 1/(1-M)$  and  $c_2 = 1/(M-1)$ . Hence  $x_j = (j-1)/(M-1)$ .

Form Equation (25) and Equation (27),  $\lim_{k \rightarrow \infty} \mathbf{P}_1^k$  can be calculated. For the stationary distribution  $\pi_\infty$ , we have

$$\begin{aligned} \pi_\infty &= \pi_0 \lim_{k \rightarrow \infty} \mathbf{P}_1^k = (1 - x_{m+1}, 0, 0, 0, \dots, 0, 0, x_{m+1}) \\ &= \left( \frac{n}{m+n}, 0, 0, \dots, 0, 0, \frac{m}{m+n} \right). \end{aligned} \quad (31)$$

Probabilities of losing and winning the game are  $n/(m+n)$  and  $m/(m+n)$  respectively.

If  $1 - 4ac > 0$ , the characteristic roots are  $r_1 = a/c$  and  $r_2 = 1$  form Equation (30). Since they are different, the general solution would be

$$x_j = c_1 r_1^j + c_2 r_2^j = c_1 \left( \frac{a}{c} \right)^j + c_2.$$

From Equation (24),  $c_1 = 1/[(a/c)^M - a/c]$  and  $c_2 = -(a/c)/[(a/c)^M - a/c]$ . Hence,  $x_j = [(a/c)^{j-1} - 1]/[(a/c)^{M-1} - 1]$ .

Similarly, the stationary distribution  $\pi_\infty$  is

$$\begin{aligned}\pi_\infty &= \pi_0 \lim_{k \rightarrow \infty} \mathbf{P}_1^k = (1 - x_{m+1}, 0, 0, 0, \dots, 0, 0, x_{m+1}) \\ &= \left( \frac{1 - (c/a)^m}{1 - (c/a)^{m+n}}, 0, 0, \dots, 0, 0, \frac{(a/c)^m - 1}{(a/c)^{m+n} - 1} \right).\end{aligned}\quad (32)$$

Probabilities of losing and winning the game are  $[1 - (c/a)^m]/[1 - (c/a)^{m+n}]$  and  $[(a/c)^m - 1]/[(a/c)^{m+n} - 1]$  respectively.

The probabilities of losing and winning the game of these two cases are the same as the result of classical methods<sup>[9,10]</sup>, demonstrating that our approach is suitable.

## 4.2 An Expansion of the Gambler's Ruin Problem

Let us consider a numeric example. It is an expansion of the gambler's ruin problem. The difference is that the bet of each game is not just one unit of money. Suppose an unfair game that the gambler starts with 100 units of money, whose rival have 99 units of money. The initial distribution of him can be written as a  $1 \times 200$  vector,  $\pi_0 = (0, 0, \dots, 0, 1, 0, 0, \dots, 0)$ . Only the 101st element is 1.

In each game, probabilities of money changed for this gambler corresponds to Table 1.

**Table 1 Probabilities of money changed in each game**

Number of bet changes	-3	-2	-1	0	+1	+2	+3
Probability	1/42	1/7	1/5	1/4	1/3	1/20	0

For example, the probability of losing two units of money is 1/7.

Let  $\mathbf{P}_2$  be the transition matrix of this example, whose size is  $200 \times 200$ .  $\mathbf{P}_2$  could be written as

$$\mathbf{P}_2 = \begin{pmatrix} 1 & \mathbf{0} & 0 \\ \mathbf{R}_1 & \mathbf{T}_2 & \mathbf{R}_2 \\ 0 & \mathbf{0} & 1 \end{pmatrix},$$

where  $\mathbf{R}_1 = (11/30, 1/6, 1/42, 0, 0, \dots, 0)^\top$  and  $\mathbf{R}_2 = (0, 0, 0, \dots, 0, 0, 1/20, 23/60)^\top$ .  $\mathbf{T}_2$  is a  $198 \times 198$  Toeplitz matrix whose first row and first column are  $(1/4, 1/3, 1/20, 0, 0, \dots, 0)$  and  $(1/4, 1/5, 1/7, 1/42, 0, 0, \dots, 0)^\top$  respectively.

Let  $\mathbf{x}_1$  be the eigenvector with eigenvalue 1 of  $\mathbf{P}_2$ , where  $\mathbf{x}_1 = (x_1, x_2, x_3, \dots, x_{200})^\top$ . Then

$$(\mathbf{P}_2 - \mathbf{I})\mathbf{x}_1 = \mathbf{0}.\quad (33)$$

$\mathbf{x}_1$  is a non-trivial solution of this linear equations. There are some methods to solve this question, e.g. the singular value decomposition<sup>[11]</sup>.

Form Equation (24), we could let  $x_1 = 0$  and  $x_{200} = 1$ . Therefore, Equation (33) can be written as

$$(\mathbf{T}_2 - \mathbf{I})\mathbf{y}_1 = -\mathbf{R}_2, \quad (34)$$

where  $\mathbf{y}_1 = (x_2, x_3, \dots, x_{189}, x_{199})^T$ . There is a unique solution to this equation, and  $\mathbf{y}_1$  is easy to calculate. This is our method. Then  $\lim_{k \rightarrow \infty} \mathbf{P}_2^k$  can be calculated and the stationary distribution could be figured out from any initial distribution, e.g. the gambler has  $m$  units of money and his rival has  $199 - m$  units of money. For simplicity, we only compute probabilities of losing and winning the game starting from 100 units of money.

To verify the correctness of the proposed approach, Equation (7) is still used to compute the probabilities. The computed results of  $x_{101}$  are all  $5.6010 \times 10^{-8}$  for the two listed methods. For this gambler, probabilities of defeat and victory are  $1 - 5.6010 \times 10^{-8}$  and  $5.6010 \times 10^{-8}$  respectively. The result of the proposed approach is the same as the general method mentioned in Equation (7), indicating the correctness of our sample method.

In fact, there is a close relationship between the simple method and the general one. If we multiply the inverse of  $\mathbf{T}_2 - \mathbf{I}$  at both sides of Equation (34), it will be similar to Equation (7). Hence this approach is a variant of the general method. So that in Equation (7), the same technique can also be taken to avoid matrix inversion.

## §5. Conclusions

In this paper, we proposed an easy calculation method to deal with the stationary distribution for absorbing Markov chains with two absorbing states. In the models defined by us, the norm of eigenvalues of the transition matrix are not greater than 1 and the algebraic multiplicity of eigenvalue 1 is 2. If the transient matrix is diagonalizable, the transition matrix is diagonalizable as well and we could use one eigenvector to calculate the stationary distribution quickly. We have used this method to derive probabilities of the classic gambler's ruin problem. The result is consistent with other methods. Furthermore, for complex expansions, our approach is still effective. In fact, it is a variant of the conventional method. Hence, for all absorbing Markov chains, similar techniques can be taken to avoid computing the inverses of transition matrices.

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## 仅运用一个特征向量计算可对角化转移矩阵的 吸收马尔可夫链的平稳分布

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**摘 要:** 吸收马尔可夫链是一种重要的统计模型, 广泛地被用于众多学科中的算法建模, 如在数字图像处理、网络分析等中. 为了得到该模型的平稳分布, 通常需要计算转移矩阵的逆, 但是对于大型矩阵来说这仍然是比较困难且耗费计算量的. 在本文中, 对于含有两个吸收态的马尔可夫链, 当其转移矩阵可对角化时, 我们提出了一种简单方法来计算其平稳分布. 在该方法中仅仅需要计算特征值 1 对应的一个特征向量即可. 我们运用该方法, 从矩阵的角度推导出了赌徒破产问题的相关概率. 同时本方法也能够处理该问题的复杂扩展形式. 事实上, 本方案是对处理吸收马尔可夫链的通用方法的一个变种, 因此在通用方法中也能够采用类似的技术来避免矩阵求逆运算.

**关键词:** 随机游走; 吸收马尔可夫链; 平稳分布; 赌徒破产问题

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