Edgeworth Expansions and Power Loss of Tests*

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Abstract

This paper deals with the Edgeworth expansions and power loss of tests for the one-sample problem. The first-order asymptotic theory, second order efficiency and power loss are given. The tests based on $L-$, $R-$, $U-$statistics and combined $L-$statistics are studied.

Keywords: Edgeworth expansions, Power loss of tests, $L-$test, $R-$test, $U-$test, combined $L-$test.

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§1. First-order asymptotic theory

Let $\{P_\theta : \theta \in \Theta \cup R_1\}$ be a family of probability measures on a measurable space$(\aleph, \mu)$ having densities $p(x, \theta)$ with respect to a $\sigma-$finite measure v. Assuming without loss of generality that $\Theta \cap R_1$ contains an interval $[0, \sigma]$, $\sigma > 0$. Suppose we have independent and identically distributed $\aleph-$valued observations $(X_1, \cdots, X_n)$ distributed according to $\{P_\theta, \theta \in \Theta \cap R_1\}$. Our problem is to test the hypothesis

$$H_0: \ \theta = 0 \qquad \text{against} \qquad H_1: \ \theta > 0.$$

We denote $P_{n,0}$ and $P_{n,1}$ the joint distributions of $(X_1, \cdots, X_n)$ under H_0 and H_1 respectively. The respective expectations will be denoted by $E_{n,0}$ and $E_{n,1}$.

It is well-known that for a fixed test size $\alpha \in (0,1)$ and a fixed alternative θ the power of every reasonable test Ψ_n will tend to 1, i.e.

$$\lim_{n\to\infty} E_{n,1}\Psi_n = 1.$$

For every $\theta > 0$. This result is not sufficiently informative for the comparison of tests performance because such an evaluation would require knowledge of the rate of convergence of their powers to 1. However, this is a complicated matter and we will not consider this problem here.

Usually, the following Pitman's approach is used: the test size $\alpha \in (0,1)$ remains fixed but instead of a fixed alternative $\theta > 0$ we consider so-called local or contiguous alternatives $\{\theta_n\}$ for which $\theta_n \to 0$ as $n \to 0$ at such rate that the power tends to a limit which lies strictly between α and 1. Under natural regularity conditions, it can be easily shown that the class of these sequences is the class of sequences θ_n for which

$$\lim_{n\to\infty} \sqrt{n}\theta_n = t$$

for some constant t with $0 < t < \infty$.

So for any $0 < t \le C$, $C > 0$ we will consider testing

$$H_0: \ \theta = 0 \qquad \text{against} \qquad H_{n,1}: \ \theta = \tau t. \tag{1.1}$$

Throughout the paper we use the abbreviation

$$\tau = n^{-\frac{1}{2}}.$$

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Obviously, we have densities

$$p_{n,0}(x) = \prod_{i=1}^{n} p_0(x_i) \qquad \text{and} \qquad p_{n,t}(x) = \prod_{i=1}^{n} p_{\tau t}(x_i). \tag{1.2}$$

Denoted by

$$\beta_n(t) = E_{n,t}\Psi_n, \tag{1.3}$$

the power of a test Ψ_n for H_0 against the local alternative $H_{n,t}$. Considered as a function of t, this sequence converges for every reasonable test to a monotone continuous function assuming its values in (0,1).

Consider the log-likelihood ratio

$$\Lambda_n(t) = \log \frac{dP_{n,t}}{dP_{n,0}} = \log \frac{p_{n,t}}{p_{n,0}}.$$

Then by (1.2)

$$\Lambda_n(t) = \sum_{i=1}^{n} [l_{\tau t}(X_i) - l_0(X_i)], \tag{1.4}$$

where $l_\theta(x) = \log p(x, \theta)$. By the Taylor series expansion,

$$l_{\tau t}(X_i) - l_0(X_i) = \tau t l^{(1)}(X_i) + \frac{1}{2}(\tau t)^2 l^{(2)}(X_i) + \cdots. \tag{1.5}$$

Here and in what follows the kth derivative of a function with respect to θ will be denoted by the superscript k. For a function of θ at $\theta = 0$ the argument θ will be often suppressed, e.g.,

$$l^{(2)}(x) = \frac{\partial^2}{\partial\theta^2} l_\theta(x)|_{\theta=0}.$$

Denote

$$L_n^{(1)} = \tau \sum_{i=1}^{n} l^{(1)}(X_i), \qquad L_n^{(2)} = \tau \sum_{i=1}^{n} [l^{(2)}(X_i) - E_0 l^{(2)}(X_1)], \qquad \cdots. \tag{1.6}$$

The sums are centered by the corresponding E_0–expectations; the first sum contains no centering because

$$E_0 l^{(1)}(X_1) = 0.$$

Further, denote by I the Fisher information

$$I = E_0(l^{(1)}(X_1))^2.$$

It is well known that

$$E_0 l^{(2)}(X_1) = -I.$$

With this notation, putting (1.5) into (1.4) yields

$$\Lambda_n(t) = tL_n^{(1)} - \frac{1}{2}t^2 I + \frac{1}{2}\tau t^2 L_n^{(2)} + \cdots. \tag{1.7}$$

The first two terms in the right-hand side of (1.7) express the local asymptotic normality (LAN) of the family of distributions.

The omitted terms in (1.7) include the nonrandom term

$$\frac{1}{6}\tau t^3 E_0 l^{(3)}(X_1)$$

and the terms of higher order than τ.

The Neyman-Pearson test, i.e., the most powerful size-α test for H_0 against $H_{n,t}$ rejects H_0 when

$$\Lambda_n(t) > c_{n,t}$$

with $c_{n,t}$ defined by (assuming continuity of the corresponding distribution)

$$P_{n,t}(\Lambda_n(t) > c_{n,t}) = \alpha.$$

Using (1.7) and the Central Limit Theorem we obtain

$$L(\Lambda_n(t)|H_0) \to N\Big(-\frac{1}{2}t^2 I, t^2 I\Big). \tag{1.8}$$

Hence

$$c_{n,t} \to c_t = t\sqrt{I}u_\alpha - \frac{1}{2}t^2 I, \tag{1.9}$$
$$\Phi(u_\alpha) = 1 - \alpha.$$

The power of this most powerful test is

$$\beta_n^*(t) = P_{n,t}\{\Lambda_n(t) > c_{n,t}\}. \tag{1.10}$$

It is well known from the LAN theory that

$$L(\Lambda_n(t)|H_{n,t}) \to N\Big(\frac{1}{2}t^2 I, t^2 I\Big) \tag{1.11}$$

Thus (1.9)-(1.11) yield

$$\beta_n^*(t) \to \beta(t) = \Phi(t\sqrt{I}u_\alpha). \tag{1.12}$$

Note that $\beta_n^*(t)$, known as the envelope power function (i.e. the supremum over all size-α tests of the power at τt), is not the power of a single test. The envelope power function renders a standard for evaluating the power function of any particular test. For each $t > 0$ it is the power of the most powerful test against $H_{n,t}$ based on $\Lambda_n(t)$. Thus it provides an upper bound for the power of any test for H_0 against

$$H_1: \quad t > 0.$$

It is well known that there are many (first order) asymptotically efficient tests, i.e., tests whose power function $\beta_n(t)$ converges to the same limit as $\beta_n^*(t)$. So are, for example, tests based on $L_n^{(1)}$, on $\Lambda_n(t_0)$ with an arbitrary $t_0 > 0$, on the maximum likelihood estimator $\widehat{\theta}_n$, on a certain linear combination of order statistics; on a certain $U-$statistics; for θ location parameter there are asymptotically efficient rank tests. Hence there is an abundance of tests fulfilling

$$\beta_n(t) \to \beta(t), \qquad t > 0 \tag{1.13}$$

i.e., of tests which are most powerful for H_0 against $H_{n,t}$ up to an error $o(1)$ for every $t > 0$. They can be compared with each other by higher order terms of their power.

§2. Second order efficiency

Typically, an asymptotically efficient test statistic (suitably normalized) has the score function $L_n^{(1)}$ as its leading term, so that it has the form

$$T_n = L_n^{(1)} + \tau Q_n + \cdots, \tag{2.1}$$

with Q_n bounded in probability. For example $\Lambda_n(t_0)$ is equivalent to

$$T_n = L_n^{(1)} + \frac{1}{2}\tau t_0 L_n^{(2)}.$$

For rank statistics ($R-$statistics) and linear combinations of order statistics ($L-$statistics) Q_n can be written as a quadratic functional of the empirical process (centered and normalized empirical distribution function).

In 70-ies for the power functions $\beta_n(t)$ of various asymptotically efficient tests an expansion in τ to terms of order τ^2 was obtained. The purpose was to study the deficiencies of the corresponding tests, which we will briefly discuss later on. Writing down such expansions in an explicit form required very involved calculations. For "parametric" test statistics first a "stochastic expansion" of the form (2.1), but containing also the τ^2term was derived. It was used to obtain Edgeworth expansions for the distributions of T_n under H_0 and $H_{n,t}$. For rank statistics a different technique based on a certain conditioning was used by Albers, Bickel and Van Zwet (1976) and Bickel and Van Zwet (1978). The Edgeworth expansion under H_0 was used to obtain an expansion in τ for the critical value a_n defined by

$$P_{n,0}\{T_n > a_n\} = \alpha.$$

Then the Edgeworth expansion for

$$\beta_n(t) = P_{n,t}\{T_n > a_n\}$$

was derived by the substitution of the expansion for a_n into the Edgeworth expansion under $H_{n,t}$. The Edgeworth expansions for $\beta_n^*(t)$ with error terms $o(\tau)$ and $o(\tau^2)$ have been obtained independently by Chibisov (1973) and Pfanzagl (1973).

Though the Edgeworth expansions for the distributions of various asymptotically efficient test statistics and of $\Lambda_n(t)$ differ by terms of order τ, it was observed that their powers $\beta_n(t)$ differ from each other and from $\beta_n^*(t)$ by $o(\tau)$ (and typically by $O(\tau^2)$), so that "first-order efficiency implies second-order efficiency", the later meaning that the power agrees with $\beta_n^*(t)$ up to terms of order τ. The approach of comparing the expansions for $\beta_n^*(t)$ and $\beta_n(t)$ described above gave no insight into the nature of this phenomenon. A simple and intuitively clear proof of this general property was given by Bickel, Chibisov and Van Zwet (1981).

The idea was, first, to treat directly the difference

$$\beta_n^*(t) - \beta_n(t)$$

and secondly, to adjust the test statistic to the log-likelihood ratio (rather than to adjust test statistics and the log-likelihood ratio to $L_n^{(1)}$), so that the difference

$$\Delta_{n,t} \equiv S_{n,t} - \Lambda_n(t) \tag{2.2}$$

is small. For example, (2.1) as a test statistic is equivalent to

$$S_{n,t} = tT_n - \frac{1}{2}t^2 I.$$

Note that this transformation does not influence the test function, and hence, the power and then

$$\Delta_{n,t} = -\tau\Big(\frac{1}{2}\tau t^2 L_n^{(2)} + tQ_n\Big) + \cdots . \tag{2.3}$$

§3. Power loss

The difference

$$\beta_n^*(t) - \beta_n(t)$$

is closely related to the deficiency of the corresponding test, which is the number of additional observations needed for this test to achieve the same power as the most powerful test. This notion was introduced by Hodges and Lehmann (1970). Deficiencies of various tests were extensively studied in 70-ies by Albers, Bickel and Van Zwet (1976) for rank tests, by Chibisov (1983), Pfanzagl (1980) for "parametric" tests.

When the limit

$$r(t) = \lim_{n\to\infty} n(\beta_n^*(t) - \beta_n(t)) \tag{3.1}$$

exists, the asymptotic deficiency is finite and can be directly expressed through this limit. We will not state this relationship here. Rather, we will directly deal with the Quantity (3.1), which we will refer to as the power loss. This quantity was actually the object of the studies on deficiency. As we pointed out, its derivation was very involved.

An elaboration of the argument given in previous Section leads to the following formula for the power loss. Suppose that

$$\Delta_{n,t} = S_{n,t} - \Lambda_{n,t}$$

as in (2.3) is of order τ in a somewhat stronger sense then it was meat before. Namely, assume that

$$(\sqrt{n}\Delta_{n,t}, \Lambda_n(t))$$

converges in distribution under $P_{n,0}$ to a certain bivariate random variable. Denoting

$$\Pi_{n,t} = \sqrt{n}\Delta_{n,t}.$$

We write it as

$$(\Pi_{n,t}, \Lambda_n(t)) \xrightarrow{P_{n,0}} (\Pi, \Lambda). \tag{3.2}$$

In all regular cases Λ is a normal random variable. Denote its distribution function and density by $\Phi_1(x)$ and $p_{0,t}(x)$. Let c_t be the limiting critical value defined by

$$\Phi_1(c_t) = \alpha.$$

Then

$$r(t) = \lim_{n\to\infty} n(\beta_n^*(t) - \beta_n(t)) = \frac{1}{2}e^{c_t}p_{0,t}(c_t)\mathsf{Var}\,[\Pi|\Lambda = c_t]. \tag{3.3}$$

Note that

$$e^{c_t}p_{0,t}(c_t) = p_{1,t}(c_t).$$

Where $p_{1,t}(x)$ is the limiting density of $\Lambda_{n,t}(t)$ under $P_{n,t}$ and

$$c_t = t\sqrt{I}u_\alpha - \frac{1}{2}t^2I, \qquad p_{0,t}(x) = \frac{1}{t\sqrt{I}}\phi\Big(\frac{2x+t^2I}{2t\sqrt{I}}\Big), \tag{3.4}$$

$$p_{1,t}(x) = \frac{1}{t\sqrt{I}}\phi\Big(\frac{2x-t^2I}{2t\sqrt{I}}\Big), \qquad \Phi_1(x) = \Phi\Big(\frac{2x+t^2I}{2t\sqrt{I}}\Big). \tag{3.5}$$

These relations imply

$$r(t) = \frac{1}{2t\sqrt{I}}\phi(u_\alpha - t\sqrt{I})\mathsf{Var}\,[\Pi|\Lambda = c_t]. \tag{3.6}$$

This formula was proved by Chibisov (1985) for statistics admitting a stochastic expansion in terms of i.i.d. sums (which is typical for "parametric" problems). Bening (1995), (1997) and Zhao and Bening (1997) proved the formula (3.6) for rank statistics, Linear combinations of order statistics and U−statistics.

§4. Tests based on L−, R−, and U−statistics

Let $X_1, \cdots, X_n$ be independent identically distributed random observations with distribution function $F(x,\theta)$ and density $p(x,\theta)$, θ ranging over an open set $\Theta \subset R^1$ containing 0. Let the hypothesis

$$H_0: \quad \theta = 0 \tag{4.1}$$

be tested against a sequence of local alternatives

$$H_{n1}: \quad \theta = \tau t, \qquad 0 < t \le C,\ C > 0. \tag{4.2}$$

Where $\tau = n^{-\frac{1}{2}}$. We will write $F(x)$ for the hypothesized distribution function $F(x,0)$.

§4.1 L-test

Consider an asymptotically efficient L-test based on

$$T_{n2} = \tau \sum_{i=1}^{n} b_{in} X_{i:n}, \tag{4.3}$$

where $(X_{1:1}, \cdots, X_{n:n})$ are the order statistics of $(X_1, \cdots, X_n)$

$$b_{in} = n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_1(s)ds, \qquad J_k(s) = (l^{(k)}(x))'|_{x=F^{-1}(s)}, \qquad k \in N. \tag{4.4}$$
$$F^{-1}(s) = \inf\{x : F(x) \geq s\}$$

and a prime denoting differentiation with respect to x.

Given $\alpha \in (0,1)$, we need to find

$$r_2(t) = \lim_{n\to\infty} n(\beta_n^*(t) - \beta_{n2}(t)),$$

where $\beta_{n2}(t)$ and $\beta_n^*(t)$ are the powers of the size $\alpha \in (0,1)$ test based on T_{n2} and on

$$\Lambda_n(t) = \sum_{i=1}^{n} \log \frac{p(X_i, \tau t)}{p(X_i, 0)} \tag{4.5}$$

respectively. We have

$$\begin{aligned} r_2(t) &= \lim_{n\to\infty} n(\beta_n^*(t) - \beta_{n2}(t)) = \frac{1}{2} e^b \mathrm{Var}\,[\Pi_2|\Lambda = b]p(b) \\ &= \frac{t}{8\sqrt{I}} \phi(u_\alpha - t\sqrt{I}) \mathrm{Var}\,[K_2 - tL_2|L_1 = u_\alpha\sqrt{I}]. \end{aligned} \tag{4.6}$$

The latter conditional variance has form:

$$\mathrm{Var}\,[K_2 - tL_2|L_1 = u_\alpha\sqrt{I}] = v_0 + v_1 t + v_2 t^2,$$

where

$$\begin{aligned}
v_0 &= 2(I_2^2 - 2I_1I_2 + I_{112}) + 4(1 - u_\alpha^2)(I_1^2 - I_{111}), \\
v_1 &= 4u_\alpha(I_0I_1 - I_{011}), \\
v_2 &= I_{001} - I_0^2, \\
I_i &= \int_0^1 J_{2-i}^{(i)}(s)\mu^{i+1}(s)dF^{-1}(s), \qquad i = 0,1, \\
I_2 &= \int_0^1 J_1'(s)s(1-s)dF^{-1}(s), \\
I_{ijl} &= \int_0^1\int_0^1 J_{2-i}^{(i)}J_{2-j}^{(j)}(t)\mu^{i-l+1}(s)\mu^{j-l+1}(t)K^l(s,t)dF^{-1}(s)dF^{-1}(t), \qquad i,j = 0,1;\ l = 1,2; \\
K(s,t) &= \min(s,t) - st, \\
\mu(s) &= \frac{1}{\sqrt{I}} \int_0^s l^{(1)}(F^{-1}(t))dt.
\end{aligned}$$

§4.2 R-test

Assume now that θ is a location parameter, $p(x,\theta) = p(x-\theta)$, and density $p(x)$ is symmetric, $p(-x) = p(x)$. Consider an asymptotically efficient rank test (R-test) for testing H_0 against H_{n1} (cf. (4.1) and (4.2)) based on

$$T_{n3} = \tau \sum_{i=1}^{n} a(R_i^+)\mathrm{sgn}(X_i), \tag{4.7}$$

where $(R_1^+,\cdots,R_n^+)$ is the vector of $(|X_1|,\cdots,|X_n|)$ and

$$a(i)=I\Big(\frac{i}{n+1}\Big)\quad with\quad I(s)=l^{(1)}\Big(F^{-1}\Big(\frac{1+s}{2}\Big)\Big). \tag{4.8}$$

Denote the powers of the size $\alpha\in(0,1)$ tests based on T_{n3} and $\Lambda_n(t)$ by $\beta_{n3}(t)$ and $\beta_n^*(t)$ respectively. So we have

$$\begin{aligned} r_3(t)&=\lim_{n\to\infty} n(\beta_n^*(t)-\beta_{n3}(t))=\frac{1}{2}e^b\mathrm{Var}\,[\Pi_3|\Lambda=b]p(b)\\ &=\frac{t}{8\sqrt{I}}\phi(u_\alpha-t\sqrt{I})\mathrm{Var}\,[K_3-tL_2|L_1=u_\alpha\sqrt{I}]. \end{aligned} \tag{4.9}$$

The latter conditional variance can be calculated

$$\mathrm{Var}\,[K_3-tL_2|L_1=u_\alpha\sqrt{I}]=w_0+w_1t+w_2t^2, \tag{4.10}$$

where

$$\begin{aligned} w_0&=4\int_0^1(I'(s))^2s(1-s)ds+\frac{4}{I}(u_\alpha^2-1)\int_0^1\int_0^1 I'(s)I'(t)I(s)I(t)K(s,t)dsdt,\\ w_1&=-\frac{2u_\alpha}{\sqrt{I}}\int_0^1\int_0^1 I'(s)I(s)g(t)K(s,t)dsdt,\\ w_2&=\frac{1}{4}\int_0^1\int_0^1 g(s)g(t)K(s,t)dsdt,\\ g(s)&=\frac{l^{(3)}(F^{-1}((1+s)/2))}{p(F^{-1}((1+s)/2))}. \end{aligned}$$

§4.3 $U-$test

Finally, let us now consider an asymptotically efficient $U-$test for testing H_0 versus H_{n1} (cf. (4.1) and (4.2)) based on

$$T_{n4}=\sum_{1\le i<j\le n} h(X_i,X_j) \tag{4.11}$$

where $\Psi(x,y)$ is measurable and symmetric in its two arguments i.e. $\Psi(x,y)=\Psi(y,x)$, real function,

$$E_0[\Psi(X_1,X_2)|X_1]=0\qquad a.s. \tag{4.12}$$

and

$$h(x,y)=l^{(1)}(x)+l^{(1)}(y)+\Psi(x,y). \tag{4.13}$$

Denote the powers of the size $\alpha\in(0,1)$ tests based on T_{n4} and $\Lambda_n(t)$ by $\beta_{n4}(t)$ and $\beta_n^*(t)$ respectively. Then we have

$$\begin{aligned} r_4(t)&=\lim_{n\to\infty} n(\beta_n^*(t)-\beta_{n4}(t))=\frac{1}{2}e^b\mathrm{Var}\,[\Pi_4|\Lambda=b]p(b)\\ &=\frac{t}{8\sqrt{I}}\phi(u_\alpha-t\sqrt{I})\mathrm{Var}\,[K_4-tL_2|L_1=u_\alpha\sqrt{I}]. \end{aligned} \tag{4.14}$$

The latter conditional variance can be calculated

$$\mathrm{Var}\,[K_4-tL_2|L_1=u_\alpha\sqrt{I}]=4(1-u_\alpha^2)I_{(1)}^2+t4u_\alpha I_{(1)}(I_{(2)}-\sqrt{I})+t^2(I_{(3)}-I_{(2)}^2), \tag{4.15}$$

where

$$\begin{aligned} I_{(1)}&=I^{-1}E_0\Psi(X_1,X_2)l^{(1)}(X_1)l^{(1)}(X_2),\\ I_{(2)}&=I^{-\frac{1}{2}}E_0l^{(2)}(X_1)l^{(1)}(X_1),\\ I_{(3)}&=E_0(l^{(2)}(X_1))^2-I^2. \end{aligned}$$

§5. Combined L–tests

Consider a testing problem in which the total sample has been divided $m \geq 2$ sub-samples. Suppose that for each of these sub-samples a separate asymptotically efficient L–statistics can be obtained and the best combination of these statistics is then compared to the ordinary undivided asymptotically efficient L–statistic. One easily sees that, under natural Conditions, splitting causes no first order efficiency loss. Hence it becomes interesting to derive second order results and it would be nice to obtain more precise results on effect of splitting. This subject and related ones have received considerable attention in the literature. For earlier works we refer to Van Zwet and Oosterhoff (1967), Albers and Akritas (1987). One- and two-sample combined rank tests have been considered in Albers (1988), (1997). Note again that the used method was essentially based on asymptotic expansions. Although the basic ideas underlying these papers are simple, the proofs are highly technical matter. The method we used for proving the main theorem carries over Le Cam's approach to higher order asymptotic and based on the likelihood ratio properties.

Let $X_n = (X_1, \cdots, X_n)$ be independent identically distributed random observations with distribution function $F(x,\theta)$ and density $p(x,\theta)$, θ ranging over an open set $\Theta \subset R^1$ containing 0. Let the hypothesis

$$H_0: \quad \theta = 0$$

be tested against a sequence of local alternatives

$$H_{n,1}: \quad \theta = \tau t, \qquad 0 < t \leq C, \ C > 0$$

where $\tau = n^{-\frac{1}{2}}$. We will write $F(x)$ for the hypothesized distribution function $F(x,\theta)$.

Consider an asymptotically efficient L–test based on

$$T_{n2} = \tau \sum_{i=1}^{n} b_{in} X_{i:n} \tag{5.1}$$

where $(X_{1:n}, \cdots, X_{n:n})$ are the order statistics of $(X_1, \cdots, X_n)$

$$\begin{aligned} &l(x,\theta) = \log p(x,\theta), \qquad l^{(i)}(x) = \frac{\partial^i}{\partial\theta^i} \log p(x,\theta)|_{\theta=0}, \qquad i = 1, 2, \cdots, \\ &b_{in} = n\int_{(i-1)/n}^{i/n} J_1(s)ds, \qquad J_k(s) = (l^{(k)}(x))'|_{x=F^{-1}(s)}, \qquad k \in N, \\ &F^{-1}(s) = \inf\{x : F(x) \geq s\}, \end{aligned} \tag{5.2}$$

and a prime denoting differentiation with respect to x.

Given $\alpha \in (0,1)$, denote $\beta_{n2}(t)$ and $\beta_n^*(t)$ the powers of the size α tests based on T_{n2} and on

$$\Lambda_n(t) = \sum_{i=1}^{n} \log \frac{p(X_i, \tau t)}{p(X_i, 0)}$$

respectively. Now suppose that our sample has been split into $m \geq 2$ sub-samples of m_l, $l = 1, \cdots, m$,

$$(X_1, \cdots, X_{n_1}), \cdots, (X_{n-n_m+1}, \cdots, X_n), \qquad n_1 + n_2 + \cdots + n_m = n,$$

and

$$\frac{n_l}{n} \to \gamma_l > 0, \quad l = 1, \cdots, m; \qquad \sum_{l=1}^{m} \gamma_l = 1, \quad \text{as } n \to \infty.$$

For each of these samples we obtain the L–statistics

$$T_{n2}^{(l)}, \qquad l = 1, \cdots, m$$

as in (5.1) and (5.2).

Consider the combined L-statistic

$$\overline{T}_{n2} = \sum_{l=1}^{m} T_{n2}^{(l)}$$

and the level $\alpha \in (0,1)$ test based on $\overline{T}_{n2}$. Let $\overline{\beta}_{n2}(t)$ be the power of this combined L-test.

Using above arguments we can find the following limits

$$\overline{r}_2^{(m)}(t) = \lim_{n\to\infty} n(\beta_{n2}(t) - \overline{\beta}_{n2}(t)), \tag{5.3}$$

$$r_2^m(t) = \lim_{n\to\infty} n(\beta_{n2}^*(t) - \overline{\beta}_{n2}(t)). \tag{5.4}$$

References

[1] Chibisov, D.M., Asymptotic expansions and deficiencies of tests, In: *Proceed. of the Intern. Congress of Math. Warszawa*, 1063–1079, 1983.

[2] Chibisov, D.M., On calculating the deficiency of some asymptotically efficient tests, *Theory Probab. Appl.*, **30**(1985), 269–288. (in Russian).

[3] Chibisov, D.M., An asymptotic expansion for the distribution of a statistic admitting a stochastic expansion Ⅰ, Ⅱ, *Theory Probab. Appl.*, **25**(1980), 732–744, **26**(1981), 1–12.

[4] Pfanzagl, J and W. Wefelmeyer, An asymptotically complete class of tests, *Z. Wahrsch. verw. Gebiete*, **45**(1978), 49–72.

[5] Pfanzagl, J., Asymptotic expansions in the parametric statistical theory, In: *Developments In Stat.*, **3**(1980), 1–97. (P.R. Krishnaiah ed.), New York, Academic Press.

[6] P.J. Bickel, D.M. Chibisov and W.R. Van. Zwet, On efficient of first and second order, *Internat. Statist. Reiew*, **49**(1981), 169–175.

[7] V.E. Bening, A formula for deficiency: one sample $L-$ and $R-$tests, Ⅰ, Ⅱ, *Math. Methods Statist.*, **44(2)**(1995), 167–188, **44(3)**(1995), 274–293.

[8] V.E. Bening and D.M. Chibisov, Higher order asymptotic optimality in testing problems with nuisance parameters, *Mathematical Methods of Statistics*, **8(2)**(1999), 142–165.

[9] Zhao Xuanmin and V.E. Bening, A method of asymptotic expansions based on the Logarithem of the likelihood ration, *Mathematical Statistics and Applied Probability*, **12(1)**(1997), 58–67.

[10] V.E. Bening, On asymptotic efficiency of some adaptive tests in the one-sample problem, *Vestink Mosk. Univ. Komput. Math. i Kibern.*, **2**(1996), 45–52.

[11] V.E. Bening, On the power of some adaptive tests in the one-sample problem, *Vestink Mosk. Univ. Komput. Math. i Kibern.*, **1**(1997), 20–24.

[12] V.E. Bening, Convergence rate estimates under alternatives for $L-$ and $R-$statistics, *Vestink Mosk. Univ. Komput. Math. i Kibern.*, **4**(1995), 23–31.

[13] V.E. Bening, On asymptotic efficiency and deficiency of some adaptive tests in the one-sample problem, *Journal of Mathematical Science*, **81(5)**(1996), 2894–2899.

[14] W. Albers, Asymptotic expansions for the power of adaptive rank tests in the one-sample problem, *Lecture Notes in Math.*, **821**(1979), 108–158.

检验的渐近展开和功效损失

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本文研究统计假设检验问题中的渐近展开和功效损失, 给出一阶渐近展开, 二阶效率和功效损失, 并且研究了建立在 $L-$, $R-$, $U-$统计量及组合 $L-$统计量上的检验问题.

关键词: 渐近展开, 功效损失, $L-$检验, $R-$检验, $U-$检验, 组合 $L-$检验.

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