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ASYMPTOTIC PROPERTIES OF THE EVLP ESTIMATION FOR SUPERIMPOSED EXPONENTIAL SIGNALS IN NOISE

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Abstract

This paper studies the model of superimposed exponential sigdals in noise:

$$Y_j(t)=\sum_{k=1}^{p}a_{kj}\lambda_k^t+e_j(t),\quad t=0,\ 1,\ \cdots,\ n-1,\ j=1,\ N$$

where $\lambda_1, \cdots, \lambda_q$ are unknown complex parameters with module 1, $\lambda_{q+1}, \cdots, \lambda_p$ are unknown complex parameters with module 1, $\lambda_{q+1}, \cdots, \lambda_p$ are unknown complex parameters with module less than 1, $\lambda_1, \cdots, \lambda_p$ are assumed distinct, p assumed known and q unknown. a_{kj}, $k=1, \cdots, p$, $j=1, \cdots, N$ are unknown complex parameters. $e_j(t)$, $t=0,1, \cdots, n-1$, $j=1, \cdots, N$, are i.i.d. complex random noise variables such that

$$Ee_1(0),\ E|e_1(0)|^2=\sigma^2,\ 0<\sigma^2<\infty,\ E|e_1(0)|^4<\infty$$

and σ^2 is unknown. This paper gives:

1. A strong consistent estimate of q;
2. Strong consistent estimates of $\lambda_1, \cdots, \lambda_q$, σ^2 and $|a_{kj}|$, $k\leq q$;
3. Limiting distributions for some of these estimates;
4. A proof of non-existence of consistent estimates for λ_k and a_{kj}, $k>q$.
5. Adiscussion of the case that $N\to\infty$

§ 1. Introduction

In signal processing, the following model is extensively used:

$$Y_j(t)=\sum_{k=1}^{p}a_{kj}\lambda_k^t+e_j(t),\tag{1.1}$$

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$$t=0, 1, \cdots, n-1, j=1, 2, \cdots, N$$

where $\lambda_1, \cdots, \lambda_p$ are unknown complex parameters with module not greater than one, and are assumed distinct from each other, a_{kj}, $k=1, 2, \cdots, p$, $j=1, 2, \cdots, N$, are unknown complex parameters, $e_j(t)$, $t=0, 1, \cdots, n-1$, $j=1, \cdots, N$, are i.i.d. complex random noise variables such that

$$Ee_1(0)=0, \ Ee_1(0)\overline{e_1(0)}=\sigma^2, \ 0<\sigma^2<\infty, \tag{1.2}$$

$$E|e_1(0)|^4<\infty, \tag{1.3}$$

where σ^2 is unknown. Throughout this paper, $i=\sqrt{-1}$, $\bar{A}$, A' and A^* denote the complex conjugate, the tra nspose and the complex conjugate of the transpose of a matrix A respectively.

The model (1.1) can be viewed either as an ordinary time series (single-experiment for $N=1$, multiple-experiment for $N>1$) with uniform sampling, or as a model for a linear uniform narrow-band array with multiple plane waves present, and each measurement vector (the "snapshot") $\boldsymbol{Y}_j=(Y_j(0), \cdots, Y_j(n-1))'$ represents the output from n individual sensors.

The primary interest in this model is to estimate the parameter vector $\boldsymbol{\lambda}=(\lambda_1, \cdots, \lambda_p)'$ based on the data $\{\boldsymbol{Y}_j, j=1, \cdots, N\}$. In some investigations, for example [1], [2] and [3], it is assumed that the vector $\boldsymbol{a}_j=(a_{1j}, a_{2j}, \cdots, a_{pj})'$, $j=1, \cdots, N$, are i.i.d. random vectors with a common mean vector zero and covariance matrix $R=E\boldsymbol{a}_j\boldsymbol{a}_j^*$. In other studies, for example [4], it is assumed that a_{kj}, the complex amplitude of the k-th signal in the j-th snapshot, is simply an unknown constant, and it is desired to estimate these constants based on the data. We shall adopt the latter assumption in this paper.

Various methods for estimating the parameters $\boldsymbol{\lambda}$ and $\boldsymbol{a}_j'$'s are proposed in the literature. If $\boldsymbol{\lambda}$ were known, the least squares (LS) method would give the following estimate of $\boldsymbol{a}_j$:

$$\hat{\boldsymbol{a}}_j=(A^*(\boldsymbol{\lambda})A(\boldsymbol{\lambda}))^{-1}A^*(\boldsymbol{\lambda})\boldsymbol{Y}_j, \quad j=1, 2, \cdots, N, \tag{1.4}$$

where

$$A(\boldsymbol{\lambda})=\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_p \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_p^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_p^{n-1} \end{bmatrix}.$$

From this consideration, some authors suggested that, after obtaining some estimate $\hat{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$, one substitutes $\hat{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda}$ in (1.4) to yield an estimate of $\boldsymbol{a}_j$. For estimation of $\boldsymbol{\lambda}$, Bresler and Macovski[4] derived the maximum likelihood (ML) criterion under the normality assumption on $\{e_j(t)\}$, which is just the LS criterion. Other methods are proposed such as that of Prony, Pisarenko and modifications thereof (e.g. [5], [6], [7], [8]). Not much is known about the statistical properties of these estimates. For some results in this respect, the reader is referred to Bai, Krishnaiah and Zhao[9]. They considered the case where $N=1$ and λ_k's are all of module one, suggested an equivariation linear prediction (EVLP) method to detect the number p of signals, and to estimate $\boldsymbol{\lambda}$ and σ^2. They established the strong consistency of the detection criterion and estimators, and obtained the limiting distributions of related estimators. Analysis and comparison for some estimates of $\boldsymbol{\lambda}$ are also made.

In this paper, we apply the EVLP method to the general model (1.1). This method is a modification of a classical method dating back to Prony[10]. As pointed out by Rao[11], the Prony method, which features in minimizing certain quadratic form of the observations, ignores the correlation of related linear forms therein, and the consistency of the related estimates is in doubt.

Roughly, the EVLP method can be described as follows: Consider the set

$$B_p=\left\{\boldsymbol{b}=(b_0, \cdots, b_p)': \sum_{k=0}^{p}|b_k|^2=1,\ b_p\geqslant 0,\ b_0, \cdots, b_{p-1} \text{ complex}\right\}. \tag{1.5}$$

Define a function $Q_p(\boldsymbol{b})$ as follows:

$$Q_p(\boldsymbol{b})=\frac{1}{N(n-p)}\sum_{j=1}^{N}\sum_{t=0}^{n-1-p}\left|\sum_{k=0}^{p} b_k Y_j(t+k)\right|^2, \quad \boldsymbol{b}\in B_p. \tag{1.6}$$

We can find a vector $\hat{\boldsymbol{b}}\in B_p$ such that

$$Q_p\triangleq Q_p(\hat{\boldsymbol{b}})=\min\{Q_p(\boldsymbol{b}): \boldsymbol{b}\in B_p\} \tag{1.7}$$

Let $\hat{\lambda}_1, \cdots, \hat{\lambda}_p$ be the roots of the equation

$$\sum_{k=0}^{p}\hat{b}_k z^k=0. \tag{1.8}$$

Then $\hat{\lambda}_k$ is taken as an estimate of λ_k, $k=1, \cdots, p$.

For considering the asymptotic properties of the estimates, we shall distinguish the following two cases:

Case (i): N is fixed and n tends to infinity;

Case (ii): n is fixed and N tends to infinity.

First, consider case (i). Put

$$A=\{k: |\lambda_k|=1,\ 1\leqslant k\leqslant p\},\ A^c=\{k: |\lambda_k|<1,\ 1\leqslant k\leqslant p\}. \tag{1.9}$$

Without loss of generality, we can assume that

$$A=\{1, 2, \cdots, q\} \text{ for some } q\leqslant p. \tag{1.9$'$}$$

We shall show in the sequel that there will be no consistent estimate for λ_k, $k\in A^c$, when $A^c\neq\emptyset$ (c.f. Theorem 4.1), and if $A^c\neq\emptyset$, the above procedure fails to provide consistent estimates for $\lambda_1, \cdots, \lambda_q$(c.f. Remark 3.1). In view of this, it is important to seek for a consistent estimate $\hat{q}$ of $q\triangleq\#(A)$. This enables us to use $\hat{q}$ to replace p in procedure (1.5)—(1.8) to obtain estimates of $\lambda_1, \cdots, \lambda_q$.

Having obtained estimate $\hat{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$, estimates of $\boldsymbol{a}_j$'s can be obtained by replacing $\boldsymbol{\lambda}$ by $\hat{\boldsymbol{\lambda}}$, as mentioned earlier. At a first look it would suggest that the estimates of $\boldsymbol{a}_j$'s so obtained should be consistent when $\hat{\boldsymbol{\lambda}}$ is a consistent estimate of $\boldsymbol{\lambda}$. In fact, this is not true. The reason is that in order to get consistent estimates of $\boldsymbol{a}_j$'s by this method,

$\hat{\lambda}_k^r-\lambda_k^r$ should be of the order $o_p(1)$ for $r\leqslant n-1$. But usually $\hat{\lambda}_k-\lambda_k$ is only of the order $O_p(1/\sqrt{n})$, and $\hat{\lambda}_k^{n-1}-\lambda_k^{n-1}$ cannot have the order $o_p(1)$. However, it is possible to estimate $|a_{kj}|$ consistently, where $k=1, 2, \cdots, q$, $j=1, 2, \cdots, N$.

In Section 2, we give a detection procedure for q, and give some estimates of λ_k, σ^2 and $|a_{kj}|$ for $k=1, \cdots, q$ and $j=1, 2, \cdots, N$. In Section 3, we establish the strong consistency of these procedures, and find the limiting distributions for some estimates. In Section 4, we show the non-existence of consistent estimates for λ_k and a_{kj}, where $k=q+1, \cdots, p$, $j=1, \cdots, N$. Finally, Section 5 is devoted to a brief discussion of case (ii).

The strong consistency of the LS estimation of λ_k, $k\in\Lambda$, has been established in [12].

§ 2. Detection and Estimation Procedures

In this section, it is desired to determine $q=\#(\Lambda)$, and to estimate λ_k, σ^2 and $|a_{kj}|$, $k=1, 2, \cdots, q$, $j=1, 2, \cdots, N$ (refer to (1.9) and (1.9)$'$). Throughout this section, N is fixed and n tends to infinity, and the following conditions are assumed:

$$\lambda_k\neq 1,\ k=1, 2, \cdots, q;\quad \lambda_k\neq\lambda_l \text{ for } k\neq l,\ k, l=1, \cdots, q, \tag{2.1}$$

and

$$\sum_{j=1}^{N}|a_{kj}|>0 \quad \text{for } k=1, 2, \cdots, q. \tag{2.2}$$

For detection problem, we also assume that (1.2) and (1.3) are satisfied. For $r=0$, $1, 2, \cdots, p$, define a set of complex vectors

$$B_r=\left\{b^{(r)}=(b_0^{(r)}, \cdots, b_r^{(r)})': b_r^{(r)}\geqslant 0, \text{ and } \sum_{k=0}^{r}|b_k^{(r)}|^2=1\right\} \tag{2.3}$$

and a quadric form of $b^{(r)}$:

$$Q_r(b^{(r)})=\frac{1}{N(n-r)}\sum_{j=1}^{N}\sum_{t=0\,r}^{n-1-r}\left|\sum_{k=0}^{r}b_k^{(r)}Y_j(t+k)\right|^2,\quad b^{(r)}\in B_r. \tag{2.4}$$

Put

$$Q_r=\min\{Q_r(b^{(r)}),\ b^{(r)}\in B_r\}. \tag{2.5}$$

Choose constant C_n satisfying the following conditions:

$$\lim_{n\to\infty}C_n=0,\ \lim_{n\to\infty}\sqrt{n}\,C_n/\sqrt{\log\log n}=\infty. \tag{2.6}$$

Then we find the nonnegative integer $\hat{q}\leqslant p$ minimizing

$$I(r)=\log Q_r+rC_n,\quad r=0, 1, \cdots, p, \tag{2.7}$$

and use $\hat{q}$ as an estimate of q.

Note that if $\hat{b}^{(r)}=(\hat{b}_0^{(r)}, \cdots, \hat{b}_r^{(r)})'$ satisfies

$$Q_r=\frac{1}{N(n-r)}\sum_{j=1}^{N}\sum_{t=0}^{n-1-r}\left|\sum_{k=0}^{r}\hat{b}_k^{(r)}Y_j(t+k)\right|^2,$$

then Q_r is the smallest eigenvalue of the matrix

$$\hat{\Gamma}^{(r)}=(\hat{\gamma}_{lm}^{(r)}),\quad l,\ m=0, 1, \cdots, r,$$

and $\hat{b}^{(r)}$ is the corresponding eigenvector, where

$$\hat{\gamma}_{lm}^{(r)}=\frac{1}{N(n-r)}\sum_{j=1}^{N}\sum_{t=0}^{n-1-r}\overline{Y_j(t+l)}Y_j(t+m),\quad l,\ m=0,\ 1,\ \cdots,\ r. \tag{2.8}$$

Put $\lambda_k=\exp(i\omega_k)$ for $k=1, \cdots, q$. As shown in Section 3, with probability one, we have $\hat{q}=q$ for n large. Hence, to estimate $\omega_1, \cdots, \omega_q$, without loss of generality, we can assume that $q=\#(A)$ is known. For simplicity we write $\hat{\Gamma}^{(q)}=\hat{\Gamma}$, $\hat{\gamma}_{lm}^{(q)}=\hat{\gamma}_{lm}$, etc. Let $\hat{b}=(\hat{b}_0, \cdots, \hat{b}_q)'\in B_q$ be a eigenvector of $\hat{\Gamma}$ associated with its smallest eigenvalue. Under the conditions (1.2), (2.1) and (2.2), it can be shown that with probability one for n large, the equation

$$\hat{B}(z)\triangleq\sum_{k=0}^{q}\hat{b}_k z^k=0 \tag{2.9}$$

has q roots, namely $\hat{\rho}_k\exp(i\hat{\omega}_k)$, $k=1, 2, \cdots, q$, where $\hat{\rho}_k\geqslant 0$, $\hat{\omega}_k\in(0, 2\pi)$, $k\leqslant q$.

Further, Q_q furnishes an estimate of σ^2.

To estimate $|a_{kj}|$, $k=1, \cdots, q$, $j=1, \cdots, N$, write $\hat{\lambda}_k=\exp(i\hat{\omega}_k)$, $k=1, \cdots, q$, and write approximately (1.1) as

$$\begin{bmatrix} Y_j(t+0)\\ Y_j(t+1)\\ \vdots\\ Y_j(t+q-1)\end{bmatrix}\approx\hat{A}_*\begin{bmatrix} a_{1j}\lambda_1^t\\ a_{2j}\lambda_2^t\\ \vdots\\ a_{qj}\lambda_q^t\end{bmatrix}+\begin{bmatrix} e_j(t+0)\\ e_j(t+1)\\ \vdots\\ e_j(t+q-1)\end{bmatrix} \tag{2.10}$$

$$t=0,\ 1,\ \cdots,\ n-q,\ j=1,\ \cdots,\ N,$$

where

$$\hat{A}_*=\begin{bmatrix} 1 & 1 & \cdots & 1\\ \hat{\lambda}_1 & \hat{\lambda}_2 & \cdots & \hat{\lambda}_q\\ \vdots & \vdots & & \vdots\\ \hat{\lambda}_1^{q-1} & \hat{\lambda}_2^{q-1} & \cdots & \hat{\lambda}_q^{q-1}\end{bmatrix}.$$

Put $\hat{A}_*^{-1}=\hat{M}=(\hat{\mu}_{lm})$, $l, m=1, 2, \cdots, q$. Motivated by (2.10), we propose the following estimate of $|a_{kj}|^2$, $k\leqslant q$, $j\leqslant N$:

$$|\hat{a}_{kj}|^2=\left(\frac{1}{n-q+1}\sum_{t=0}^{n-q}\left|\sum_{l=1}^{q}\hat{\mu}_{kl}Y_j(t+l-1)\right|^2-\sum_{l=1}^{q}|\hat{\mu}_{kl}|^2Q_q\right)_+, \tag{2.11}$$

where for any real x,

$$(x)_+=\begin{cases}x, & \text{if}\quad x\geqslant 0,\\ 0, & \text{if}\quad x<0.\end{cases}$$

Remark 2.1. If we consider the more general model

$$Y_j(t)=a_{0j}+\sum_{k=1}^{p}a_{kj}\lambda_k^t+e_j(t), \tag{2.12}$$

$$t=0,\ 1,\ \cdots,\ n-1,\ j=1,\ \cdots,\ N,$$

where a_{0j} is an unknown constant, $\lambda_k\neq 1$, $\lambda_k\neq\lambda_l$ for $k\neq l$, k, $l=1, \cdots, q$. We can use $\hat{a}_{0j}=\sum_{t=0}^{n-1}Y_j(t)/n$ to estimate a_{0j}. Then the above procedures of detection and estimation can be used with $Y_j(t)$ replaced by $Y_j(t)-\hat{a}_{0j}$.

§3. Asymptotic Behaviour of the Ddetection and Estimation

In this section, we establish the strong consistency of the detection and estimation procedures proposed in Section 2. The asymptotic normality of some estimates is also established. Throughout this section, N is fixed and n tends to infinity.

Some known results are needed in the following discussion. For convenience of reference, we state these as lammas.

Lemma 3.1. Let $\{X_n, n\geqslant 1\}$ be a sequence of independent real random variables such that $\sum_{n=1}^{\infty} E|X_n|<\infty$. Then $\sum_{n=1}^{\infty} X_n$ converges a.s.

Refer to Stout([13], 1974, p. 94).

Lemma 3.2. (Petrov). Let $\{X_n, n\geqslant 1\}$ be a sequence of independent real random variables with zero means. Write $B_n^2=\sum_{j=1}^{n} EX_j^2$ and $S_n=\sum_{j=1}^{n} X_j$. If

$$\liminf_{n\to\infty} B_n^2/n>0$$

and

$$E|X_j|^{2+\delta}\leqslant K<\infty, \quad j\geqslant 1$$

for some constants K and $\delta\gg 0$, then

$$\limsup_{n\to\infty} S_n/(2B_n^2\log\log B_n^2)^{1/2}=1 \quad \text{a.s.}$$

For a proof, the reader is referred to Petrov ([14], p. 306) and Stout ([13], p. 274).

Lemma 3.3. Let $A=(a_{jk})$ and $B=(b_{jk})$ be two Hermitian $p\times p$ matrices with spectrum decompositions

$$A=\sum_{k=1}^{p}\delta_k u_k u_k^*, \quad \delta_1\geqslant\delta_2\geqslant\cdots\geqslant\delta_p$$

and

$$B=\sum_{k=1}^{p}\lambda_k v_k v_k^*, \quad \lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_p.$$

Further, we assume that

$$\lambda_{n_{h-1}+1}=\cdots=\lambda_{n_h}=\tilde{\lambda}_h,\ n_0=0<n_1<\cdots<n_s=p,$$
$$\tilde{\lambda}_1>\tilde{\lambda}_2>\cdots>\tilde{\lambda}_s,$$

and that

$$|a_{jk}-b_{jk}|<\alpha, \quad j, k=1, 2, \cdots, p.$$

Then there is a constant C independent of α, such that

(i) $|\delta_k-\lambda_k|<C\alpha,\ k=1, \cdots, p,$

(ii) $\sum_{k=n_{h-1}+1}^{n_h} u_k u_k^*=\sum_{k=n_{h-1}+1}^{n_h} v_k v_k^*+G^{(h)}$ with

$$G^{(h)}=(g_{jk}^{(h)}),\ |g_{jk}^{(h)}|\leqslant C\alpha, \quad j, k=1, \cdots, p, h=1, \cdots, s.$$

Refer to [15].

Lemma 3.4. Let $\{X_n, n\geqslant 1\}$ be a sequence of i.i.d. real random variables such that $EX_1=0$ and $EX_1^2<\infty$. Let $\{a_{nk}\}$ be a double sequence of real numbers such that

$$|a_{nk}|\leqslant Ck^{-1/2} \quad \text{for all } k\geqslant 1,\ n\geqslant 1,$$

and
$$\sum_k a_{nk}^2\leqslant Cn^{-\alpha} \quad \text{for all } n\geqslant 1,$$

where $\alpha>0$ and $C<\infty$ are constants. Then we have

$$\lim_{n\to\infty}\sum_k a_{nk}X_k=0 \quad \text{a.s.}$$

Refer to Stout ([13], p. 231.)

Lemma 3.5. Let $g_n(x)$ be a sequence of K-degree polynomials with roots $x_1^{(n)}, \cdots, x_K^{(n)}$ for each n, and let $g(x)$ be a k-degree polynomials with roots $x_1, \cdots, x_k$, $k\leqslant K$. If $g_n(x)\to g(x)$ as $n\to\infty$, then after suitable rearrangement of $x_1^{(n)}, \cdots, x_K^{(n)}$, we have $x_j^{(n)}\to x_j$, $j=1, 2, \cdots, k$ and $|x_j^{(n)}|\to\infty$, $j=k+1, \cdots, K$.

See Bai([16]).

Theorem 3.1. Suppose that in the model (1.1), p is known, and the conditions (1.2), (1.3), (2.1) and (2.2) are satisfied. Then

$$\lim_{n\to\infty}\hat{q}=q \quad \text{a.s.}$$

Proof. Assume that $q=\#(\Lambda)$, $\rho=\max\limits_{q<k<p}|\lambda_k|<1$, and $\max\{|a_{kj}|,\ 1<k<p,\ j=1, 2, \cdots, N\}=K$. Under the model (1.1),

$$\begin{aligned}\hat{\gamma}_{l,m}^{(r)}&=\frac{1}{N(n-r)}\sum_{j=1}^{N}\sum_{t=0}^{n-1-r}\overline{Y_j(t+l)}Y_j(t+m)\\&=\frac{1}{N(n-r)}\sum_{j=1}^{N}\sum_{t=0}^{n-1-r}\Big(\sum_{k=1}^{p}\bar{\lambda}_k^{t+l}\bar{a}_{kj}+\overline{e_j(t+l)}\Big)\Big(\sum_{k=1}^{p}\lambda_k^{t+m}a_{kj}+e_j(t+m)\Big),\end{aligned} \tag{3.1}$$

where l, $m=0, 1, \cdots, r$, $r=0, 1, \cdots, p$. We have

$$\begin{aligned}&\frac{1}{N(n-r)}\sum_{j=1}^{N}\sum_{t=0}^{n-1-r}\sum_{k=q+1}^{p}\bar{\lambda}_k^{t+l}\bar{a}_{kj}\Big|\Big(\sum_{k=1}^{q}\lambda_k^{t+m}a_{kj}\Big|+\Big|\sum_{k=q+1}^{p}\lambda_k^{t+m}a_{kj}\Big|\Big)\\&\leqslant\frac{K^2}{n-r}(p-q)p\sum_{t=0}^{\infty}\rho^t\leqslant\frac{p^2K^2}{(n-r)(1-\rho)}=O\Big(\frac{1}{n}\Big),\quad l,\ m=0, 1, \cdots, r.\end{aligned} \tag{3.2}$$

By Lemma 3.1, $\sum\limits_{t=0}^{\infty}\rho^t|e_j(t+m)|$ converges a.s., and

$$\begin{aligned}&\frac{1}{N(n-r)}\sum_{j=1}^{N}\sum_{t=0}^{n-1-r}\sum_{k=q+1}^{p}\bar{\lambda}_k^{t+l}\bar{a}_{kj}e_j(t+m)\Big|\\&\leqslant\frac{K(p-q)}{N(n-r)}\sum_{j=1}^{N}\sum_{t=0}^{\infty}\rho^t|e_j(t+m)|=O\Big(\frac{1}{n}\Big)\quad\text{a.s.}\end{aligned} \tag{3.3}$$

By (3.1)—(3.3), with probability one we have

$$\begin{aligned}\hat{\gamma}_{lm}^{(r)}=&\sum_{k=1}^{q}\lambda_k^{m-l}\Big(\frac{1}{N}\sum_{j=1}^{N}|a_{kj}|^2\Big)\\&+\sum_{k,K=1,k\neq K}^{q}\Big(\frac{1}{N}\sum_{j=1}^{N}\bar{a}_{kj}a_{Kj}\Big)\bar{\lambda}_k^l\lambda_K^m\frac{1}{n-r}\sum_{t=0}^{n-1-r}(\bar{\lambda}_k\lambda_K)^t\\&+\sum_{k=1}^{q}\bar{\lambda}_k^{l-m}\frac{1}{N}\sum_{j=1}^{N}\bar{a}_{kj}\Big(\frac{1}{n-r}\sum_{t=0}^{n-1-r}\bar{\lambda}_k^{t+m}e_j(t+m)\Big)\\&+\sum_{k=1}^{q}\lambda_k^{m-l}\frac{1}{N}\sum_{j=1}^{N}a_{kj}\Big(\frac{1}{n-r}\sum_{t=0}^{n-1-r}\lambda_k^{t+l}\overline{e_j(t+l)}\Big)\\&+\frac{1}{N(n-r)}\sum_{j=1}^{N}\sum_{t=0}^{n-1-r}\overline{e_j(t+l)}e_j(t+m)+O\Big(\frac{1}{n}\Big)\end{aligned}$$

$$\triangleq J_1+J_{2n}+J_{3n}+J_{4n}+J_{5n}+O\left(\frac{1}{n}\right). \tag{3.4}$$

Write $\lambda_k=\exp(i\omega_k)$, $\omega_k\in(0, 2\pi)$, $k=1, 2, \cdots, q$. Since $\omega_k\neq\omega_l$ for $k\neq l$, we have

$$J_{2n}=O\left(\frac{1}{n}\right). \tag{3.5}$$

By Lemma 3.2,

$$J_{3n}=O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad \text{a.s.,} \quad J_{4n}=O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad \text{a.s.} \tag{3.6}$$

By the law of the iterated logarithm of M-dependence sequence,

$$J_{5n}=\begin{cases} O\left(\sqrt{\frac{\log\log n}{n}}\right), & \text{for } l\neq m, \\ \sigma^2+O\left(\sqrt{\frac{\log\log n}{n}}\right), & \text{for } l=m, \end{cases} \quad \text{a.s.} \tag{3.7}$$

Put

$$\Omega^{(r)}=\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \bar\lambda_1 & \bar\lambda_2 & \cdots & \bar\lambda_q \\ \bar\lambda_1^2 & \bar\lambda_2^2 & \cdots & \bar\lambda_q^2 \\ \vdots & \vdots & & \vdots \\ \bar\lambda_1^r & \bar\lambda_2^r & \cdots & \bar\lambda_q^r \end{bmatrix}, \quad \begin{array}{l} A=\operatorname{diag}\left[\frac{1}{N}\sum_{j=1}^{N}|a_{1j}|^2, \cdots, \frac{1}{N}\sum_{j=1}^{N}|a_{qj}|^2\right], \\ \Gamma^{(r)}=\sigma^2 I_{r+1}+\Omega^{(r)}A\Omega^{(r)*}. \end{array} \tag{3.8}$$

Then, by (3.4)—(3.8) we have

$$\hat\Gamma^{(r)}=\Gamma^{(r)}+O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad \text{a.s.} \tag{3.9}$$

Let $\hat\theta_1^{(r)}\geqslant\cdots\geqslant\hat\theta_{r+1}^{(r)}$ and $\theta_1^{(r)}\geqslant\cdots\geqslant\theta_{r+1}^{(r)}$ be the eigenvalues of $\hat\Gamma^{(r)}$ and $\Gamma^{(r)}$ respectively. By Lemma 3.3,

$$\hat\theta_k^{(r)}=\theta_k^{(r)}+O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad \text{a.s.,} \ k=1, \cdots, r+1. \tag{3.10}$$

Since $\operatorname{rank}(\Omega^{(r)}A\Omega^{(r)*})=\min(r+1, q)$, we have

$$\begin{aligned} &\theta_{r+1}^{(r)}>\sigma^2 \quad \text{for } r<q. \\ &\theta_{r+1}^{(r)}=\sigma^2 \quad \text{for } r\geqslant q. \end{aligned} \tag{3.11}$$

Since $Q_r=\hat\theta_{r+1}^{(r)}$, we get

$$\lim_{n\to\infty} Q_r=\theta_{r+1}^{(r)}>\sigma^2, \quad \text{a.s., for } r<q, \tag{3.12}$$

and

$$|Q_r-\sigma^2|=O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad \text{a.s. for } r\geqslant q. \tag{3.13}$$

From (2.6), (2.7), (3.12) and (3.13), it is easily shown that, with probability one for n large,

$$I(q)<I(r) \quad \text{for } r\neq q, \ 0\leqslant r\leqslant p,$$

and, by the definition of $\hat q$,

$$\hat q=q.$$

Theorem 3.1 is proved.

In the sequel we assume that q is known. For simplicity, we write $\hat\Gamma^{(q)}=\hat\Gamma$, $\hat\gamma_{lm}^{(q)}=\hat\gamma_{lm}$, $\hat b^{(q)}=\hat b$, $\Omega^{(q)}=\Omega$, etc.

Theorem 3.2. Suppose that in the model (1.1), the conditions (1.2), (2.1) and (2.2) are satisfied. Then, for appropriate ordering, we have

$$\lim_{n\to\infty} \hat{\omega}_k = \omega_k \quad \text{a.s.}, \ k=1, 2, \cdots, q,$$

and

$$\lim_{n\to\infty} Q_q = \sigma^2 \quad \text{a. s.}$$

Proof. Under the conditions of the theorem, (3.2)—(3.5) still hold. By Lemma 3.4,

$$\lim_{n\to\infty} J_{\delta_n} = \begin{cases} 0, & \text{for } l \neq m, \\ \sigma^2, & \text{for } l = m, \end{cases} \quad \text{a.s.}$$

It follows that

$$\lim_{n\to\infty} \hat{\Gamma} = \Gamma (= \sigma^2 I_{q+1} + \Omega A \Omega^*) \quad \text{a.s.} \tag{3.14}$$

Define

$$B(z) \triangleq b_q \prod_{k=1}^{q} (z-\lambda_k) \triangleq b_0 + b_1 z + \cdots + b_q z^q \tag{3.15}$$

such that $b_q > 0$ and $\sum_{k=0}^{q} |b_k|^2 = 1$. Then $\boldsymbol{b} = (b_0, \cdots, b_q)' \in B_q$ and

$$\Omega A \Omega^* \boldsymbol{b} = 0, \quad \Gamma \boldsymbol{b} = \sigma^2 \boldsymbol{b}. \tag{3.16}$$

Let $\hat{\theta}_1 \geqslant \cdots \geqslant \hat{\theta}_{q+1}$ and $\theta_1 \geqslant \cdots \geqslant \theta_{q+1}$ be the eigenvalues of $\hat{\Gamma}$ and Γ respectively. Since rank $(\Omega A \Omega^*) = q$, we have

$$\theta_1 \geqslant \cdots \geqslant \theta_q > \theta_{q+1} (= \sigma^2). \tag{3.17}$$

By (3.16) and (3.17), $\boldsymbol{b}$ is the unit eigenvector of Γ associated with the unique smallest eigenvalue of Γ. Now $\hat{\boldsymbol{b}} = (\hat{b}_0, \cdots, \hat{b}_q)' \in B_q$ is the unit eigenvector of $\hat{\Gamma}$ associated with its smallest eigenvalue Q_q. Using Lemma 3.3 and (3.14), we get

$$\lim_{n\to\infty} \hat{\boldsymbol{b}} = \boldsymbol{b} \ \text{a.s., and} \ \lim_{n\to\infty} Q_q = \sigma^2 \ \text{a.s.} \tag{3.18}$$

By Lemma 3.5, for appropriate ordering, we have

$$\lim_{n\to\infty} \hat{\rho}_k \exp(i\hat{\omega}_k) = \exp(i\omega_k) \quad \text{a.s.}, \ k=1, 2, \cdots, q,$$

which implies that

$$\lim_{n\to\infty} \hat{\rho}_k = 1 \quad \text{a.s. and} \lim_{n\to\infty} \hat{\omega}_k = \omega_k \quad \text{a.s.}, \ k=1, 2, \cdots, q.$$

Theorem 3.2 is proved.

Theorem 3.3. If (1.2), (2.1) and (2.2) hold, then

$$\lim_{n\to\infty} |\hat{a}_{kj}|^2 = |a_{kj}|^2 \ \text{a.s. for } k=1, \cdots, q, \ j=1, \cdots, N.$$

Proof. By the Theorem 3.2,

$$\lim_{n\to\infty} \hat{\lambda}_k = \lambda_k \quad \text{a.s.}, \ k=1, \cdots, q.$$

Define

$$M = (\mu_{lm}) = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_q \\ \vdots & \vdots & \vdots \\ \lambda_1^{q-1} & \lambda_2^{q-1} & \lambda_q^{q-1} \end{bmatrix}^{-1}.$$

Then we have

$$\hat{\mu}_{lm} \to \mu_{lm} \quad \text{a.s.}, \ l, \ m=1, \cdots, \ q, \tag{3.19}$$

which implies that

$$\lim_{n\to\infty} \sum_{l=1}^{q} \hat{\mu}_{kl} \lambda_m^{l-1} = \delta_{km} = \begin{cases} 1, & k=m, \\ 0, & k \neq m, \end{cases} \quad \text{a.s.} \tag{3.20}$$

By (1.1), (3.19) and (3.20), for $k=1, \cdots, q$ we have

$$\begin{aligned}
&\frac{1}{n-q+1} \sum_{t=0}^{n-q} \left| \sum_{l=1}^{q} \hat{\mu}_{kl} Y_j(t+l-1) \right|^2 \\
&= \frac{1}{n-q+1} \sum_{t=0}^{n-q} \left| \sum_{l=1}^{q} \hat{\mu}_{kl} \Big(\sum_{m=1}^{p} a_{mj} \lambda_m^{t+l-1} + e_j(t+l-1) \Big) \right|^2 \\
&\overset{\text{a.s.}}{=} \frac{1}{n-q+1} \sum_{t=0}^{n-q} \left| \sum_{m=1}^{p} \delta_{km} a_{mj} \lambda_m^t + \sum_{m=1}^{p} |a_{mj}| \cdot o(1) \right. \\
&\qquad \left. + \sum_{l=1}^{q} \mu_{kl} e_j(t+l-1) + \sum_{l=1}^{q} |e_j(t+l-1)| \cdot o(1) \right|^2 \\
&\overset{\text{a.s.}}{=} \frac{1}{n-q+1} \sum_{t=0}^{n-q} \left| a_{kj} \lambda_k^t + o(1) + \sum_{l=1}^{q} \mu_{kj} e_j(t+l-1) + \sum_{l=1}^{q} |e_j(t+l-1)| \cdot o(1) \right|^2 \\
&\overset{\text{a.s.}}{=} \frac{1}{n-q+1} \sum_{t=0}^{n-q} \Big\{ |a_{kj}|^2 + \sum_{l,m=1}^{q} \mu_{kl} \bar{\mu}_{km} e_j(t+l-1) \overline{e_j(t+m-1)} + o(1) \\
&\qquad + \sum_{l=1}^{q} \Big(|e_j(t+l-1)|^2 + |e_j(t+l-1)| \Big) \cdot o(1) + a_{kj} \lambda_k^t \sum_{l=1}^{q} \mu_{kl} e_j(t+l-1) \\
&\qquad + \bar{a}_{kj} \bar{\lambda}_k^t \sum_{l=1}^{q} \bar{\mu}_{kj} \overline{e_j(t+l-1)} \Big\}.
\end{aligned} \tag{3.21}$$

By the SLLN,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n-q+1} \sum_{t=0}^{n-q} \sum_{l,m=1}^{q} \mu_{kl} \bar{\mu}_{km} e_j(t+l-1) \overline{e_j(t+m-1)} \\
&= \sum_{l,m=1}^{q} \mu_{kl} \bar{\mu}_{km} \delta_{lm} \sigma^2 = \sum_{l=1}^{q} |\mu_{kl}|^2 \sigma^2 \quad \text{a.s.}
\end{aligned} \tag{3.22}$$

$$\lim_{n\to\infty} \frac{1}{n-q+1} \sum_{t=0}^{n-q} \left(|e_j(t+l-1)|^2 + |e_j(t+l-1)| \right) = \sigma^2 + E|e_1(o)|, \quad \text{a.s.} \tag{3.23}$$

By Lemma 3.4,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n-q+1} \sum_{t=0}^{n-q} a_{kj} \lambda_k^t \sum_{l=1}^{q} \mu_{kl} e_j(t+l-1) \\
&= \sum_{l=1}^{q} a_{kj} \mu_{kl} \lim_{n\to\infty} \frac{1}{n-q+1} \sum_{t=0}^{n-q} \lambda_k^t e_j(t+l-1) = 0 \quad \text{a.s.}
\end{aligned} \tag{3.24}$$

By (3.21)—(3.24),

$$\lim_{n\to\infty} \frac{1}{n-q+1} \sum_{t=0}^{n-q} \left| \sum_{l=1}^{q} \hat{\mu}_{kl} Y_j(t+l-1) \right|^2 = |a_{kj}|^2 + \sum_{l=1}^{q} |\mu_{kl}|^2 \sigma^2, \quad \text{a.s.} \tag{3.25}$$

From (2.11), (3.18), (3.19) and (3.25), Theorem 3.3 follows.

Remark 3.1. If $q<p$ and we estimate $\lambda_1, \cdots, \lambda_q$ using (1.5)—(1.8) directly, then we have

$$\lim_{n\to\infty} \hat{\Gamma}^{(p)} = \Gamma^{(p)} \quad \text{a.s.}$$

and $\theta_1^{(p)} \geqslant \cdots \geqslant \theta_q^{(p)} > \theta_{q+1}^{(p)} = \cdots = \theta_{p+1}^{(p)} (=\sigma^2)$ are the eigenvalues of $\Gamma^{(p)}$. All eigenvectors of

$\Gamma^{(p)}$ associated with σ^2 consist of a $(p-q)$ dimension subspace. Assume that $\hat{b}^{(p)}\in B_p$ such that

$$Q_p(\hat{b}^{(p)})=\min\{Q_p(b^{(p)}):b^{(p)}\in B_p\},$$

then $\hat{b}^{(p)}$ is the eigenvector of $\hat{\Gamma}^{(p)}$ associated with the smallest eigenvalue of $\hat{\Gamma}^{(p)}$. We do not know whether $\hat{b}^{(p)}$ converges. In general, we do not know whether there are q roots among all roots of $\sum_{k=0}^{p}\hat{b}_k^{(p)}z^k$ which tend to $\{\lambda_k,\ k=1, 2, \cdots, q\}$.

Remark 3.2. Suppose that in the model (2.12), the condition (1.2) hold. Then $\hat{a}_{0j}=\sum_{t=0}^{n-1}Y_j(t)/n$ is a strongly consistent estimate of a_{0j}. For those procedures of detection and estimation discussed in Remark 2.1, Theorem 3.1—3.3 are also true.

Finally, we establish the asymptotic normality of Q_q and $(\hat{\omega}_k,\ k=1, 2, \cdots, q)$. To this end, we assume that under the model (1.1), the conditions (2.1), (2.2) are satisfied, and $e_j(t)$, $j=1, 2, \cdots, N$, $t=0, 1, \cdots, n-1$, are i.i.d. complex variables, $e_j(t)=e_{j1}(t)+ie_{j2}(t)$, $e_{j1}(t)$ and $e_{j2}(t)$ satisfy the following conditions.

$$Ee_j(t)=0\quad Ee_{j1}^2(t)=Ee_{j2}^2(t)=\sigma^2/2,$$

$$Ee_{j1}(t)e_{j2}(t)=0 \text{ and } \mathrm{Var}(|e_j(t)|^2)=\alpha\sigma^4 \text{ with } \alpha>0. \tag{3.26}$$

Put

$$U=\begin{bmatrix} u_0 & \bar{u}_1 & \cdots & \bar{u}_q \\ u_1 & u_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \bar{u}_1 \\ u_q & \cdots & u_1 & u_0 \end{bmatrix} \tag{3.27}$$

where $u_0, u_1, \cdots, u_q$ are independent, and

(i) $u_0\sim N_r(0, \alpha\sigma^4)$,

(ii) $u_k\sim N_c(0, \sigma^4)$, $k=1, 2, \cdots, q$. (3.28)

Here N_c, N_r denote complex and real normal distribution respectively.

Define $b=(b_0, b_1, \cdots, b_q)'$ by (3.15). Put

$$A=\mathrm{diag}\left[\frac{1}{N}\sum_{j=1}^{N}|a_{1j}|^2,\ \cdots,\ \frac{1}{N}\sum_{j=1}^{N}|a_{qj}|^2\right],$$

$$\Omega=\begin{bmatrix} 1 & 1 & \cdots 1 \\ \bar{\lambda}_1 & \bar{\lambda}_2 & \cdots \bar{\lambda}_q \\ \vdots & \vdots & \vdots \\ \bar{\lambda}_1^q & \bar{\lambda}_2^q & \cdots \bar{\lambda}_q^q \end{bmatrix},$$

$$\zeta_n=\sqrt{N(n-q)}(Q_q-\sigma^2),\qquad T_{nk}=\sqrt{N(n-q)}(\hat{\rho}_k-1), \tag{3.29}$$

$$\Delta_{nk}=\sqrt{N(n-q)}(\hat{\omega}_k-\omega_k),\quad k=1, 2, \cdots, q,$$

$$T_n=(T_{n1}, \cdots, T_{nq})' \text{ and } \Delta_n=(\Delta_{n1}, \cdots, \Delta_{nq})'.$$

Write

$$B(z)=\sum_{k=0}^{q}b_kz^k,$$

$$D(\exp(i\omega))=-i\frac{d}{d\omega}B(e^{i\omega})$$

and

$$G=\mathrm{diag}[D(\exp(i\omega_1)), \cdots, D(\exp(i\omega_s))]. \tag{3.30}$$

We have the following.

Theorem 3.4. Suppose that in the model (1.1), the conditions (2.1), (2.2) and (3.26) are satisfied. Then we have

$$\zeta_n \xrightarrow{D} = b^* U b, \tag{3.31}$$

$$T_n + iA_n \xrightarrow{D} G^{-1}A^{-1}(\Omega^*\Omega)^{-1}\Omega^* U b, \tag{3.32}$$

as $n\to\infty$.

Here we quote the following

Lemma 3.6. Suppose that the condition (3.26) hold. Then

$$\frac{1}{\sqrt{n-q}}\sum_{t=0}^{n-1-q}\lambda_k^{t+l}\overline{e_j(t+l)} \xrightarrow{D} v_{kj},$$

$$k=1, 2, \cdots, q,\ j=1, 2, \cdots, N,\ l=0, 1, \cdots, q,$$

$$\frac{1}{\sqrt{n-q}}\sum_{t=0}^{n-1-q}(|e_j(t+l)|^2-\sigma^2) \xrightarrow{D} u_{0j},\quad j=1, \cdots, N,\ l=0, 1, \cdots, q,$$

$$\frac{1}{\sqrt{n-q}}\sum_{t=0}^{n-1-q}\overline{e_j(t+l)}e_j(t+m) \xrightarrow{D} u_{l-m,j},\quad j=1, \cdots, N,\ 0\leqslant m<l\leqslant q.$$

Here u_{kj}'s and v_{kj}' are independent of each other, and

(i) $u_{0j}\sim N_r(0, \alpha\sigma^4)$, $j=1, \cdots, N$.

(ii) $u_{kj}\sim N_c(0, \sigma^4)$, $j=1, \cdots, N$, $k=1, \cdots, q$. (3.33)

(iii) $v_{kj}\sim N_c(0, \sigma^2)$, $j=1, \cdots, N$, $k=1, \cdots, q$.

Refer to Lemma 4.1 in [9].

The proof of Theorem 3.4 runs along the line as in the proof of Theorem 4.1 in [9], so the details are omitted.

Remark 3.3. For the model (2.12), Theorem 3.4 applies those estimates discussed in Remark 2.1.

§ 4. Non-Existence of Consistent Estimates of $\lambda_k, k\in\Lambda^c$ and $a_1,\cdots,a_N$ when $\Lambda^c\neq\phi$

Throughout this section, N is fixed and $n\to\infty$. For non-existence of consistent estimate of λ_k, $k\in\Lambda^c$, we have the following

Theorem 4.1. Suppose that in the model (1.1), $e_j(t)$, $j=1, \cdots, N$, $t=0, 1, \cdots, n-1$, are i.i.d., $e_j(t)\sim N_c(0, \sigma^2)$ with $0<\sigma^2<\infty$, and the parameter space of $\boldsymbol{\lambda}=(\lambda_1, \cdots, \lambda_p)'$ contains at least two points $\boldsymbol{\lambda}^{(k)}=(\lambda_1^{(k)}, \cdots, \lambda_p^{(k)})'$, $(k=1, 2)$ such that for some $l(1\leqslant l\leqslant p)$, $\lambda_l^{(1)}\neq\lambda_l^{(2)}$, $|\lambda_l^{(k)}|<1$, $k=1, 2$, and $\sum_{j=1}^{N}|a_{lj}^2|>0$. Then no consistent estimate of λ_l exists as N is fixed and $n\to\infty$.

Proof. Assume $l=p$. It suffices to show that a consistent estimate of λ_p cannot exist even when $\{a_{kj}\}$ and σ^2 are known. Hence, without loss of generality, we assume $\sigma^2=2$.

Introduce the prior distribution H:

$$H(\boldsymbol{\lambda}^{(1)})=H(\boldsymbol{\lambda}^{(2)})=1/2$$

and the square loss $|d-\lambda_p|^2$. Write

$$f_k=(2\pi)^{-nN}\exp\left\{-\frac{1}{2}\sum_{j=1}^{N}\sum_{t=0}^{n-1}\left|Y_j(t)-\sum_{l=1}^{p-1}\lambda_l^t a_{lj}-(\lambda_p^{(k)})^t a_{pj}\right|^2\right\},\quad k=1,2.$$

Under the above prior distribution and loss function, the Bayesian estimate of λ_p is

$$\tilde{\lambda}_p=(f_1\lambda_p^{(1)}+f_2\lambda_p^{(2)})/(f_1+f_2).$$

Denote by $R(\tilde{\lambda}_p)$ the Bayesian risk of $\tilde{\lambda}_p$, we have

$$\begin{aligned}R(\tilde{\lambda}_p)&\geqslant\frac{1}{2}E(|\tilde{\lambda}_p-\lambda_p^{(1)}|^2|k=1)\\&=\frac{1}{2}E\left\{\left(\frac{f_2}{f_1+f_2}\right)\right\}^2\left|k=1\right\}|\lambda_p^{(1)}-\lambda_p^{(2)}|^2.\end{aligned}\tag{4.1}$$

Noticing that $Y_j(t)-\sum_{l=1}^{p-1}\lambda_l^t a_{lj}-(\lambda_p^{(1)})^t a_{pj}=e_j(t)$ when $k=1$, we have

$$\begin{aligned}\log\frac{f_2}{f_1}\geqslant&-\frac{1}{2}\sum_{j=1}^{N}\sum_{t=0}^{n-1}|a_{pj}|^2|(\lambda_p^{(1)})^t-(\lambda_p^{(2)})^t|^2\\&-\left|\sum_{j=1}^{N}\sum_{t=0}^{n-1}((\lambda_p^{(2)})^t-(\lambda_p^{(1)})^t(a_{pj}e_j(t)\right|.\end{aligned}\tag{4.2}$$

Since $|\lambda_p^{(1)}|<1$, $|\lambda_p^{(2)}|<1$, we have

$$\lim_{n\to\infty}\sum_{j=1}^{N}\sum_{t=0}^{n-1}|a_{pj}|^2|(\lambda_p^{(1)})^t-(\lambda_p^{(2)})^t|^2<\infty.\tag{4.3}$$

Also, by Lemma 3.1, the second term of the right hand side of (4.2) converges with probability one to a finite random variable. From this, (4.2) and (4.3), it follows that there exists a positive constant K such that

$$P(f_2/f_1>K|k=1)\geqslant 1/2$$

for n sufficiently large. Hence, form (4.1) we obtain

$$R(\tilde{\lambda}_p)\geqslant\frac{1}{4}\left(\frac{K}{1+K}\right)^2|\lambda_p^{(1)}-\lambda_p^{(2)}|^2>0\tag{4.4}$$

for n large.

But if ξ_n is a consistent estimate of λ_p, then define

$$\tilde{\xi}_n=\begin{cases}\lambda_p^{(1)}, & \text{if } |\xi_n-\lambda_p^{(1)}|<|\xi_n-\lambda_p^{(2)}|\\ \lambda_p^{(2)}, & \text{otherwise.}\end{cases}$$

We shall have $\tilde{\xi}_n\xrightarrow{P.}\lambda_p^{(k)}$ for $\lambda=\lambda^{(k)}$, $k=1,2$.

Since $\tilde{\xi}_n$ is bounded, by the dominated convergence theorem, we have

$$R(\tilde{\xi}_n)=1/2E(|\tilde{\xi}_n-\lambda_p^{(1)}|^2|\boldsymbol{\lambda}=\boldsymbol{\lambda}^{(1)})+1/2E(|\tilde{\xi}_n-\lambda_p^{(2)}|^2|\boldsymbol{\lambda}=\boldsymbol{\lambda}^{(2)})\to 0\tag{4.5}$$

as $n\to\infty$, where $R(\tilde{\xi}_n)$ is the Bayesian risk of $\tilde{\xi}_n$. But this contradicts (4.4) in view of the fact that $\tilde{\lambda}_p$ is the Bayesian estimate of λ_p, and the theorem is proved.

For the similar problem of $\boldsymbol{a}_j=(a_{1j},\cdots,a_{pj})'$ when $\Lambda^0\neq\emptyset$, we have the following

Theorem 4.2. Suppose that in the model (1.1), $e_j(t)$, $j=1,\cdots,N$, $t=0,1,\cdots,n-1$, are i.i.d., $e_j(t)\sim N_c(0,\sigma^2)$ with $0<\sigma^2<\infty$. Also, some component of $\boldsymbol{\lambda}$, say λ_p, has a module less than one, and $\lambda_k\neq\lambda_l$ if $k\neq l$. Then no consistent estimates can be found for $a_{p1}, \cdots, a_{pN}$.

Proof. As in the proof of Theorem 4.1, we may assume that $\lambda_1, \cdots, \lambda_p$ and a_{kj}, $k=1, \cdots, p-1$, $j=1, \cdots, N$, are known. Also without loss of generality we may assume $N=1$. Write $Y_1(t)-\sum_{k=1}^{p-1} a_{k1}\lambda_k^t = X(t)$, and for simplicity, write $\lambda_p=\lambda$, $a_p=\beta$. Then the model (1.1) is reduced as the following linear model:

$$X(t)=\beta\lambda^t+e(t), \quad t=0, 1, \cdots, n-1, \tag{4.6}$$

where λ is known and $|\lambda|<1$. It is desired to show that there is no consistent estimate for β.

Let $\hat{\beta}$ denote the *LS* estimate of β. By a theorem of Drygas[17], the consistency of $\hat{\beta}$ is equivalent to $\mathrm{Var}(\hat{\beta})\to 0$. But from

$$\hat{\beta}=\left(\sum_{t=0}^{n-1}|\lambda|^{2t}\right)^{-1}\sum_{t=0}^{n-1}\bar{\lambda}^t X(t),$$

and

$$\mathrm{Var}(\hat{\beta})=\sigma^2\left(\sum_{t=0}^{n-1}|\lambda|^{2t}\right)^{-1}\to\sigma^2(1-|\lambda|^2)\neq 0,$$

we know that $\hat{\beta}$ is not consistent.

Since $\{e(t)\}$ is a sequence of i.i.d. variables with a common normal distribution, it follows by a theorem of Ker-Chan Li[18] that there cannot exist any consistent estimate for β. The Theorem 4.2 is proved.

§ 5. The Case When n is Fixed ond $N\to\infty$

In this section, we assume that $n\geqslant p+1$ is fixed and N tends to infinity. Consider the model (1.1). Assum ethat $\lambda_k\neq\lambda_l$ if $k\neq l$ (note that the condition $|\lambda_k|=1$ can be dropped), and that (1.2) is true. We can use the EVLP method described in section 1 to obtain estimates $\hat{\lambda}_k$'s of λ_k's (refer to (1.5)—(1.8) and so on). We have the following

Theorem 5.1. Suppose that under the model (1.1), $\lambda_k\neq\lambda_l$ if $k\neq l$, and (1.2) holds. Also, $n\geqslant p+1$ is fixed and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\bar{a}_j a'_j=\Psi$$

exists, where $a'_j=(a_{1j}, a_{2j}, \cdots, a_{pj})$, $\Psi=(\psi_{lm})_1^p$ is a $p\times p$ positive definite matrix. Then, $\hat{\lambda}_1, \cdots, \hat{\lambda}_p$ and Q_p are strongly consistent estimates of $\lambda_1, \cdots, \lambda_p$ and σ^2.

Proof. By (1.5)—(1.7), Q_p is the smallest eigenvalue of the matrix $\hat{\Gamma}=(\hat{\gamma}_{lm})$, $l, m=0, 1, \cdots, p$, and $\hat{b}$ is the corresponding eigenvector, where

$$\hat{\gamma}_{lm}=\frac{1}{N(n-p)}\sum_{j=1}^{N}\sum_{t=0}^{n-1-p}\overline{Y_j(t+l)}\,Y_j(t+m), \quad l, m=0, 1, \cdots, p. \tag{5.1}$$

By(1, 1),

$$\hat{\gamma}_{lm}=\sum_{k,K=1}^{p}\frac{1}{n-p}\sum_{t=0}^{n-1-p}\bar{\lambda}_k^{t+l}\lambda_K^{t+m}\frac{1}{N}\sum_{j=1}^{N}\bar{a}_{kj}a_{Kj}$$

$$+\sum_{k=1}^{p}\frac{1}{n-p}\sum_{t=0}^{n-1-p}\bar{\lambda}_k^{t+l}\frac{1}{N}\sum_{j=1}^{N}\bar{a}_{kj}e_j(t+m)$$
$$+\sum_{k=1}^{p}\frac{1}{n-p}\sum_{t=0}^{n-1-p}\lambda_k^{t+m}\frac{1}{N}\sum_{j=1}^{N}a_{kj}\overline{e_j(t+l)}$$
$$+\frac{1}{N(n-p)}\sum_{t=0}^{n-1-p}\sum_{j=1}^{N}\overline{e_j(t+l)}\,e_j(t+m)$$
$$\triangleq I_{1N}+I_{2N}+I_{3N}+I_{4N}. \tag{5.2}$$

We have

$$\lim_{N\to\infty} I_{1N}=\sum_{k,h=1}^{p}\bar{\lambda}_k^l\lambda_h^m\frac{1}{n-p}\sum_{t=0}^{n-1-p}\bar{\lambda}_k^t\psi_{kh}\lambda_h^t,\quad l,\ m=0,\ 1,\ \cdots,\ p. \tag{5.3}$$

By Lemma 3.4,

$$\lim_{N\to\infty} I_{2N}=\lim_{N\to\infty} I_{3N}=0\quad \text{a.s.} \tag{5.4}$$

By the SLLN,

$$\lim_{N\to\infty} I_{4N}=\sigma^2\delta_{lm},\quad \text{a. s.},\ l,\ m=0,\ 1,\ \cdots,\ p. \tag{5.5}$$

Write

$$\underset{(p+1)\times p}{\Omega}=\begin{bmatrix}1 & 1 & \cdots & 1\\ \bar{\lambda}_1 & \bar{\lambda}_2 & \cdots & \bar{\lambda}_p\\ \bar{\lambda}_1^2 & \bar{\lambda}_2^2 & & \bar{\lambda}_p^2\\ \vdots & \vdots & & \vdots\\ \bar{\lambda}_1^p & \bar{\lambda}_2^p & \cdots & \bar{\lambda}_p^p\end{bmatrix},\quad \Omega_1=\operatorname{diag}[\lambda_1,\ \cdots,\ \lambda_p]. \tag{5.6}$$

By (5.2)—(5.5), we have

$$\lim_{N\to\infty}\hat{\Gamma}=\Gamma\quad \text{a.s.} \tag{5.7}$$

where

$$\Gamma=\sigma^2 I_{p+1}+\frac{1}{n-p}\sum_{t=0}^{n-1-p}\Omega\bar{\Omega}_1^t\Psi\Omega_1^t\Omega^*. \tag{5.8}$$

Noticing that rank $\left(\sum_{t=0}^{n-1-p}\Omega\bar{\Omega}_1^t\Psi\Omega_1^t\Omega^*\right)=p$, we can finish the proof by repeating the argument used in the proof of Theorem 3.2.

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带噪声的叠加指数信号的 EVLP 估计的渐近性质

本文研究了如下的带噪声中的指数信号模型

$$Y_j(t)=\sum_{k=1}^{p}a_{kj}\lambda_k^t+e_j(t)\quad t=0,\ 1,\ \cdots,\ n-1,\ j=1,\ 2,\cdots,N$$

其中 $\lambda_1, \lambda_2, \cdots, \lambda_q$ 是未知的模为 1 的复参数, $\lambda_{q+1}, \cdots, \lambda_p$ 是未知的模小于 1 的复参数. 并假设 $\lambda_1,\ \lambda_2, \cdots, \lambda_p$ 不相同, p 已知, q 未知, $a_{kj}(k=\overline{1,\ p},\ j=\overline{1,\ N})$ 为未知的复参数. $e_j(t)\,(t=\overline{0,\ n-1},\ j=\overline{1,\ N})$ 为独立同分布的复随机噪声变量, 且有
其中 σ^2 未知,

$$Ee_1(0)=0,\ E|e_1(0)|^2=\sigma^2\ 0<\sigma^2<\infty,\ E|e_1(0)|^4<\infty$$

本文给出了

1. q 的强相合估计、
2. $\lambda_1,\ \lambda_2,\ \cdots, \lambda_q,\ \sigma^2$ 及 $|a_{kj}|\,(k\leqslant q)$ 的强相合估计;
3. 上述某些估计的极限分布;
4. λ_k 及 $a_{kj}(k>q)$ 不存在相合估计的证明;
5. $N\to\infty$ 情形的讨论。