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Discrete-Time Quantum Random Walks on the N-ary Tree *

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Abstract: We study discrete-time quantum random walks on the N-ary tree by a framework for discrete-time quantum random walks, this framework has no need for coin spaces, it just choose the evolution operator with no constraints other than unitarity, and contain path enumeration using regeneration structures and z transform. As a result, we calculate the generating function of the amplitude at the root in closed form.

Keywords: quantum random walk; z transform; path enumeration; N-ary tree

2020 Mathematics Subject Classification: 05C81; 81P16

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§1. Introduction

Since Aharonov et al. [1] presented quantum random walks for the first time, the emergence of quantum random walks on the graph plays a crucial role. Quantum random walks as a quantum version of classical random walks, the standard approaches to deal with quantum random walks come from classical random walks. Unlike classical random walks, quantum random walks require the role of unitary operators.

Quantum random walks can be divided into discrete-time quantum random walks (DTQW) and continuous-time quantum random walks (CTQW). About applications of quantum random walks see [2–5]. In 2012, Venegas-Andraca [6] reviewed the theoretical research and development of DTQW and CTQW. Inui et al. [7] and Machida [8] also studied quantum random walks in one-dimensional spaces in 2005 and 2013, respectively. In recent years, there are also some new developments about quantum random walks. For example, Wang and Ye [9] presented some new results concerning quantum Bernoulli noises, and constructed a time-dependent quantum random walk with infinitely many degrees of

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freedom in 2016 and Wang et al.[10] also introduced an open quantum random walk based on the quantum Bernoulli noise in 2018.

In general, the state space of quantum random walks consist of the product space of a coin space and a position space. We observed the most direct relationship between quantum random walks and classical random walks[2,3], similar to the specific representation of classical memory-2 random walks, we present a general DTQW framework. However, this framework has no need for coin spaces, in this paper, the quantum random walk is constructed by choosing the evolution operator with no constraints other than unitarity. It is also unifying of other approaches, quantum computation and algorithmic uses[11–13], notably coined and Szegedy's[14].

We briefly summarize the contents of this paper. In Section 2, we present a framework for DTQW by the specific representation of classical memory-2 random walks and introduce path enumeration using regeneration structures and z transform. In Section 3, we choose the evolution operator with no constraints other than unitarity, under the DTQW framework, consider a pure state at an arbitrary level in an N-ary tree, we construct a symmetric DTQW on an N-ary tree and calculate its amplitude at root via path enumeration using regeneration structures and z transform.

§2. A Framework for Discrete-Time Quantum Random Walks

In this section, we take memoried random walks and "coin" discrete-time quantum random walks as a motivation.

Definition 1 A random walk in one-dimensional space with property that the next step depends on the direction of the previous step is called a memoried random walk.

Remark 2 The description of Definition 1 is as follows: on a line, if the walker came to a site i from the site $i-1$, the probability to go to $i+1$ is p (to maintain the direction), while the probability to go to $i-1$ is $1-p$ (reverse the direction).

Before we give the definition of a standard coined random walk on a d-regular graph with n vertices, we state the state space of the random walk, one is a tensor space of two Hilbert spaces: an auxiliary ("coin") space $\mathscr{H}_A$ spanned by d states $|a\rangle$, in which a unitary operator $\boldsymbol{C}$ mixes components and a space of vertices $\mathscr{H}_V$ spanned by n states $|v\rangle$.

Definition 3[15] In a tensor space $\mathscr{H}_A \otimes \mathscr{H}_V$, if the random walk under the action of the evolution operator $\boldsymbol{U}$, there is $\boldsymbol{U}(|a\rangle \otimes |v\rangle) = \boldsymbol{S}(\boldsymbol{C} \otimes \boldsymbol{I})(|a\rangle \otimes |v\rangle)$, where $\boldsymbol{S}$ is called the shift operator which the specific action is $\boldsymbol{S} = |\uparrow\rangle\langle\uparrow| \otimes \sum_j |j+1\rangle\langle j| + |\downarrow\rangle\langle\downarrow| \otimes \sum_j |j-1\rangle\langle j|$ on a cycle with the $\mathscr{H}_A$ basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ and $\boldsymbol{I}$ is an identity operator, then the random walk is called the standard coined random walk on a d-regular graph with n vertices.

Giving an example.

Example 4 Consider a standard coined random walk on a d-regular graph with n vertices, if a unitary operator $\boldsymbol{C}$ is

$$\boldsymbol{C} = \begin{pmatrix} \sqrt{p} & \sqrt{1-p} \\ \sqrt{1-p} & -\sqrt{p} \end{pmatrix},$$

we have

(i) evolving the state "up",

$$\begin{aligned} \boldsymbol{S}(\boldsymbol{C} \otimes \boldsymbol{I}) |\uparrow\rangle \otimes |i\rangle &= \boldsymbol{S}(\sqrt{p} |\uparrow\rangle \otimes |i\rangle + \sqrt{1-p} |\downarrow\rangle \otimes |i\rangle) \\ &= \sqrt{p} |\uparrow\rangle \otimes |i+1\rangle + \sqrt{1-p} |\downarrow\rangle \otimes |i-1\rangle, \end{aligned}$$

(ii) evolving the state "down",

$$\boldsymbol{S}(\boldsymbol{C} \otimes \boldsymbol{I}) |\downarrow\rangle \otimes |i\rangle = \sqrt{1-p} |\uparrow\rangle \otimes |i+1\rangle - \sqrt{p} |\downarrow\rangle \otimes |i-1\rangle.$$

Remark 5 As the square of the amplitude is the probability, Example 4 is an equal probability random walk when $p = 1/2$.

1) A Representation for Classical Memory-2 Random Walks

Definition 6 In a Hilbert space, a random walk with property that the next step depends on the current state and the previous state is called a classical memory-2 random walk.

Remark 7 Definition 6 is the generalization of Definition 1 in two-dimension space.

Similarly, consider the evolution of a classical memory-2 random walk, such as a memory-2 Markov process, the evolution operator (the Markov tensor) is denoted by $\mathscr{M}$ which is the third-rank operator. In classical probability theory, a memory-2 Markov process can be represented by a probability distribution $\mu(t)$ of dimension 2. Since the space of a memory-2 Markov process has dimension n and each state is labeled by two indices (the site the walker came from and the current site), so the probability distribution is

$$\mu(t)=\begin{pmatrix}\mu_{0,0}(t) & \mu_{0,1}(t) & \cdots & \mu_{0,n-1}(t)\\ \mu_{1,0}(t) & \mu_{1,1}(t) & \cdots & \mu_{1,n-1}(t)\\ \vdots & \vdots & \ddots & \vdots\\ \mu_{n-1,0}(t) & \mu_{n-1,1}(t) & \cdots & \mu_{n-1,n-1}(t)\end{pmatrix}.$$

In order to facilitate the function of the evolution operator $\mathscr{M}$, $\mu(t)$ is replaced by a column of rows r_i or a row of columns c_i.

Definition 8 [16] Let $p_{ij|k}$ be the conditional probability for the transition $j\to k$, given that the random walk came to j from i. $j=0,1,\cdots,n-1$, the transition probability matrix $\boldsymbol{P}_j$ is defined as

$$\boldsymbol{P}_j=\begin{pmatrix}p_{0,j|0} & p_{0,j|1} & \cdots & p_{0,j|n-1}\\ p_{1,j|0} & p_{1,j|1} & \cdots & p_{1,j|n-1}\\ \vdots & \vdots & \ddots & \vdots\\ p_{n-1,j|0} & p_{n-1,j|1} & \cdots & p_{n-1,j|n-1}\end{pmatrix},$$

and the evolution operator $\mathscr{M}=[\boldsymbol{P}_0\ \boldsymbol{P}_1\ \cdots\ \boldsymbol{P}_{n-1}]$.

It is easy to see that such a construction makes sense. Moreover, we have

Proposition 9 [16] Let the mapping φ: $\mu(t)\mapsto\mu(t+1): r_j(t+1)=c_j^{\mathrm{T}}(t)\boldsymbol{P}_j$, for each $j=0,1,\cdots,n-1$. Consider the cycle of a space $\{0,1,\cdots,n\}$ with identified ends (0 and n), with only nearest-neighbor transitions, $(j\pm1,j)\to(j,j\pm1)$, take the classical memory-2 random walk. We have

(i) $(j-1,j)\to(j,j+1)$, with probability p,

(ii) $(j-1,j)\to(j,j-1)$, with probability $1-p$.

Moreover, the transition probability matrix

$$\boldsymbol{P}_j=\begin{pmatrix}\ddots &&&&&&\\ & 1 &&&&&\\ && 1-p & 0 & p &&\\ && 0 & 1 & 0 &&\\ && p & 0 & 1-p &&\\ &&&&& 1 &\\ &&&&&& \ddots\end{pmatrix}.$$

2) Path Counting and Regeneration Sums

We give an N-ary tree as shown in the following figure (Figure 1).

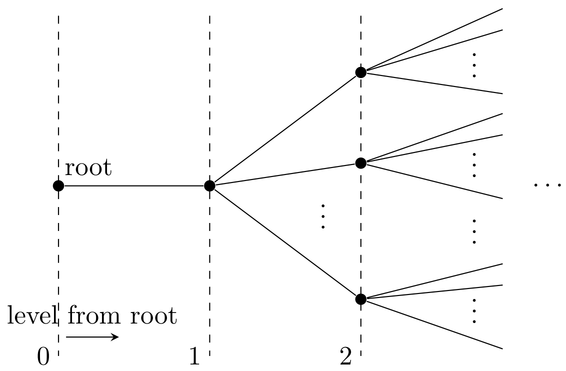

Figure 1 An N-ary tree, the level is infinite, at level 0, we mark it as a root.

A quantum random walk has a variety of paths, all paths that are at the root at time t have the following structure. The amplitude for the path which touch the root for the first time at one point (step s) is denoted by $h_n(s)$. The amplitude for the path which touch the root for the first time from the level n and the remainder of the walk is a root-to-root path is denoted by $\boldsymbol{H}_n(t)$. $\boldsymbol{G}(t-s)$ is the amplitude for the path which touch the root then go out in the tree, eventually coming back to the root at step t, possibly touching it multiple times in the process. As shown in the following figure (Figure 2).

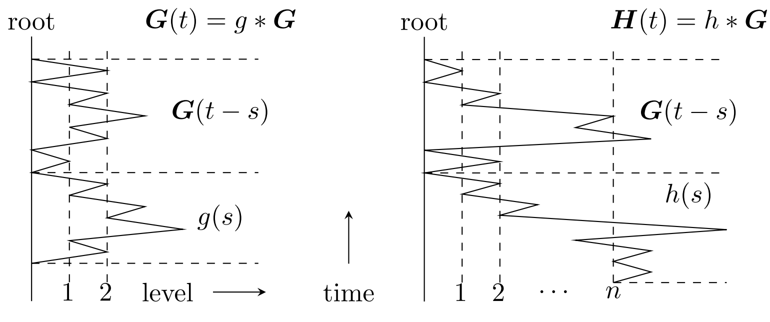

Figure 2 Figure interpretation of $h_n(s)$, $\boldsymbol{G}(t-s)$, $\boldsymbol{H}_n(t)$, $\boldsymbol{G}(t)$, $g(s)$. Where peaks are positions furthest from the root.

Fact 10 Starting from level n, the relationship among $\boldsymbol{H}_n(t)$, $h_n(s)$ and $\boldsymbol{G}(t-s)$ can be described by the following convolution,

$$\boldsymbol{H}_n(t) = \sum_{s \geqslant n}^{t} h_n(s)\boldsymbol{G}(t-s). \tag{1}$$

Remark 11 For the $n = 0$ case, $\boldsymbol{G}(t) = \boldsymbol{H}_0(t)$.

As shown in the figure above (Figure 2), $\boldsymbol{G}(t)$ is the amplitude the path which after touch the root for the first time, the remainder of the random walk is a root-to-root path. $g(s)$ (called a simple loop) is the amplitude the path which go from the root into the tree and back to it (reaching it again for the first time).

By Remark 11, similar to $h_n(s)$, we review the relationship among $\boldsymbol{G}(t)$, $g(s)$ and $\boldsymbol{G}(t-s)$.

Proposition 12 [16] The relationship among $\boldsymbol{G}(t)$, $g(s)$ and $\boldsymbol{G}(t-s)$ can be described by the following convolution,

$$\boldsymbol{G}(t) = \sum_{s=0}^{t} g(s)\boldsymbol{G}(t-s) + \delta_0(t). \tag{2}$$

The regeneration sums of Fact 10 and Proposition 12 is a simple way to calculate, we need to introduce z transform $\widehat{f}(z) = \sum_{t=0}^{\infty} f(t)z^t$, $|z| < 1$. Then we have

Proposition 13 [16] For (1) and (2), using z transform, we have

$$\widehat{\boldsymbol{H}}_n(z) = \widehat{h}_n(z)\widehat{\boldsymbol{G}}(z) \quad \text{and} \quad \widehat{\boldsymbol{G}}(z) = \widehat{g}(z)\widehat{\boldsymbol{G}}(z) + 1 \Rightarrow \widehat{\boldsymbol{H}}_n(z) = \widehat{h}_n(z)\frac{1}{1-\widehat{g}(z)}. \tag{3}$$

§3. Quantum Random Walks on an N-ary Tree

In this section, we mainly calculate the generating function of amplitude at the root of the quantum random walk on the N-ary tree. Constructing the evolution operator

$$\boldsymbol{U}_j = \begin{pmatrix} \boldsymbol{U}_j^{\widetilde{s}} & 0 \\ 0 & \boldsymbol{I} \end{pmatrix},$$

with

$$\boldsymbol{U}_j^{\widetilde{s}} = \frac{1}{\sqrt{(N-1)^2+N}} \begin{pmatrix} N-1 & a & a & \cdots & a \\ a & N-1 & a & \cdots & a \\ a & a & N-1 & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ a & a & a & \cdots & N-1 \end{pmatrix},$$

where $a = \mathrm{e}^{2\pi i/3}$, $\boldsymbol{U}_j^{\widetilde{s}}$ is a local unitary operator which operate in the subspace of nodes. It is not hard to prove that $\boldsymbol{U}_j$ is a unitary operator.

Lemma 14 In a node of an N-ary tree, the action of operator $\widehat{\boldsymbol{U}}\widehat{\boldsymbol{X}}$ is shown in the following figure (Figure 3), $\widehat{\boldsymbol{U}} = \sum\limits_{j=0}^{N-1} \Pi_j \otimes \boldsymbol{U}_j$, where $\Pi_j = |j\rangle\langle j|$, $\widehat{\boldsymbol{X}} : |i\rangle \otimes |j\rangle \mapsto |j\rangle \otimes |i\rangle$, consider a pure state $|\psi_0\rangle = |i_2\rangle \otimes |j\rangle$, we have

$$\begin{aligned}|i_2\rangle \otimes |j\rangle \to |j\rangle \otimes \Big[&\frac{a}{\sqrt{(N-1)^2+N}}|i_1\rangle + \frac{N-1}{\sqrt{(N-1)^2+N}}|i_2\rangle + \frac{a}{\sqrt{(N-1)^2+N}}|i_3\rangle \\ &+ \cdots + \frac{a}{\sqrt{(N-1)^2+N}}|i_{N+1}\rangle\Big].\end{aligned}$$

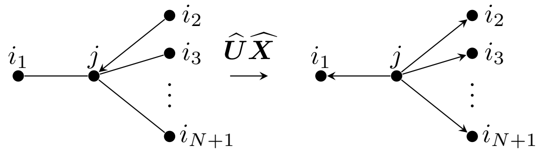

Figure 3 Evolution of a pure state $|\psi_0\rangle = |i_2\rangle \otimes |j\rangle$ in the subspace. A node of the tree is labelled as j, which connected to it as i_1 (to its left, toward the root), i_2 (to its right, away from the root), i_3 (to its right, away from the root), $\cdots$, i_{N+1} (to its right, away from the root).

Proof With $\widehat{\boldsymbol{U}} = \sum\limits_{j=0}^{N-1} \Pi_j \otimes \boldsymbol{U}_j$ and $\widehat{\boldsymbol{X}} : |i\rangle \otimes |j\rangle \mapsto |j\rangle \otimes |i\rangle$, we have

$$\begin{aligned}|\psi_1\rangle = \widehat{\boldsymbol{U}}\widehat{\boldsymbol{X}}|\psi_0\rangle &= \Big(\sum_{i\in S} |i\rangle\langle i| \otimes \boldsymbol{U}_i\Big)\widehat{\boldsymbol{X}}|i_2\rangle \otimes |j\rangle \\ &= |j\rangle \otimes \frac{1}{\sqrt{(N-1)^2+N}}(a, N-1, a, a, \cdots, a, 0, \cdots)^{\mathsf{T}}.\end{aligned}$$

Thus the state is evolved by $\widehat{\boldsymbol{U}}\widehat{\boldsymbol{X}}$ to the superposition,

$$\begin{aligned}|i_2\rangle \otimes |j\rangle \to |j\rangle \otimes \Big[&\frac{a}{\sqrt{(N-1)^2+N}}|i_1\rangle + \frac{N-1}{\sqrt{(N-1)^2+N}}|i_2\rangle + \frac{a}{\sqrt{(N-1)^2+N}}|i_3\rangle \\ &+ \cdots + \frac{a}{\sqrt{(N-1)^2+N}}|i_{N+1}\rangle\Big].\end{aligned}$$

This completes the proof. □

Corollary 15 Summarizing the law of the random walk as follows (the direction and coefficient of the next step):

When the direction of the previous step is away from the root, it can

(i) turn back, with the coefficient $(N-1)/\sqrt{(N-1)^2+N}$ (left turn);

(ii) continue, with the coefficient $Na/\sqrt{(N-1)^2+N}$ (right step).

When the direction of the previous step is toward the root, it can

(i) turn away, with the coefficient $[(N-1)+(N-1)a]/\sqrt{(N-1)^2+N}$ (right turn);

(ii) continue, with the coefficient $a/\sqrt{(N-1)^2+N}$ (left step).

The appearance of reflection is an interesting situation, it complicates the calculation of the amplitude.

Remark 16 There is a special case that not following the above classification, whenever paths touch the root their next step can only be a turn back, with the coefficient 1.

Lemma 17 [4] The number of paths with k peaks from $(0,0)$ to $(2n,0)$ on a lattice in two dimensions, taking only Northeast or Southeast steps, is given by Narayana numbers

$$\boldsymbol{N}(n,k)=\frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Fact 18 A simple loop must take an even number of steps, it starts at the root, having arrived to it from the first node, and it can only step back onto the first node, so the coefficient for this step is 1, paths with k peaks take k left turns and $k-1$ right turns. Loops of t steps need take $(t-2)/2-(k-1)$ right, as well as left.

Lemma 19 [16] For the z transform $\widehat{h}_n(s)$ of $h_n(s)$, and the z transform $\widehat{h}_1(s)$ of $h_1(s)$. Through the following figure (Figure 4), the relationship between $\widehat{h}_n(s)$ and $\widehat{h}_1(s)$ is

$$\widehat{h}_n(z)=[\widehat{h}_1(z)]^n. \tag{4}$$

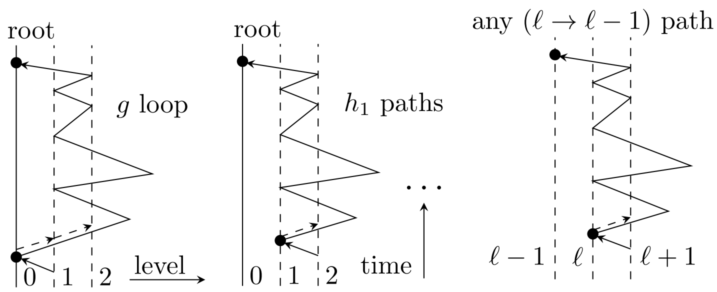

Figure 4 Consider paths from the nth level that reach the root for the first time in s steps (namely with amplitude $h_n(s)$), the first step is to arrive to level $n-1$ (with amplitude h_1), the second step is to arrive to level $n-2$ (with amplitude h_1) and so forth, until the root is hit.

Lemma 20 Consider a discrete-time quantum random walk on the N-ary tree, the relationship between $g(s+1)$ and $h_1(s)$ can be described by the following equation,

$$h_1(s) = \frac{a}{\sqrt{(N-1)^2+N}}\delta(s-1) + \frac{(N-1)(1+a)}{Na} g(s+1) \times \mathbf{1}_{s\in\{3,5,\cdots\}}.$$

Proof Consider a path which start at level 1 and find it way to the root for the first time in more than one step, as shown in Figure 4, a loop $g(s)$ differs from $h_1(s)$ as follows: the first step of $g(s)$ that with coefficient 1 and the next step of $g(s)$ (for paths with $t > 2$) is a right step. However, $h_1(s)$ is already at level 1 and the next step of $h_1(s)$ is a right turn. Thus we divide the expression for $g(s)$ by the coefficient $Na/\sqrt{(N-1)^2+N}$ associated with the second step of $g(s)$, and multiply the expression for $h_1(s)$ by the coefficient $[(N-1)+(N-1)a]/\sqrt{(N-1)^2+N}$, we also divide the expression for $g(s)$ by the coefficient 1 associated with the first step of $g(s)$. In the special case $s=1$, the coefficient of $h_1(1)$ is $a/\sqrt{(N-1)^2+N}$. This path takes one step more as compared to $g(s)$, we make use of the expression for $g(s+1)$, starting from $s=3$, we have

$$h_1(s) = \frac{a}{\sqrt{(N-1)^2+N}}\delta(s-1) + \frac{(N-1)(1+a)}{Na} g(s+1) \times \mathbf{1}_{s\in\{3,5,\cdots\}}.$$

This completes the proof. □

Theorem 21 A discrete-time quantum random walk on the N-ary tree which the generating function of the amplitude at the root is

$$\boldsymbol{H}_n(t) = \frac{(-a)^n}{2\pi i}\frac{1}{N^n} \times \oint_{|z|=r} \frac{\{(N-1)\widehat{g}(z) - [\sqrt{(N-1)^2+N}\,]z^2\}^n}{1-\widehat{g}(z)}\frac{\mathrm{d}z}{z^{t+n+1}}.$$

Proof By Fact 18, a simple loop with k peaks for $t \geqslant 4$ steps, bears the coefficient is

$$\begin{aligned}
&\Big[\frac{N-1}{\sqrt{(N-1)^2+N}}\Big]^k \Big[\frac{(N-1)(1+a)}{\sqrt{(N-1)^2+N}}\Big]^{k-1} \Big[\frac{a}{\sqrt{(N-1)^2+N}}\Big]^{t/2-k} \Big[\frac{Na}{\sqrt{(N-1)^2+N}}\Big]^{t/2-k} \\
={}& \frac{(N-1)^k}{\sqrt{(N-1)^2+N}^{\,t-1}}[(N-1)(1+a)]^{k-1}(Na^2)^{t/2-k} \\
={}& \frac{(N-1)^k}{\sqrt{(N-1)^2+N}^{\,t-1}}\Big[\frac{(N-1)(1+a)}{Na^2}\Big]^{k-1}(Na^2)^{k-1}(Na^2)^{t/2-k} \\
={}& \frac{N-1}{\sqrt{(N-1)^2+N}^{\,t-1}}\Big[-\frac{(N-1)^2}{N}\Big]^{k-1}(Na^2)^{t/2-1},
\end{aligned}$$

where $a = \mathrm{e}^{2\pi i/3}$. Note that the $t=2$ case, their coefficient is $1\times(N-1)/\sqrt{(N-1)^2+N}$, since the amplitude at the root at time t is computed as a sum of the amplitudes of all possible paths that are at the root at that time, thus the amplitude of a loop $g(t)$ with k peaks is

$$\begin{aligned} g(t) = {} & \frac{N-1}{\sqrt{(N-1)^2+N}}\delta_0(t-2) \\ & + \sum_{k=1}^{(t-2)/2} \frac{N-1}{\sqrt{(N-1)^2+N}^{t-1}} \Big[-\frac{(N-1)^2}{N}\Big]^{k-1} (Na^2)^{t/2-1} \boldsymbol{N}\Big(\frac{t-2}{2}, k\Big). \end{aligned}$$

Using Lemma 17, with $t = 2m+2$, we have

$$\begin{aligned} g(2m+2) = {} & \frac{N-1}{\sqrt{(N-1)^2+N}}\delta_0(m) \\ & + \frac{N-1}{m\sqrt{(N-1)^2+N}} \Big[\frac{Na^2}{(N-1)^2+N}\Big]^m \sum_{k=0}^{m-1} \Big[-\frac{(N-1)^2}{N}\Big]^k \binom{m}{k+1}\binom{m}{k}. \end{aligned}$$

Employing

$$\sum_{k=0}^{m-1} (\alpha\beta)^k \binom{m}{k+1}\binom{m}{k} = \frac{1}{2\pi}\int_0^{2\pi} (1+\alpha \mathrm{e}^{ix})^m (1+\beta \mathrm{e}^{-ix})^m \frac{\mathrm{e}^{-ix}}{\alpha}\mathrm{d}x,$$

where α, β are constants. Under the constraint $\alpha\beta = -(N-1)^2/N$, we have

$$\begin{aligned} g(2m+2) = {} & \frac{N-1}{\sqrt{(N-1)^2+N}}\delta_0(m) + \frac{1}{2\pi}\frac{N-1}{m\sqrt{(N-1)^2+N}} \Big[\frac{Na^2}{(N-1)^2+N}\Big]^m \\ & \times \int_0^{2\pi} \Big[\frac{N-(N-1)^2}{N} + \alpha \mathrm{e}^{ix} + \beta \mathrm{e}^{-ix}\Big]^m \frac{\mathrm{e}^{-ix}}{\alpha}\mathrm{d}x. \end{aligned} \tag{5}$$

Using z transform of $g(2m+2)$, since loops take even number of steps and $\widehat{g}(z)_{t=0} = g(0) = 0$, with $t = 2m+2$,

$$\widehat{g}(z) = \sum_{t=0}^{\infty} g(t)z^t = g(0) + g(2)z^2 + \sum_{t=4,6,\cdots} g(t)z^t = \widehat{g}(z)_{m=0} + \sum_{m=1}^{\infty} g(m)z^{2m+2}.$$

Since the transform of δ is 1, and (5) becomes

$$\begin{aligned} \widehat{g}(z) = {} & \frac{N-1}{\sqrt{(N-1)^2+N}}z^2 + \frac{N-1}{2\pi\sqrt{(N-1)^2+N}}z^2 \int_0^{2\pi} \frac{\mathrm{e}^{-ix}}{\alpha} \\ & \times \Big\{ \sum_{m=1}^{\infty} \frac{1}{m} \Big[\frac{Na^2}{(N-1)^2+N}\Big]^m \Big[\frac{N-(N-1)^2}{N} + \alpha \mathrm{e}^{ix} + \beta \mathrm{e}^{-ix}\Big]^m z^{2m} \Big\}\mathrm{d}x. \end{aligned}$$

We make use of $\sum_{n=1}^{\infty} x^n/n = -\ln(1-x)$, $|x| < 1$, and putting $\alpha = -\beta = (N-1)/\sqrt{N}$, we have

$$\widehat{g}(z) = \frac{N-1}{\sqrt{(N-1)^2+N}}z^2 + \frac{1}{2\pi}\frac{\sqrt{N}}{\sqrt{(N-1)^2+N}}z^2 \int_0^{2\pi} \mathrm{e}^{-ix}$$

$$\times \Big\{ -\ln \Big\{ 1 - \frac{Na^2z^2}{(N-1)^2+N} \Big[\frac{N-(N-1)^2}{N} + \frac{(N-1)(\mathrm{e}^{ix}-\mathrm{e}^{-ix})}{\sqrt{N}} \Big] \Big\} \Big\} \mathrm{d}x.$$

Using $\omega = \mathrm{e}^{-ix}$, we have

$$\begin{aligned}
\widehat{g}(z) &= \frac{N-1}{\sqrt{(N-1)^2+N}} z^2 - \frac{1}{2\pi} \frac{\sqrt{N}}{\sqrt{(N-1)^2+N}} z^2 \\
&\quad \times \int_0^{2\pi} (-i\mathrm{e}^{-ix}) \frac{-\dfrac{Na^2z^2(N-1)(i\mathrm{e}^{ix}+i\mathrm{e}^{-ix})}{[(N-1)^2+N]\sqrt{N}}}{1 - \dfrac{Na^2z^2}{(N-1)^2+N} \Big[\dfrac{N-(N-1)^2}{N} + \dfrac{N-1}{\sqrt{N}} (\mathrm{e}^{ix} - \mathrm{e}^{-ix}) \Big]} \mathrm{d}x \\
&= \frac{N-1}{\sqrt{(N-1)^2+N}} \Bigg\{ z^2 - z^4 \frac{1}{2\pi i} \frac{Na^2}{(N-1)^2+N} \\
&\quad \times \oint_{|\omega|=1} \frac{\dfrac{1}{\omega} + \omega}{1 - \dfrac{Na^2z^2}{(N-1)^2+N} \Big[\dfrac{N-(N-1)^2}{N} + \dfrac{N-1}{\sqrt{N}} \Big(\dfrac{1}{\omega} - \omega \Big) \Big]} \mathrm{d}\omega \Bigg\}.
\end{aligned}$$

Using the Residue theorem to transform $\widehat{g}(z)$, we have

$$\widehat{g}(z) = \frac{N-1}{\sqrt{(N-1)^2+N}} z^2 \Big\{ 1 - \frac{1}{(N-1)^2+N} \Big[\frac{A\sqrt{B} - A + Ca^2z^2}{2(N-1)} \Big] \Big\},$$

where

$$\begin{aligned}
&A = N[(N-1)^2+N]^2, \qquad B = 1 + \frac{2[(N-1)^2-N]a^2z^2}{(N-1)^2+N} + a^4z^4, \\
&C = N[(N-1)^4 - N^2].
\end{aligned}$$

By Lemma 20, we have

$$h_1(s) = \frac{a}{\sqrt{(N-1)^2+N}} \delta(s-1) + \frac{(N-1)(1+a)}{Na} g(s+1) \times \mathbf{1}_{s \in \{3,5,\cdots\}}.$$

Using z transform of $h_1(s)$, we have

$$\begin{aligned}
\widehat{h}_1(z) &= \frac{az}{\sqrt{(N-1)^2+N}} + \frac{(N-1)(1+a)}{Na} \Big[\frac{1}{z} \sum_{t=3,5,\cdots} g(t+1) z^{t+1} \Big] \\
&= \frac{[\sqrt{(N-1)^2+N}\,]az}{N} - \frac{(N-1)a}{N} \frac{\widehat{g}(z)}{z}. \qquad (6)
\end{aligned}$$

With (3), (4) and (6), we have

$$\widehat{\boldsymbol{H}}_n(z) = \Big(-\frac{a}{N} \Big)^n \Big\{ \frac{(N-1)\widehat{g}(z) - [\sqrt{(N-1)^2+N}\,]z^2}{z} \Big\}^n \frac{1}{1 - \widehat{g}(z)}.$$

Then we use the inverse transform of z which is

$$\boldsymbol{H}_n(t) = \frac{1}{2\pi i}\oint_{|z|=r}\frac{\widehat{\boldsymbol{H}}_n(t)}{z^{t+1}}\mathrm{d}z,$$

the generating function of the amplitude at the root is obtained,

$$\begin{aligned}\boldsymbol{H}_n(t) &= \frac{1}{2\pi i}\Big(-\frac{a}{N}\Big)^n\oint\frac{1}{z^{t+1}}\Big\{\frac{(N-1)\widehat{g}(z)-[\sqrt{(N-1)^2+N}\,]z^2}{z}\Big\}^n\frac{\mathrm{d}z}{1-\widehat{g}(z)}\\ &= \frac{(-a)^n}{2\pi i}\frac{1}{N^n}\oint_{|z|=r}\frac{\{(N-1)\widehat{g}(z)-[\sqrt{(N-1)^2+N}\,]z^2\}^n}{1-\widehat{g}(z)}\frac{\mathrm{d}z}{z^{t+n+1}}.\end{aligned}$$

This completes the proof. □

§4. Conclusion Remarks

For the discrete-time quantum random walk on the N-ary tree, the construction of the walk is simple, but the calculation is complicated by a boundary. Based on the discrete-time quantum random walk framework, and combine path enumeration and z transform, we compute the generating function of the amplitude at the root. Quantum random walks as a general tool for exploration and modeling of physical systems, the above study establishes the basis for further study of amplitude numerical calculation.

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N 叉树上的离散时间量子随机行走

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摘 要: 通过离散时间量子随机行走的框架, 我们研究了在 N 叉树上的离散时间量子随机行走, 该框架不需要硬币空间, 仅仅只需要选择一个除了酉性再无其它限制的演化算子, 并且包含了使用再生结构的轨道枚举和 z 变换. 作为结果, 我们在封闭形式中计算了在根处的振幅的生成函数.

关键词: 量子行走; z 变换; 轨道枚举; N 叉树

中图分类号: O211.9; O413.1