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Estimation of Parameters For A Dependent Bivariate Weibull Distribution*

YE CINAN
(Nanjing University of Sci. & Tech., Nanjing, 210014)

Abstract

Consider a dependent bivariate Weibull model whose survival function is $\overline{F}(x_1,x_2) = \exp\{-[(x_1^{1/\alpha}/\theta_1)^{1/\delta} + (x_2^{1/\alpha}/\theta_2)^{1/\delta})^{1/\delta}]^{\delta}\}$ with $x_i > 0, \theta_i > 0, (i=1,2), \alpha > 0$ and $0 < \delta \le 1$. Based on the data of both components and series systems experiments under type I censoring, estimators of the unknown parameters $\theta_1, \theta_2, \alpha$ and δ are given, their asymptotic properties are discussed also. A simulation result is given, too.

Keywords: Bivariate Weibull Distribution, Type I Censoring, Estimation, Asymptotic Property.

AMS Subject Classification: 62N05, 62F10.

§1. Introduction

In reliability study and life testing, the Weibull distribution is perhaps the most widely used lifetime distribution model. For statistical analysis of multivariate lifelime distribution, independence between components is assumed usually. However, in practice, there exists dependence between components very often, which shows that investigating dependent multivariate lifetime distribution is very meaningful. In 1979, Larry Lee [2] proposed a dependent bivariate Weibull distribution whose survival function is $\overline{F}(x_1,x_2) = \exp\{-[(x_1^{1/\alpha}/\theta_1)^{1/\delta} + (x_2^{1/\alpha}/\theta_2)^{1/\delta}]^{\delta}\}, (x_1 > 0, x_2 > 0)$ with unknown parameters $\theta_1 > 0, \theta_2 > 0, \alpha > 0$ and $0 < \delta \le 1$. We henceforth refer to it as LBVW $(\theta_1, \theta_2, \alpha, \delta)$. Hougaard (1986) [1] showed that LBVW is indeed a meaningful physical model. If conditionally given the random factor $Z = z$, the components lives X_1 and X_2 are independently Weibull distributed with failure rate $z\theta_1^{-1/\delta}x_1^{1/(\alpha\delta)-1}/(\alpha\delta)$ and $z\theta_2^{-1/\delta}x_2^{1/(\alpha\delta)-1}/(\alpha\delta)$, and if Z has the positive stable distribution with parameter δ, given by the Laplace transform $\mathsf{E}[\exp(-sZ)] = \exp(-s^{\delta})$, then the unconditional distribution of (X_1, X_2) is LBVW $(\theta_1, \theta_2, \alpha, \delta)$. It is easy to see that X_1 and X_2 are independent if and only if dependence parameter $\delta = 1$. The complicated form of the density of LBVW poses the major difficulty for statistical inference. In this paper, both component and system (series) life test data under type I censoring are used to do statistical inference. In section 2, some properties of LBVW are given. In section 3, estimators of $(\theta_1, \theta_2, \alpha, \delta)$ are proposed, and the joint asymptotic distribution of those estimators are derived. In section 4, a simulation result is given.

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§2. Properties of LBVW

Suppose that (X_1, X_2) follows LBVW $(\theta_1, \theta_2, \alpha, \delta)$. Then (X_1, X_2) has the following properties:

1. The marginal survival functions of X_1 and X_2 are $\overline{F}_i(x_i) = \exp(-x_i^{1/\alpha}/\theta_i)$ $(i = 1, 2)$, respectively.

2. (X_1, X_2) can be represented in terms of independent random variables as follows: $X_1 = \theta_1^\alpha U^{\alpha\delta} S^\alpha$ and $X_2 = \theta_2^\alpha (1-U)^{\alpha\delta} S^\alpha$, where U and S are independent, $U \sim U(0,1)$ and S has a mixture of gamma distributions with density function $h(s) = [(1-\delta) + s\delta]e^{-s}, (s > 0)$ (see [2]) for the proof).

3. Let $T = \min(X_1, X_2)$ and $D = I_{[X_1 \le X_2]}(x_1, x_2)$. Then $\overline{F}_T(t) = \exp(-t^{1/\alpha}/\theta_T)$, where $\theta_T = (\theta_1^{-1/\delta} + \theta_2^{-1/\delta})^{-\delta}$, and T and D are independent. In fact, by property 2, $\mathsf{P}\{T > t|D = 1\} = \mathsf{P}\{\min(X_1, X_2) > t|X_1 < X_2\} = \mathsf{P}\{X_1 > t|X_1 < X_2\} = \mathsf{P}\{\theta_1^\alpha U^{\alpha\delta} S^\alpha > t|U < \theta_2^{1/\delta}/(\theta_1^{1/\delta} + \theta_2^{1/\delta})\}$. Denoting $\mu = \theta_2^{1/\delta}/(\theta_1^{1/\delta} + \theta_2^{1/\delta})$. It follows that $\mathsf{P}\{T > t|D = 1\} = \mu^{-1}\mathsf{P}\{\theta_1^\alpha U^{\alpha\delta} S^\alpha > t, U < \mu\} = \mu^{-1}\int_0^\mu \mathsf{P}\{\theta_1^\alpha U^{\alpha\delta} S^\alpha > t|U = u\}dF_U(u)$. Computing shows that $\mathsf{P}\{T > t|D = 1\} = \exp\{-t^{1/\alpha}/\theta_T\} = \mathsf{P}\{T > t\}$. It can be proved by the same method that $\mathsf{P}\{T > t|D = 0\} = \mathsf{P}\{T > t\}$, which implies that T and D are independent.

4. From property 2, it follows that $\ln(X_1/X_2) = \alpha\delta\ln(U/(1-U)) + \alpha\ln(\theta_1/\theta_2)$, which implies that $\ln(\theta_1/\theta_2)$ follows the logistic distribution with location parameter $\alpha\ln(\theta_1/\theta_2)$ and scale parameter $\alpha\delta$, since $\ln(U/(1-U))$ follows the standard logistic distribution.

5. Using property 2, we obtain that, for $k_1 > 0$ and $k_2 > 0$

$$\mathsf{E}(X_1^{k_1}, X_2^{k_2}) = \theta_1^{\alpha k_1}\theta_2^{\alpha k_2}\Gamma(\delta\alpha k_1 + 1)\Gamma(\delta\alpha k_2 + 1)\Gamma(\alpha(k_1 + k_2) + 1)/\Gamma(\delta\alpha(k_1 + k_2) + 1), \tag{2.1}$$

from which it follows that $\mathsf{E}(X_i) = \theta_i^\alpha\Gamma(\alpha + 1)$, $D(X_i) = \theta_i^{2\alpha}[\Gamma(2\alpha + 1) - \Gamma^2(\alpha + 1)], (i = 1, 2)$, $\mathsf{Cov}(X_1, X_2) = (\theta_1\theta_2)^\alpha[\Gamma^2(\alpha\delta + 1)\Gamma(2\alpha + 1)/\Gamma(2\alpha\delta + 1) - \Gamma^2(\alpha + 1)]$.

6. Differentiating both sides of the formula (2.1) with respect to k_1, one gets

$$\begin{aligned}
&\mathsf{E}(X_1^{k_1}X_2^{k_2}\ln X_1)\\
&= \theta_1^{\alpha k_1}\alpha(\ln\theta_1)\theta_2^{\alpha k_2}\Gamma(\delta\alpha k_1 + 1)\Gamma(\delta\alpha k_2 + 1)\Gamma(\alpha(k_1 + k_2) + 1)/\Gamma(\delta\alpha(k_1 + k_2) + 1)\\
&+ \theta_1^{\alpha k_1}\theta_2^{\alpha k_2}\Gamma'(\delta\alpha k_1 + 1)\delta\alpha\Gamma(\delta\alpha k_2 + 1)\Gamma(\alpha(k_1 + k_2) + 1)/\Gamma(\delta\alpha(k_1 + k_2) + 1)\\
&+ \theta_1^{\alpha k_1}\theta_2^{\alpha k_2}\Gamma(\delta\alpha k_1 + 1)\Gamma(\delta\alpha k_2 + 1)\Gamma'(\alpha(k_1 + k_2) + 1)\alpha/\Gamma(\delta\alpha(k_1 + k_2) + 1)\\
&- \theta_1^{\alpha k_1}\theta_2^{\alpha k_2}\Gamma(\delta\alpha k_1 + 1)\Gamma(\delta\alpha k_2 + 1)\Gamma(\alpha(k_1 + k_2) + 1)\Gamma'(\delta\alpha(k_1 + k_2) + 1)\delta\alpha/\Gamma^2(\delta\alpha(k_1 + k_2) + 1).
\end{aligned} \tag{2.2}$$

Settin $k_1 = 0$ in both sides of (2.2) and simplifying it one gets

$$\begin{aligned}
\mathsf{E}(X_2^{k_2}\ln X_1) &= \alpha(\ln\theta_1)\theta_2^{\alpha k_2}\Gamma(\alpha k_2 + 1) + \theta_2^{\alpha k_2}\Gamma'(1)\delta\alpha\Gamma(\alpha k_2 + 1)\\
&+ \theta_2^{\alpha k_2}\Gamma'(\alpha k_2 + 1)\alpha - \theta_2^{\alpha k_2}\delta\alpha\Gamma(\alpha k_2 + 1)\psi(\delta\alpha k_2 + 1),
\end{aligned} \tag{2.3}$$

here ψ is digamma function whose definition is $\psi(x) = \Gamma'(x)/\Gamma(x)$, and whose some related values are $\psi(1) = -\gamma, \psi'(1) = \pi^2/6, \Gamma'(1) = \Gamma(1)$, $\Gamma''(1) = \psi'(1) + \psi^2(1)$, where $\gamma = 0.5772\ldots$ is the Euler constant.

Setting $k_2 = 0$ in the expression (2.3), one obtains $\mathsf{E}(\ln X_1) = (\ln\theta_1 - \gamma)\alpha$. Similarly, the following holds: $\mathsf{E}(\ln X_2) = (\ln\theta_2 - \gamma)\alpha$. Repeating the above method, one obtains $D(\ln X_1) =$

$\alpha^2\pi^2/6$, $\mathrm{Cov}(\ln X_1, X_2) = \alpha^2\pi^2(1-\delta^2)/\alpha$, and $Corr(\ln X_1, \ln X_2) = 1-\delta^2$.

§3. Estimators of $\theta_1, \theta_2, \alpha$ and δ

Suppose that (X_1, X_2) follows LBVW $(\theta_1, \theta_2, \alpha, \delta)$, and X_1 and X_2 are lifetime of components A and B, respectively. Under type I censoring, consider life test of l_1 units of component A and l_2 units of component B independently, respectively. The truncated times are X_{10} and X_{20} respectively. Denote $I_i = I_{[X_i \le X_{i0}]}$, $V_i = X_i I_i + X_{i0}(l - I_i) = \min(X_i, X_{i0}), (i = 1,2)$. As the results of the above tests, we get two samples: $(V_{1i}, I_{1i})(i = 1, \ldots, l_1)$ and $\{V_{2j}, I_{2j}\}, (j = 1, \ldots, l_2)$. Let $K_i = \sum_{j=1}^{l_1} I_{ij}, (i = 1,2)$. Besides componets life test, consider life test of two-component series system also under type I censoring. Suppose that life test with m prototypes of the series system is conducted, and the truncated time is T_0. The test of components and the test of system are independently conducted. Denote $I_T = I_{[T \le T_0]}$ and $V_T = TI_T + T_0(1 - I_T) = \min(T, T_0)$. So we obtain sample of size m $\{V_{Tj}, I_{Tj}\}, (j = 1, \ldots, m)$. The another version of this sample is $T_{(1)} \le T_{(2)} \le \cdots \le T_{(N)} \le T_0$, where $N = \sum_{j=1}^{m} I_{T_j}$ is the number of failures occuring in $[0, T_0]$. For those observable failure times $T_{(1)}, \ldots, T_{(N)}$, we define failure indicator $D_{(j)} = 1(0)$ if the j-th system failure is caused by the failure of component $A(B)(j = 1, 2, \ldots, N)$.

For the component X_1 the likelihood function of $\{V_{1i}, I_{1i}\}(i = 1, \ldots, l_1)$ is

$$L = (\alpha\theta_1)^{-k_1} \prod_{i=1}^{l_1} V_{1i}^{(1/\alpha - 1)I_{1i}} \exp(-\sum_{i=1}^{l_1} V_{1i}^{1/\alpha}/\theta_1), \tag{3.1}$$

from which we can find the MLE $\widetilde{\alpha}$ and $\widetilde{\theta}_1$. The relationship between $\widetilde{\alpha}$ and $\widetilde{\theta}_1$ is $\widetilde{\theta}_1 = \sum_{i=1}^{l_1} V_{1i}^{1/\widetilde{\alpha}}/K_1$. Since $\widetilde{\alpha}$ does not have closed form, in practice a numerical procedure must be used to determine $\widetilde{\alpha}$, and then $\widetilde{\theta}_1$. Also since $\widetilde{\alpha}$ does not have closed form, it is impossible to discuss the property of $(\widetilde{\alpha}, \widetilde{\theta}_1)$. By computing (see [4]) $\mathrm{E}[I_1(\ln X_{10} - \ln X_1)] = \alpha p_1 h(p_1)$, where $p_1 = \mathrm{P}(X_1 \le X_{10}), h(p) = \ln[-\ln(1-p)] - I_{1p}$, $I_{1p} = p^{-1}\int_0^p \ln[-\ln(1-t)]dt$. On the basis of the above fact we propose estimators of α and θ_1 as follows

$$\widehat{\alpha}_1 = \sum_{i=1}^{l_1}(\ln X_{10} - \ln X_{1i})I_{1i}/(K_1 h(\widehat{p}_1)), \qquad \widehat{\theta}_1 = \sum_{i=1}^{l_1} V_{1i}^{1/\widehat{\alpha}}/K_1, \tag{3.2}$$

where $\widehat{p}_1 = K_1/l_1$. In order that $\widehat{\alpha}_1$ and $\widehat{\theta}_1$ make sence, assume that at least one, but not all, components fail during the time interval $[0, X_{10}]$, that is, $0 < K_1 < l_1$. It is clear that $\mathrm{P}\{K_1 = k_1 | 0 < K_1 < l_1\} = \binom{l_1}{k_1} p_1^{k_1} q_1^{l_1 - k_1}/(1 - p_1^{l_1} - q_1^{l_1})$ with $q_1 = 1 - p_1, k_1 = 1, \ldots, l_1 - 1$. We refer to this distribution as quasi-binomial one denoted by $\widetilde{B}(l_1, p_1)$. Similarly we propose the estimator of θ_2 as follows

$$\widehat{\theta}_2 = \sum_{i=1}^{l_2} V_{2i}^{1/\widehat{\alpha}_2}/K_2, \tag{3.3}$$

where $\widehat{\alpha}_2 = \sum_{i=1}^{l_2}(\ln X_{20} - \ln X_{2i})I_{2i}/(K_2 h(\widehat{p}_2)), \widehat{p}_2 = K_2/l_2$, and $0 < K_2 < l_2$. As for estimation of δ, we obtain estimator of θ_T first as follows :

$$\widehat{\theta}_T = \frac{1}{N}\sum_{j=1}^{m} V_{Tj}^{1/\widehat{\alpha}_T}, \tag{3.4}$$

where $\widehat{\alpha}_T = \sum_{j=1}^{m}(\ln T_0 - \ln T_j)I_{Tj}/(Nh(\widehat{p}_T))$, $\widehat{p}_T = N/m$, and $0 < N < m$. Then solve the equation $\widehat{\theta}_T^{-1/\delta} = \widehat{\theta}_1^{-1/\delta} + \widehat{\theta}_2^{-1/\delta}$ to get estimator $\widetilde{\delta}$ of δ. Here a numerical procedure must be used . It is difficult to discuss the property of $\widetilde{\delta}$. Now we derive another estimator of δ. Since $\ln(X_1/X_2)$ follows logistic distribution with location parameter $\alpha\ln(\theta_1/\theta_2)$ and scale parameter $\alpha\delta$,

$$\mathsf{P}\{X_1 \le X_2\} = \mathsf{P}\{\ln(X_1/X_2) \le 0\} = \{1 + \exp[\ln(\theta_1/\theta_2)/\delta]\}^{-1} = \theta_1^{-1/\delta}(\theta_1^{-1/\delta} + \theta_2^{-1/\delta})^{-1} \triangleq \mu$$

and $\mathsf{P}\{X_1 > X_2\} = 1 - \mu$, from which it follows that $\delta = [2\ln\theta_T - \ln\theta_1 - \ln\theta_2]/\ln[\mu(1-\mu)]$. Defining $\overline{D} = \sum_{j=1}^{N} D_{(j)}/N$ and noting $\mathsf{P}\{D_{(j)} = 1\} = \mathsf{P}\{X_{1j} \le X_{2j}\} = \mu$, we take $\overline{D}$ as an estimator of μ, and therefore propose moment type estimator of δ as follows:

$$\begin{aligned}\widehat{\delta} &= (2\ln\widehat{\theta}_T - \ln\widehat{\theta}_1 - \ln\widehat{\theta}_2)/\ln[\overline{D}(1-\overline{D})] \\ &= \Big[2\ln\Big(\sum_{j=1}^{m} V_{Tj}^{1/\widehat{\alpha}_T}/N\Big) - \ln\Big(\sum_{i=1}^{l_1} V_{1i}^{1/\widehat{\alpha}_1}/K_1\Big) - \ln\Big(\sum_{i=1}^{l_2} V_{2i}^{1/\widehat{\alpha}_2}/K_2\Big)\Big] \Big/ \ln[\overline{D}(1-\overline{D})].\end{aligned} \tag{3.5}$$

Any linear combinations of $\widehat{\alpha}_1$, $\widehat{\alpha}_2$ and $\widehat{\alpha}_T$, $c_1\widehat{\alpha}_1 + c_2\widehat{\alpha}_2 + c_3\widehat{\alpha}_T$, with $c_i \ge 0, i = 1,2,3$ and $\sum_{i=1}^{3} c_i = 1$ can be emploied as an estimator of α. Now we choose, for example, $\widehat{\alpha}_T$ as an estimator of α, and discuss the asymptotic property of $(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\alpha}_T, \widehat{\delta})$. We present some lemmas before deriving the joint asymptotic distribution of $(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\alpha}_T, \widehat{\delta})$.

Lemma 3.1 $\overline{D} \xrightarrow{\text{a.s}} \mu, \quad (m \to \infty)$.

From lemma 3.1 it follows that as $m \to \infty, \overline{D}(1-\overline{D}) \xrightarrow{\text{a.s}} \mu(1-\mu)$, which guarantees that $\widehat{\delta}$ makes sense if m is large enough.

Lemma 3.2 Suppose that the sequence of random variables $Y_m \xrightarrow{\text{L}} Y (m \to \infty)$, and for every positive integer m, $N_m \sim \widetilde{B}(m,p)(0 < p < 1)$. Then $Y_{N_m} \xrightarrow{\text{L}} Y (m \to \infty)$.

The proofs of Lemma 3.1 and Lemma 3.2 are similar to that of Lemma 3.1 and Lemma 3.2 in [5]. We omit it here.

Lemma 3.3 Put $W_{1m} = \sqrt{m}(\overline{D} - \mu), W_{2m} = \sqrt{m}(\sum_{j=1}^{m} V_{Tj}^{1/\widehat{\alpha}_T}/N - \theta_T), W_{3m} = \sqrt{m}(\widehat{\alpha}_T - \alpha)$ for every positive integer m. Then $\mathsf{Cov}(W_{1m}, W_{2m}) = 0$ and $\mathsf{Cov}(W_{1m}, W_{3m}) = 0$.

Proof To prove this lemma, we need the standard decomposition of covariance: $\mathsf{Cov}(W_{1m}, W_{3m}) = \mathsf{E}\{\mathsf{Cov}[(W_{1m}, W_{3m})|N]\} + \mathsf{Cov}[\mathsf{E}(W_{1m}|N), \mathsf{E}(W_{3m}|N)]$. Given $N (0 < N < m)$, property 3 of LBVW shows that $D_{(j)}$ and I_{Tj} are independent $(j = 1, \ldots, N)$ and therefore $\overline{D}$ and $\widehat{\alpha}_T$ are independent, which implies that $\mathsf{Cov}[(W_{1m}, W_{3m})|N] = 0$. On the other hand, $\mathsf{E}(W_{1m}|N) = \mathsf{E}[\sqrt{m}(\sum_{j=1}^{N} D_{(j)}/N - \mu)|N] = 0$, since $\mathsf{E}[D_{(j)}] = \mu$ for every fixed j. Thus we arrive at the following expression: $\mathsf{Cov}(W_{1m}, W_{3m}) = 0$. Similarly, we can prove the expression $\mathsf{Cov}(W_{1m}, W_{2m}) = 0$. This completes the proof of this lemma.

Suppose that $s = l_1 + l_2 + m$, and $l_1/s \to v_1, l_2/s \to v_2$ and $m/s \to v_0$ as $s \to \infty$, where $0 < v_0, v_1, v_2 < 1$ and $v_0 + v_1 + v_2 = 1$. Denote

$$\sigma_i^2(\alpha) = \alpha^2\{I_{2pi} - p_i I_{1pi}^2 + p_i^3(1-p_i)[h'(p_i)]^2\}/[p_i h(p_i)]^2,$$

$$\sigma_i^2(\theta_i) = \theta_i^2/p_i + \theta_i^2(1 + I_{1pi} + \ln\theta_i)^2\sigma_i^2(\alpha)/\alpha^2$$

$$\sigma_T^2(\alpha) = \alpha^2\{I_{2pT} - p_T I_{1p_T}^2 + p_T^3(1-p_T)[h'(p_T)]^2\}/[p_T h(p_T)]^2,$$

$$\sigma_T^2(\theta_T) = \theta_T^2/p_T + \theta_T^2(1 + I_{1p} + \ln\theta_T)^2\sigma_T^2(\alpha)/\alpha^2,$$

$$\sigma_{34}^{**} = -\theta_T(1 + I_{1p_T} + \ln\theta_T)\sigma_T^2(\alpha)/\alpha,$$
$$a_{41} = -[\theta_1 \ln(\mu(1-\mu))]^{-1}, \quad a_{42} = -[\theta_2 \ln(\mu(1-\mu))]^{-1}, \quad a_{44} = 2[\theta_T \ln(\mu(1-\mu))]^{-1},$$
$$a_{45} = -(2\ln\theta_T - \ln\theta_1 - \ln\theta_2)(1-2\mu)[\ln(\mu(1-\mu))]^{-2}[\mu(1-\mu)]^{-1}.$$

The following theorem shows that $(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\alpha}_T, \widehat{\delta})'$ has asymptotic normal distribution.

Theorem 3.2 $(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\alpha}_T, \widehat{\delta})'$ is asymptotically normally distributed:

$$\sqrt{s}(\widehat{\theta}_1 - \theta_1, \widehat{\theta}_2 - \theta_2, \widehat{\alpha}_T - \alpha, \widehat{\delta}_T - \delta)' \xrightarrow{L} N(0, \Sigma), \quad (s \to \infty).$$

where

$$\Sigma = (\sigma_{ij})_{4\times 4}, \sigma_{11} = \sigma_i^2(\theta_1)/v_1,$$
$$\sigma_{22} = \sigma_2^2(\theta_2)/v_2, \qquad \sigma_{33} = \sigma_T^2(\alpha)/v_0,$$
$$\sigma_{44} = \sigma_1^2(\theta_1)a_{41}^2/v_1 + \sigma_2^2(\theta_2)a_{42}^2/v_2 + \sigma_T^2(\theta_T)a_{44}^2/v_0 + \mu(1-\mu)a_{45}^2/(p_T v_0),$$
$$\sigma_{12} = \sigma_{21} = \sigma_{13} = \sigma_{31} = \sigma_{23} = \sigma_{32} = 0, \qquad \sigma_{14} = \sigma_{41} = \sigma_1^2(\theta_1)a_{41}/v_1,$$
$$\sigma_{24} = \sigma_{42} = \sigma_2^2(\theta_2)a_{42}/v_2, \qquad \sigma_{34} = \sigma_{43} = \sigma_{34}^{**}a_{44}/v_0.$$

Proof According to the central limit theorem,

$$\sum_{j=1}^{m}(D_{(j)} - \mu)/\sqrt{m\mu(1-\mu)} \xrightarrow{L} N(0,1), \quad (m \to \infty).$$

Therefore, by Lemma 3.2,

$$\sum_{j=1}^{m_\bullet}(D_{(j)} - \mu)/\sqrt{N\mu(1-\mu)} \xrightarrow{L} N(0,1), \quad (m \to \infty),$$

which means that

$$\sqrt{m}(\overline{D} - \mu) = (\sqrt{m}/N)\sum_{j=1}^{N}(D_{(j)} - \mu) \xrightarrow{L} N(0, \mu(1-\mu)/p_T), \ (m \to \infty). \tag{3.6}$$

Noting the independence of X_{1i}, X_{2i} and T_i, by Lemma 3.3 and [4], we have

$$\sqrt{s}(\widehat{\theta}_1 - \theta_1, \widehat{\theta}_2 - \theta_2, \widehat{\alpha}_T - \alpha, \widehat{\theta}_T - \theta_T, \overline{D} - \mu)' \xrightarrow{L} N(\underset{\sim}{0}, \Sigma^*), \quad (s \to \infty),$$

where five-dimesional vector $\underset{\sim}{0} = (0,0,0,0,0)'$, $\Sigma^* = (\sigma_{ij}^*)_{5\times 5}$, $\sigma_{ii}^* = \sigma_i^2(\theta_i)/v_i, (i = 1,2)$, $\sigma_{33}^* = \sigma_T^2(\sigma)/v_0$, $\sigma_{44}^* = \sigma_T^2(\theta_T)/v_0$, $\sigma_{34}^* = \sigma_{43}^* = \sigma_{34}^{**}/\nu_0$, $\sigma_{55}^* = \mu(1-\mu)/(p_T v_0)$, and all of other σ_{ij}^*'s being equal to zero. Using Delta method and noting that

$$\frac{\partial(\theta_1, \theta_2, \alpha, \delta)}{\partial(\theta_1, \theta_2, \alpha, \theta_T, \mu)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ a_{41} & a_{42} & 0 & a_{44} & a_{45} \end{bmatrix} \triangleq M,$$

it follows that

$$\sqrt{s}(\widehat{\theta}_1 - \theta_1, \widehat{\theta}_2 - \theta_2, \widehat{\alpha}_T - \alpha, \widehat{\delta} - \delta)' \xrightarrow{L} N(0, \Sigma), \quad (s \to \infty),$$

where $\Sigma = M\Sigma^* M'$ stated as this theorem.

§4. Simulation Result

According to property 2 of LBVW, our simulation scheme is the following:

1. Generate four independent groups of quasi-random numbers of $U(0,1)$ from computer as follows: (1) $u_{11}, u_{12}, \ldots, u_{1l_1}$, (2) $u_{21}, u_{22}, \ldots, u_{2l_2}$, (3) $u_{31}, u_{32}, \ldots, u_{3m}$, (4) $u_{41}, u_{42}, \ldots, u_{4m}$.

2. Using the expression $X_i = (-\theta_i \ln U)^\alpha$ to generate two groups of quasi-random numbers of X_i with survival function $\exp(-X_i^{1/\alpha}/\theta_i)$ from $u_{ij} (i = 1,2; j = 1, \ldots, l_i)$ as follows: (1) $x_{11}, x_{12}, \ldots, x_{1l_1}$, (2) $x_{21}, x_{22}, \ldots, x_{2l_2}$.

3. By Von Neumann's method [3], generating quasi-random numbers of S with density function $[(1-\delta) + s\delta]e^{-s}$ from $u_{3j} (j = 1, 2, \ldots, m)$ as follows: $s_{31}, s_{32}, \ldots, s_{3m}$.

4. According to the expressions $X_1 = \theta_1^\alpha U^{\alpha\delta} S^\alpha$ and $X_2 = \theta_2^\alpha (1-U)^{\alpha\delta} S^\alpha$, generating quasi-random pairs of (X_1, X_2), with LBVW distribution from s_{3j} and $u_{4j} (j = 1, \ldots, m)$ as follows: $(\widetilde{x}_{11}, \widetilde{x}_{21}), (\widetilde{x}_{12}, \widetilde{x}_{22}), \ldots, (\widetilde{x}_{1m}, \widetilde{x}_{2m})$, futhermore, by $T = \min(X_1, X_2)$, generating quasi-numbers of $T: \ t_1, \ldots, t_m$.

5. According to expressions (3.2), (3.3), (3,4) and (3.5), from $x_{11}, \ldots, x_{1l_1}; x_{21}, \ldots, x_{2l_2}$ and $t_1, \ldots, t_m$, obtain the estimators $\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\alpha}_T$ and $\widehat{\delta}$, respectively.

6. Repeat the above process n times.

For $l_1 = l_2 = m = 100$, $X_{10} = 1.5, X_{20} = 1.35$, $T_0 = 1.4, \theta_1 = 1.2, \theta_2 = 1.4, \alpha = 1.15, \delta = 0.1, n = 100$, the results are $\widehat{\theta}_1 = \sum_{n=1}^{100} \widehat{\theta}_{1n}/100 = 1.2029$, $\widehat{\theta}_2 = \sum_{n=1}^{100} \widehat{\theta}_{2n}/100 = 1.3995$, $\widehat{\alpha}_T = \sum_{n=1}^{100} \widehat{\alpha}_{Tn}/100 = 1.1474$, $\widehat{\delta} = \sum_{n=1}^{100} \widehat{\delta}_n/100 = 0.1010$. It is easy to see that these results are satisfactory.

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二元相依Weibull分布的参数估计

叶 慈 南

(南京理工大学, 南京, 210014)

考虑生存函数为$\overline{F}(x_1, x_2) = \exp\{-[(x_1^{1/\alpha}/\theta_1)^{1/\delta} + (x_2^{1/\alpha}/\theta_2)^{1/\delta}]^\delta\}$, $x_i > 0, \theta_i > 0, i = 1, 2, \alpha > 0, 0 < \delta \le 1$的二元相依Weibull分布. 基于在I型截尾情形下两个元件与串联系统的寿命试验数据, 本文给出了未知参数$\theta_1, \theta_2, \alpha$和$\delta$的估计, 并讨论了这些估计的渐近性质. 本文还给出了随机模拟的结果.

关键词: 二元 Weibull分布, I型截尾, 估计, 渐近性质.

学科分类号: 212.1, 213.2.