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Some Measures of the Severity of Ruin and the Cost of Recovery in the Classical Risk Model Disturbed by Diffusion*

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Abstract

In this paper we consider the maximal severity of ruin and the cost of recovery which are proposed by Picard (1994). We will improve and extend the results in Picard (1994) from the classical risk model to the model disturbed by diffusion.

Keywords: Maximal severity of ruin, cost of recovery, local martingale, Itô's formula.

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§ 1. Introduction

We consider the compound Poisson model disturbed by diffusion. Let $(R_t)_{t\geq 0}$ denote the surplus model of an insurance company and R_t can be written as

$$R_t = R_0 + ct + \sum_{k=1}^{N_t} W_k + B_t, \qquad t \geq 0.$$

Thus the constant R_0 is the insure's initial capital; the premiums are received continuously at a constant rate c per unit time; $(N_t)_{t\geq 0}$ is a Poisson process with mean per unit time λ; $(W_k)_{k\geq 1}$ are independent random variables with common distribution function $P(x)$ which has density function $p(x)$, mean value μ and $P(0) = 0$, and $(B_t)_{t\geq 0}$ is a Wiener process which has mean value zero and variance $2Dt$, $D > 0$ for fixed t; $(N_t)_{t\geq 0}$, $(B_t)_{t\geq 0}$ and $(W_k)_{k\geq 1}$ are mutually independent. We assume that

$$c - \mu\lambda > 0.$$

If necessary, we shall suppose that for $\alpha \geq 2$, $m_\alpha \widehat{=} \mathsf{E}(W_k^\alpha)$ exists. As usual we define the time of the ruin as $T = \inf\{t | t > 0, R_t < 0\}$, the first time of the surplus trajectory up crosses the level zero as $T' = \inf\{t | t > T, R_t > 0\}$, the probability of ruin as $\Psi(u) = \mathsf{P}(T < \infty | R_0 = u)$ and the survival probability as $\Phi(u) = 1 - \Psi(u)$. The considered problems are that when ruin has occurred, for continuing activities the company needs a loan to pay its debt, and then will pay the loan by its activities. Theoretically speaking, when $c - \lambda\mu > 0$, the loan can be pay by the activities of the

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company because the number of the surpluses of the company will have been more than any given number sooner or latter (cf. Doney (1991) for example). On the other hand, practically speaking, if the amount of the loan is too large or alloted loan time is too long, the company will be not able to get an essential loan from a loaner. Therefore before deciding to continue activities or not, the company should investigate that how deep and how long the company will stay in the red if activities are continued. It means that an index $T'-T$ and another index $\max\limits_{T\le t\le T'}|R_t|$, and even more some quantities which stand for common effects of $[T,T']$ and $\{|R_t|, T\le t\le T'\}$ such as $\int_T^{T'}|R_s|\mathrm{d}s$ need to be evaluated. The problems are studied by Picard (1994) with respect to the classical risk model. He consider the quantities such as

$$M=\max_{T\le t\le T'}|R_t|,\qquad I=\int_T^{T'}|R_t|\mathrm{d}t$$

and the more general index $\int_T^{T'}h(R_t)\mathrm{d}t$ where the function h is given (for example $h(x)=x^2$). According to Picard (1994) M is called the maximal severity of ruin, and I is the cost of recovery. The approach is made in Picard (1994) by looking for a martingale, which is based on an intuitive fact that within finite time all sample pathes of the surplus of the classical risk model are uniformly upper-bounded. In this paper, the compound Poisson model disturbed by diffusion $(R_t)_{t\ge0}$ can be considered as a surplus process of the company with random errors $(\sigma B_t)_{t\ge0}$ which are caused by the activities of the company. Due to Wiener process's strange behavior, with strictly positive probabilities Wiener process can reach any given levels within finite time, so that the above fact is not valid if the classical risk model is disturbed by diffusion. Motivated by the idea of Picard (1994), we will make approach by using Itô's formula to look for a local martingale, which is in control of the Wiener process's behavior. In this paper it merits our attention that there is a special term $D(f'(x))^2$ in the expression of the function $g(x)$, which is irregular in study of the case that is disturbed by diffusion, so that some final calculation results in this paper are influenced by both the regular term $Df''(x)$ and the irregular term $D(f'(x))^2$. Also due to the Wiener process's behavior when give some sufficient condition for existence of $\int_T^{T'}|R_t|^k\mathrm{d}t$, we first have to prove that existence of the nth order conditional moment of $T'-T$ and existence of the nth order conditional moment of M are dependent on existence of the nth order moment of W_k. In order to calculate some conditional moments, in theorem 3 we propose some formulae which are different from those of Picard (1994). In addition, in Remark 1 and Remark 2 of the following sections we indicate two omissions in Picard (1994), which are apt to be oversighted.

§2. The Maximal Severity

Theorem 1 For $M=\max\{|R_t|, T\le t\le T'\}$, we have

$$\mathsf{P}(M\le z|R_0=u, T<\infty)=\frac{\Psi(u)-\Psi(u+z)}{\Psi(u)(1-\Psi(z))},\qquad z>0;\tag{2.1}$$

$$\mathsf{P}(M=0|R_0=u,T<\infty)=\frac{\Psi_d(u)}{\Psi(u)}; \tag{2.2}$$

$$\mathsf{P}(M\le z|T<\infty,|R_T|=y)=\frac{1-\Psi(z-y)}{1-\Psi(z)},\qquad z\ge y>0, \tag{2.3}$$

where $\Psi_d(u)$ denote the probability of ruin from initial surplus u and caused by oscillation.

Proof We can prove the results (2.1) and (2.3) using the same argument as in the proof of theorem 1 of Picard (1994). In (2.1) letting $z\to 0^+$ and then using L'Hospital's rule and the formulae (1.6) and (4.7) of Dufresne et al. (1991), we directly get (2.2). #

The formula (2.2) and Corollary 1 below reflect difference between the classical risk model and the classical risk model disturbed by diffusion.

Corollary 1

$$\Pi(y)\,\hat{=}\,\mathsf{P}(|R_t|\le y\ \text{ for }\ T\le t\le T'|T<\infty,|R_T|=y)=0,\qquad y>0. \tag{2.4}$$

Proof Let $z=y$ in (2.3) and note that $\Psi(0)=1$. #

Corollary 2

$$\mathsf{E}(M|T<\infty,|R_T|=y)=y+\int_y^\infty\frac{\Psi(z-y)-\Psi(z)}{1-\Psi(z)}\mathrm{d}z,\qquad y>0; \tag{2.5}$$

$$\mathsf{E}(M|T<\infty,R_0=u)=\frac{1}{\Psi(u)}\int_0^\infty\frac{\Psi(u+z)-\Psi(u)\Psi(z)}{1-\Psi(z)}\mathrm{d}z. \tag{2.6}$$

Proof Using (2.3) and (2.4), we derive (2.5) as follows

$$\begin{aligned}\mathsf{E}(M|T<\infty,|R_T|=y)&=-\int_y^\infty z\mathrm{d}\Big(1-\frac{1-\Psi(z-y)}{1-\Psi(z)}\Big)+y\Pi(y)\\&=-z\Big(1-\frac{1-\Psi(z-y)}{1-\Psi(z)}\Big)\Big|_y^\infty+\int_y^\infty\Big(1-\frac{1-\Psi(z-y)}{1-\Psi(z)}\Big)\mathrm{d}z\\&=y+\int_y^\infty\frac{\Psi(z-y)-\Psi(z)}{1-\Psi(z)}\mathrm{d}z.\end{aligned}$$

The formula (2.6) follows from (2.1). #

Remark 1 In the case of the classical risk model obviously $M\ge y$, which implies corresponding formula in Picard (1994) is not valid. The expression of that formula should be

$$\mathsf{E}(M|T<\infty,|R_T|=y)=y+\int_y^\infty\frac{\Psi(z-y)-\Psi(z)}{1-\Psi(z)}\mathrm{d}z,$$

which is corresponding to (2.5) and can be derived by following the proof of formula (2.5).

§ 3. The Cost of Recovery

Let Y_t denote $\sum\limits_{k=1}^{N_t}W_k$, $\mathcal{F}_t=\sigma\{R_s,0\le s\le t\}$ which is the σ-field generated by R_s for $0\le s\le t$ and $G=\sigma\{B_s,s<\infty\}$ which is the σ-field generated by B_s for $s<\infty$. The lemma stated below is used in the following station.

Lemma 1 Y_t, N_t and B_t are $\mathcal{F}_t$ measurable.

Proof Set $\Delta R_s = R_s - R_{s-}$. We see that $Y_t = -\sum_{0\le s\le t} \Delta R_s$ and

$$N_t = \inf\{n; 0 < t_1 < \cdots < t_n \le t \text{ and } 0 < Y_{t_1} < \cdots < Y_{t_n}\}.$$

The above expressions imply the validity of the conclusions for Y_t and N_t, and consequently for B_t. #

We now look for a convenient local martingale $(U_t, \mathcal{F}_t)_{t\ge 0}$ by using Itô's formula. Let f and g be two real functions (g continuous, f of class C^2) that are connected by the relation

$$g(x) = -\lambda - cf'(x) + D(f'(x))^2 - Df''(x) + \lambda \mathsf{E}[\exp(f(x) - f(x-W))], \qquad x \in \Re, \tag{3.1}$$

where $\Re$ denotes the real set and the random variable W has the distributions function $P(x)$.

Theorem 2 With the functions f and g stated above, we choose

$$U_t = \exp\Big(-f(R_t) - \int_0^t g(R_s)\mathrm{d}s\Big). \tag{3.2}$$

If for any $a \in \Re$, $-f$, $-g$, $e^{-f}|cf' - D(f')^2 + Df''|$ admit an upper bound on $(-\infty, a]$, then $(U_t, \mathcal{F}_t)_{t\ge 0}$ is a local martingale.

Proof Applying Itô's formula (cf. Ikeda et al. (1981)) to the semimartingale U_t with (3.1) yields

$$\begin{aligned} U_t &= \exp(-f(0)) + \int_0^t U_{s-}[-cf'(x)]\mathrm{d}s + \int_0^t U_{s-}[-f'(x)]\mathrm{d}B_s \\ &\quad + \int_0^t U_{s-}[-g(x)]\mathrm{d}s + \int_0^t U_{s-}[D(f'(x))^2 - Df''(x)]\mathrm{d}s \\ &\quad + \int_0^t U_{s-}\{\exp[-f(x - W_{N_{s-}+1}) + f(x)] - 1\}\mathrm{d}N_s|_{x=R_{s-}} \\ &= \exp(-f(0)) + \int_0^t U_{s-}[-f'(x)]\mathrm{d}B_s - \int_0^t \lambda U_{s-}\mathsf{E}\{\exp[-f(x-W) + f(x)] - 1\}\mathrm{d}s \\ &\quad + \int_0^t U_{s-}\{\exp[-f(x - W_{N_{s-}+1}) + f(x)] - 1\}\mathrm{d}N_s|_{x=R_{s-}}. \end{aligned} \tag{3.3}$$

Let

$$\tau_n = \inf\{t : |B_t| > n\}, \qquad n = 1, 2, \cdots.$$

From the conclusion of Lemma 1 we know that τ_n is $(\mathcal{F}_t)_{t\ge 0}$–stopping time and it is well known that $\tau_n < \infty$ (for example see Revuz & Yor (1991)). For any positive number r which is less than t and a positive integer m, let $r_k \,\hat{=}\, r + k(t-r)/m$, $k = 1, 2, \cdots, m$. For any positive integer n noting that τ_n is also measurable on the σ-field G and using the conditions stated in Theorem 2, we have

$$\begin{aligned}
&\mathsf{E}\Big[\Big(\int_{r\wedge\tau_n}^{t\wedge\tau_n} U_{s-}\{\exp[-f(x-W_{N_{s-}+1})+f(x)]-1\}\mathrm{d}N_s|_{x=R_{s-}}\Big)\Big|\mathcal{F}_r\Big]\\
=\ &\lim_{m\longrightarrow+\infty}\sum_{k=1}^{m}\mathsf{E}\{[U_{(r_k\wedge\tau_n)-}\{\exp[-f(x-W_{N_{(r_k\wedge\tau_n)-}}+1)+f(x)]-1\}\\
&\cdot(N_{r_{k+1}\wedge\tau_n}-N_{r_k\wedge\tau_n})|_{x=R_{(r_k\wedge\tau_n)-}}]|\mathcal{F}_r\}\\
=\ &\lim_{m\longrightarrow+\infty}\sum_{k=1}^{m}\mathsf{E}[I_{(\tau_n>r_k)}U_{r_k^-}\mathsf{E}(\{\exp[-f(x-W_{N_{r_k^-}+1})+f(x)]-1\}\\
&\cdot(N_{r_{k+1}\wedge\tau_n}-N_{r_k\wedge\tau_n})|_{\mathcal{F}_{r_k^-}\vee G})|_{x=R_{r_k^-}}|\mathcal{F}_r]\\
=\ &\lim_{m\longrightarrow+\infty}\sum_{k=1}^{m}\mathsf{E}[I_{(\tau_n>r_k)}U_{r_k^-}(\mathsf{E}\{\exp[-f(x-W)+f(x)]-1\}|_{x=R_{r_k^-}})\\
&\cdot\mathsf{E}(N_{r_{k+1}\wedge z}-N_{r_k\wedge z})|_{z=\tau_n}|\mathcal{F}_r]\\
=\ &\lim_{m\longrightarrow+\infty}\sum_{k=1}^{m}\mathsf{E}[I_{(\tau_n>r_k)}U_{r_k^-}\mathsf{E}(\exp[-f(x-W)+f(x)]-1)|_{x=R_{r_k^-}}\\
&\cdot\lambda(r_{k+1}\wedge\tau_n-r_k\wedge\tau_n)|\mathcal{F}_r]\\
=\ &\mathsf{E}\Big(\int_{r\wedge\tau_n}^{t\wedge\tau_n} U_{s-}\mathsf{E}\{\exp[-f(x-W)+f(x)]-1\}\lambda\mathrm{d}s|_{x=R_{s-}}|\mathcal{F}_r\Big).
\end{aligned}\tag{3.4}$$

The equality (3.4) means the terms

$$\Big(\int_0^t U_{s-}\{\exp[-f(x-W_{N_{s-}+1})+f(x)]-1\}\mathrm{d}N_s\\
-\lambda\int_0^t U_{s-}\mathsf{E}\{\exp[-f(x-W)+f(x)]-1\}\mathrm{d}s|_{x=R_{s-}}|\mathcal{F}_r\Big)_{t\geq 0}$$

is a local martingale, and so does $(U_t,\mathcal{F}_t)_{t\geq 0}$ from (3.3). #

Corollary 3 In addition to the foregoing hypotheses on f and g, we suppose that $f(0)=0$ and $f\geq 0$, $g\geq 0$ on $\Re_-$, then

$$\mathsf{E}\Big[\exp\Big(-\int_T^{T'} g(R_s)\mathrm{d}s\Big)\Big|T<\infty, R_T\Big]=\exp(-f(R_T)).\tag{3.5}$$

Proof Using the optional sampling theorem to the local martingale $(V_t,\mathcal{F})_{t\geq T}$, where $V_t = U_t\exp\Big(\int_0^T g(R_s)\mathrm{d}s\Big)$, we have

$$\mathsf{E}(V_{T'\wedge\tau_n\wedge t}|\mathcal{F}_T)I_{(\tau_n\wedge t>T)}=V_T I_{(\tau_n\wedge t>T)}.$$

But for $T\leq s\leq T'$, $R_s\leq 0$, which with the hypotheses on f and g means that $V_{T'\wedge\tau_n\wedge t}$ is uniformly bounded for $\tau_n\wedge t\geq T$, thus by letting $\tau_n\wedge t\to\infty$ and using the dominated convergence theorem, we get

$$\mathsf{E}(V_{T'}|\mathcal{F}_T, T<\infty)=V_T.$$

Consequently by the strong Markov property of $(R_s)_{s\geq 0}$, we have

$$\mathsf{E}(V_{T'}|T<\infty, R_T)=V_T$$

which is (3.5) due to $R_{T'}=0$, and $f(0)=0$. #

In order to use (3.5) at best we usually choose f to be a polynomial form as in the following lemmas.

Lemma 2 Supposing that $m_k=\mathsf{E}(W^k)$ exits for $k=1,2,\cdots,n,$, we choose ϵ, $0<\epsilon<\min\{1,(c-\lambda\mu)/(\lambda Kn!n)\}$, where $K=\max\{\lambda|b_1|/(c-\lambda\mu),m_2,\cdots,m_n\}$, then when $f(x)=\sum_{j=1}^{n}b_jx^j$, where $b_1\neq 0$, $b_j(-1)^j\geq 0$ for any j, $|b_j|\leq \epsilon^{n-1}|b_1|$ for $j=2,3,\cdots,n$, and g is given by (3.1), the corollary of Theorem 2 is valid.

Proof It is enough to verify the conditions in the corollary of Theorem 2. Following the proof of the lemma in Picard (1994), one can verifies the conditions of the corollary if below inequality

$$(f'(x))^2\geq f''(x) \quad \text{on} \quad \Re_-$$

is valid. In fact, under the conditions of the Lemma 2 the inequality (3.4) in Picard (1994) is still valid when we replace M by K, which leads to

$$(f'(x))^2\geq |b_1||f'(x)|\geq \frac{c-\lambda\mu}{\lambda K}|f'(x)|\geq |f''(x)|\geq f''(x). \qquad \#$$

Let $\mathsf{E}_c(\cdots)$ denote $\mathsf{E}_c(\cdots|T<\infty,R_T)$ in the following parts.

Lemma 3 For any $k=1,2,\cdots,n$

(i) $\mathsf{E}_c(T'-T)^k$ exists when m_k exists,

(ii) E_cM^k exists when m_k exists,

where $M=\max\{|R_t|,T\leq t\leq T'\}$ and $m_1\,\widehat{=}\,\mu$.

Proof (i) Let $f(x)=-\alpha x$, $\alpha>0$. From (3.1) we have $g(x)=c\alpha+D\alpha^2+\lambda(\widehat{p}(\alpha)-1)$, where $\widehat{p}(\alpha)=\int_0^\infty e^{-\alpha y}\mathrm{d}P(y)$. Let

$$\eta=\varphi(\alpha)\,\widehat{=}\,c\alpha+D\alpha^2+\lambda(\widehat{p}(\alpha)-1). \tag{3.6}$$

When m_k exists, it is easy to see that the kth order derivative function $\varphi^{(k)}(\alpha)$, $0\leq\alpha\leq 1$ exists, continues, is bounded and has $\varphi'(\alpha)\not\equiv 0$, $0\leq\alpha\leq 1$. We can conclude that $\eta=\varphi(\alpha)$, $0\leq\alpha\leq 1$ has a unique inverse $\alpha=\varphi^{-1}(\eta)$, and by mathematical induction for $k\geq 1$ we get

$$(\varphi^{-1}(\eta))^{(k)}=\frac{G(\varphi^{(i)}(\alpha),1\leq i\leq k)}{(\varphi'(\alpha))^{2k-1}}\Big|_{\alpha=\varphi^{-1}(\eta)},$$

where $G(x_i,1\leq i\leq k)$ is a polynomial of the variables x_i, $1\leq i\leq k$. By using Lemma 2 and the corollary of Theorem 2 to $f(x)=-\alpha x$ we obtain

$$\mathsf{E}_c[\exp(-\eta(T'-T))]=\exp(-\varphi^{(-1)}(\eta)|R_T|), \qquad \eta\geq 0. \tag{3.7}$$

Taking the kth order derivatives with respect to η on the two side in (3.7), and using Fatou's lemma and the formulae as above, for $k \geq 1$ we have

$$\begin{aligned} \mathsf{E}_c(T'-T)^k &= \lim_{\eta\to 0+} \mathsf{E}_c[(T-T')^k \exp(-\eta(T'-T))] \\ &= \lim_{\eta\to 0+} [\exp(-\varphi^{-1}(\eta)|R_T|)]^{(k)} \cdot (-1)^k \\ &= \exp(-\varphi^{-1}(\eta)|R_T|)^{(k)} \cdot (-1)^k|_{\eta=0} < \infty. \end{aligned}$$

Thus $\mathsf{E}_c(T'-T)^k$ exists if m_k exists.

(ii) Using the formula (2.3) and integrating by parts, for $y = |R_T|$, we have

$$\begin{aligned} \mathsf{E}_c M^k &= -\int_y^\infty z^k \mathrm{d}\Big(1-\frac{\phi(z-y)}{\phi(z)}\Big) \\ &= -z^k\Big(1-\frac{\phi(z-y)}{\phi(z)}\Big)\Big|_y^\infty + k\int_y^\infty z^{k-1}\Big(1-\frac{\phi(z-y)}{\phi(z)}\Big)\mathrm{d}z \\ &\leq -z^k\Big(1-\frac{\phi(z-y)}{\phi(z)}\Big)\Big|_y^\infty + \frac{k}{\phi(y)}\cdot\int_y^\infty z^{k-1}(\psi(z-y)-\psi(z))\mathrm{d}z. \end{aligned} \tag{3.8}$$

Since $\mathsf{E}_c M^k \geq 0$, it is enough to prove $\int_y^\infty z^{k-1}(\psi(z-y)-\psi(z))\mathrm{d}z < \infty$. We see that

$$\begin{aligned} &\int_y^\infty z^{k-1}(\psi(z-y)-\psi(z))\mathrm{d}z \\ = &\int_0^\infty (z+y)^{k-1}\psi(z)\mathrm{d}z - \int_0^\infty z^{k-1}\psi(z)\mathrm{d}z + \int_0^y z^{k-1}\psi(z)\mathrm{d}z \\ = &\sum_{i=1}^{k-1} C_{k-1}^i y^i \int_0^\infty z^{k-1-i}\psi(z)\mathrm{d}z + \int_0^y z^{k-1}\psi(z)\mathrm{d}z. \end{aligned} \tag{3.9}$$

(3.9) has indicated the conclusion is valid for $k = 1$. When $k \geq 2$ from Corollary 1 in Doney (1991), we know that

$$\int_0^\infty e^{-sz}\psi(z)\mathrm{d}z = \frac{1}{s} - \frac{c-\mu\lambda}{\varphi(s)} = \frac{D+\lambda s^{-2}(\widehat{\rho}(s)-1+\mu s)}{c+Ds+\lambda s^{-1}(\widehat{\rho}(s)-1)},$$

which yields

$$\int_0^\infty z^{k-2}\psi(z)\mathrm{d}z = \lim_{s\to 0+}\Big[\frac{D+\lambda s^{-2}(\widehat{p}(s)-1+\mu s)}{c+Ds+\lambda s^{-1}(\widehat{p}(s)-1)}\Big]^{(k-2)}. \tag{3.10}$$

(3.10) indicates $\int_0^\infty z^{k-2}\psi(z)\mathrm{d}z$ exists if $\lim_{s\to 0+}[(1/s^2)\cdot(\widehat{p}(s)-1+\mu s)]^{(k-2)}$ exists. It is easy to see that $(1/s^2)\cdot(\widehat{p}(s)-1+\mu s)$ is the laplace transform of the function $\int_x^\infty (1-P(t))\mathrm{d}t$, with integrating by parts we have

$$\begin{aligned} \lim_{s\to 0+}\Big[\frac{1}{s^2}(\widehat{p}(s)-1+\mu s)\Big]^{(k-2)} &= \lim_{s\to 0+}\int_0^\infty e^{-sx}\cdot(-x)^{k-2}\int_x^\infty (1-P(t)\mathrm{d}t\mathrm{d}x \\ &= (-1)^{k-2}\int_0^\infty x^{k-2}\cdot\int_x^\infty (1-P(t))\mathrm{d}t\mathrm{d}x \\ &= \frac{(-1)^k}{k(k-1)}m_k. \end{aligned} \tag{3.11}$$

From (3.9)–(3.11) we conclude (ii) is valid. #

From the conclusion of Lemma 3, we obtain he following result:

Corollary 4 When m_k exists, $\mathsf{E}_c\int_T^{T'} R_t^{k-1}\mathrm{d}t$ exists.

Proof Using Lemma 3 and Hölder inequality, we have

$$\begin{aligned}\Big|\mathsf{E}_c\int_T^{T'} R_t^{k-1}\mathrm{d}t\Big| &\le \mathsf{E}_c\int_T^{T'}|R_T|^{k-1}\mathrm{d}t \le \mathsf{E}_c M^{k-1}(T-T')\\ &\le (\mathsf{E}_c M^k)^{(k-1)/k}(\mathsf{E}_c(T'-T)^k)^{1/k}<\infty. \qquad \#\end{aligned}$$

Theorem 3 Let $f_i(x)=\sum\limits_{j=1}^{n_i} b_j^{(i)}x^j$, $i=1,2$ are chosen as in Lemma 2, and for α_1, $\alpha_2\ge 0$, let

$$\begin{aligned}g(x,\alpha_1,\alpha_2) &= -\lambda-\sum_{i=1}^2\alpha_i(cf_i'(x)+Df_i''(x))\\ &\quad +\lambda\mathsf{E}\Big\{\exp\Big[\sum_{i=1}^2\alpha_i(f_i(x)-f_i(x-W))\Big]\Big\}+\sum_{i=1}^2 D(\alpha_i f'(x))^2.\end{aligned}$$

If $m_{n_1+n_2}$ exists, then

$$\begin{aligned}&\mathsf{E}_c\Big[\prod_{i=1}^2\Big(\int_T^{T'} g'_{\alpha_i}(R_t,0,0)\mathrm{d}t-\lambda\int_T^{T'}\mathsf{E}\prod_{i=1}^2(f_i(x)-f_i(x-W))\Big|_{x=R_t}\mathrm{d}t\\ &-2D\int_T^{T'} f_1(R_t)f_2(R_t)\mathrm{d}t\Big]=f_1(R_T)\cdot f_2(R_T).\end{aligned}\tag{3.12}$$

Proof Let $f=\alpha_1f_1+\alpha_2f_2$, then $g(x)=g(x,\alpha_1,\alpha_2)$. Obviously $g(x,0,0)=0$ and for $x\le 0$

$$\begin{aligned}&|g'_{\alpha_i}(x,\alpha_1,\alpha_2)|\\ \le\ & c|f_i'(x)|+D|f_i''(x)|+\lambda\mathsf{E}[f_i(x-W)-f_i(x)]+2D\alpha_i(f_i'(x))^2, \qquad i=1,2.\end{aligned}\tag{3.13}$$

Using Hölder inequality and Lemma 2, we see that

$$\begin{aligned}\mathsf{E}_c\Big[\int_T^{T'}(f_i'(R_t))^2\mathrm{d}t\Big]^{1/2} &\le \mathsf{E}_c[(T'-T)^{1/2}M^{n_i-1}n_i^2|b_1^{(i)}|]\\ &\le (\mathsf{E}_cM^{n_i})^{(n_i-1)/n_i}(\mathsf{E}_c(T'-T)^{n_i/2})^{1/n_i}n_i^2|b_1^{(i)}|\\ &< \infty.\end{aligned}\tag{3.14}$$

It is well known that

$$2axe^{-ax^2}\le\Big(\frac{2a}{e}\Big)^{1/2},\tag{3.15}$$

for $a\ge 0$ and $x\ge 0$, espcially, letting $x=\alpha_i$, $a=D\int_T^{T'}(f_i'(R_s))^2\mathrm{d}s$ and using (3.14), we have

$$\begin{aligned}&\mathsf{E}_c\Big[\exp\Big(-\alpha_i^2D\int_T^{T'}(f_i'(R_s))^2\Big)\mathrm{d}t\Big]\\ \le\ & D\Big(\frac{2}{e}\Big)^{1/2}n_i^2|b_1^{(i)}|(\mathsf{E}_cM^{n_i})^{(n_i-1)/n_i}(\mathsf{E}_c(T'-T)^{n_i/2})^{1/n_i}.\end{aligned}\tag{3.16}$$

Note that the first three terms on the right side of (3.13) is a polynomial of degree $n_i - 1$ with respect to x and independent of α_i. With the dominated convergence theorem and (3.13)–(3.16), we are allowed to write

$$\begin{aligned}&\frac{\partial}{\partial\alpha_i}\mathsf{E}_c\Big[\exp\Big(-\int_T^{T'} g(R_t,\alpha_1,\alpha_2)\mathrm{d}t\Big)\Big]\\ =\ &\mathsf{E}_c\Big[-\exp\Big(-\int_T^{T'} g(R_t,\alpha_1,\alpha_2)\mathrm{d}t\Big)\int_T^{T'} g'_{\alpha_i}(R_t,\alpha_1,\alpha_2)\mathrm{d}t\Big].\end{aligned}\tag{3.17}$$

By a similar analysis we can further use differentiation with respect to α_j $(j\neq i)$ and then we get

$$\begin{aligned}&\frac{\partial^2}{\partial_i\partial_j}\mathsf{E}_c\Big\{\exp\Big(-\int_T^{T'} g(R_t,\alpha_1,\alpha_2)\mathrm{d}t\Big)\Big\}\\ =\ &\mathsf{E}_c\Big\{\exp\Big(-\int_T^{T'} g(R_t,\alpha_1,\alpha_2)\mathrm{d}t\Big)\Big[\prod_{i=1}^{2}\int_T^{T'} g'_{\alpha_i}(R_t,\alpha_1,\alpha_2)\mathrm{d}t\\ &-\lambda\int_T^{T'}\mathsf{E}\Big\{\exp\Big[\sum_{i=1}^{2}\alpha_i(f_i(x)-f_i(x-W))\Big]\prod_{i=1}^{2}(f_i(x)-f_i(x-W))\Big\}\Big|_{x=R_t}\mathrm{d}t\Big]\\ &-2D\int_T^{T'} f_1(R_t)f_2(R_t)\mathrm{d}t\Big\}.\end{aligned}\tag{3.18}$$

Using Lemma 2 and letting $\alpha_i=0,\ i=1,2$, we get (3.12).　　#

Remark 2　In Picard (1994) since $g''_{r^2}(x,r)$ includes the term $(f(x)-f(x-W))^2$ which is a polynomial of degree $2n-2$ with repeat to x, we see that the condition that m_n exists is not enough to the validity of the formula (3.9) in Picard (1994). In fact an enough condition is that m_{2n} exists.

Corollary 5　If m_n exists, then

$$(c-\lambda\mu)\mathsf{E}_c(T'-T)=-R_T\tag{3.19}$$

and

$$\begin{aligned}&(c-\lambda\mu)\mathsf{E}_c\Big[\int_T^{T'} R_t^{j-1}\mathrm{d}t+\lambda\sum_{k=2}^{j}\binom{j}{k}m_k\mathsf{E}_c\Big(\int_T^{T'} R_t^{j-k}\mathrm{d}t\Big)\Big]\\ &+D(j-1)j\cdot\mathsf{E}_c\Big(\int_T^{T'} R_t^{j-2}\mathrm{d}t\Big)=-R_T^j,\qquad j=2,\cdots,n.\end{aligned}\tag{3.20}$$

Proof　Letting $i=1$ and then $\alpha_1=\alpha_2=0$ in (3.17) we have

$$\mathsf{E}_c\Big[\int_T^{T'} g'_{\alpha_1}(R_t,0,0)\mathrm{d}t\Big]=f_1(R_T).\tag{3.21}$$

Note that

$$\begin{aligned}g'_{\alpha_i}(x,0,0)\ &=\ -cf'_i(x)-Df''_i(x)-\lambda\sum_{r=1}^{n_i}(-1)^r\frac{m_r}{r!}f_i^{(r)}(x)\\ &=\ -(c-\lambda\mu)f'_i(x)-Df''_i(x)\\ &\quad-\lambda\sum_{r=2}^{n_i}(-1)^r\frac{m_r}{r!}\sum_{j=r}^{n_i}j(j-1)\cdots(j-r+1)\cdot b_j^{(i)}x^{j-r},\qquad i=1,2.\end{aligned}\tag{3.22}$$

With $i=1$ we substitute (3.22) in (3.21) and then the identification of terms in $b_j^{(1)}$ leads to (3.19) and (3.20). #

Corollary 6

$$\mathsf{E}_c\Big(\int_T^{T'}|R_t|\mathrm{d}t\Big)=\frac{R_T^2}{2l}+\frac{h}{2l^2}|R_T|,$$
$$\mathsf{Var}_c(T'-T)=\frac{h}{l^3}|R_T|,$$
$$\mathsf{Cov}_c\Big(T'-T,\int_T^{T'}|R_t|\mathrm{d}t\Big)=\frac{h}{2l^3}R_T^2+\Big(\frac{h^2}{l}+\frac{\lambda m_3}{2}\Big)\frac{|R_T|}{l^3},$$
$$\mathsf{Var}_c\Big(\int_T^{T'}|R_t|\mathrm{d}t\Big)=\frac{h|R_T|^3}{3l^3}+\Big(\frac{h^2}{l}+\frac{\lambda m_3}{2}\Big)\frac{R_T^2}{l^3}+\Big(\frac{5h^3}{4l^2}+\frac{4\lambda m_3 h}{3l}+\frac{\lambda m_4}{4}\Big)\frac{|R_T|}{l^3},$$
$$\mathsf{E}_c\Big(\int_T^{T'}R_t^2\mathrm{d}t\Big)=\frac{|R_T|^3}{3l}+\frac{hR_T^2}{2l^2}+\Big(\frac{h^2}{2l^2}+\frac{\lambda m_3}{3}\Big)\frac{|R_T|}{l^2},$$

where l denotes $c-\lambda\mu$ and h denotes $2D+\lambda m_2$.

Proof Use (3.20) for $j=2$ and $j=3$. In (3.12) put $f_1(x)=f_2(x)=b_1x+b_2x$ and identify. #

Remark 3 Replacing $2D+\lambda m_2$ by λm_2 with all of the formulae in this corollary we obtain correspondent formulae in Picard (1994).

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带干扰风险模型的破产严重性及其恢复代价的测量

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本文针对带干扰风险模型考虑了由 Picard (1994) 引进的破产最大严重程度和恢复所需代价的概念以及对它们的测量问题，并给出了相应于 Picard (1994) 的各种公式的明确表达.

关 键 词: 破产的最大严重程度，恢复所需代价，局部鞅，伊藤公式.

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