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The Average Run Lengths of Control Charts for Stable Lévy Processes*

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Abstract

As we know that the average run length (ARL) is an extensively used measure in statistical process control (SPC) for evaluating and comparing the detection performance of various control charts. In this paper we not only present the asymptotic estimation of the ARL for the exponentially weighted moving average (EWMA), generalized EWMA (GEWMA) and generalized likelihood ratio (GLR) control charts but also compare the detection performance by the numerical simulation among the four charts: EWMA, GEWMA, GLR and CUSUM in detecting the mean change of a stable Lévy process.

Keywords: Change point detection, average run length, stable Lévy process, EWMA, GEWMA, GLR, CUSUM.

AMS Subject Classification: 93C83.

§ 1. Introduction

The problem of quick detection of change point in a stochastic system has many important applications, including industrial quality control, automated fault detection in controlled dynamical systems, segmentation of signals, and so on. To deal with the problem, various control charts, such as Shewhart chart, the CUSUM, EWMA, GEWMA and GLR control charts, etc., were proposed. Many studies on these control charts have been conducted by Crowder (1989), Lucas and Saccucci (1990), Montgomery and Mastrangelo (1991), Baxley (1995), Lai (1995), Reynolds (1996), Box and Luceno (1997), Ramirez (1998), Hawkins and Olwell (1998), Luceno (1999), Mastrangelo and Brown (2000), Jiang, Tsui and Woodall (2000), Jones, Champ and Rigdon (2001), Shu, Apley and Tsung (2002), and Han and Tsung (2004).

As can be seen that all research work in the above literature is based on an assumption that the observation stochastic system (or process) is subject to the normal distribution or the variance of the process is finite. In fact, there are many stochastic systems such as the bankroll

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trading volume in the finance network, the return rate of stock market, temperature distributions in unclear reactors and the volume of annual rainfall, which are neither normal or have finite variance. These stochastic systems are usually subject to a stable distribution with infinite variance, which is called a stable Lévy process. Thus, there are two interesting problems in the following which have received minimal attention. How the popular control charts such as the EWMA, CUSUM and GLR can be effectively used in the stable Lévy process? What is the performance of the control charts for monitoring the change point in the process? One way to deal with the second problem is to estimate the ARL of the control chart, since the estimation of the ARL can establish the clear asymptotic relations between the ARL, control limit and statistical properties of the observation processes.

Although an estimation of the ARL for the CUSUM chart has been given in [19], the study of ARL for the EWMA, GEWMA and GRL control charts in detecting the mean changes of a stable Lévy process is lack. The main purpose of this paper is to present the asymptotic estimation of the ARL for the EWMA, GEWMA and GRL control charts and give a numerical comparison of the three charts in detecting the mean shifts of the stable Lévy process.

§ 2. Estimation of the ARL — Some Inequalities

We first give the definition of a stable Lévy process (see ref. [1]).

Definition 1 We call $X \sim S_\alpha(\sigma, \beta, \mu)$ to be a stable random variable, if there exists parameter $0 < \alpha \le 2$, $\sigma \ge 0$, $-1 \le \beta \le 1$, and real number μ, such that the character function of the random variable has the following form:

$$\mathsf{E}[\exp\{i\theta X\}] = \begin{cases} \exp\Big\{ -\sigma^\alpha|\theta|^\alpha\Big(1 - i\beta(\mathrm{sign}\theta)\tan\dfrac{\pi\alpha}{2}\Big) + i\mu\theta\Big\}, & \text{when} \quad \alpha \ne 1; \\ \exp\Big\{ -\sigma|\theta|\Big(1 + i\beta\dfrac{2}{\pi}(\mathrm{sign}\theta)\ln|\theta|\Big) + i\mu\theta\Big\}, & \text{when} \quad \alpha = 1. \end{cases} \tag{1}$$

Definition 2 A stable Lévy process is a Lévy process $X = \{X(t), t \ge 0\}$ (i.e. X is stochastically continuous, has independent and stationary increments) in which each $X(t)$ is a stable random variable.

Next we write the definitions of the three control charts, EWMA, GEWMA, and GRL in the following.

EWMA:

$$T_{\mathrm{E}}(c) = \inf\{n \ge 1 : \overline{W}_n(r) \ge c\}, \tag{2}$$

$$\overline{W}_n(r) = \frac{W_n(r)}{\sigma_{W_n}} = \frac{\sqrt{2-r}}{\sqrt{r[1-(1-r)^{2n}]}}\sum_{i=0}^{n-1} r(1-r)^i X_{n-i}, \tag{3}$$

$$W_n(r) = rX_n + (1-r)W_{n-1}(r), \qquad W_0(r) = 0, \tag{4}$$

where $0 < r \le 1$, σ_{W_n} is the standard variance of $W_n(r)$.

GLR:

$$T_{\rm GL}(c)=\inf\Big\{n\geq 1:\max_{1\leq k\leq n}\Big(\sum_{i=n-k+1}^{n}X_i/k^{1/2}\Big)\geq c\Big\}. \tag{5}$$

Han and Tsung (2004) proposed to replace the weight parameter r by $1/(1+i)$ in the EWMA so that $r(1-r)^i$ gets maximum. Thus we have

GEWMA:

$$T_{\rm GE}(c)=\inf\Big\{n\geq 1:\max_{1\leq k\leq n}\Big[\frac{\sqrt{2-1/k}}{\sqrt{(1/k)\cdot[1-(1-1/k)^{2n}]}}\sum_{i=0}^{n-1}\frac{1}{k}\Big(1-\frac{1}{k}\Big)^iX_{n-i}\Big]\geq c\Big\}. \tag{6}$$

We list two known lemmas (see ref. [16]) in the following which will be used in proving the main results.

Lemma 1 X_1,X_2 are two independent variables, $X_i\sim S_\alpha(\sigma_i,\beta_i,\mu_i)$, $i=1,2$, then

$$X_1+X_2\sim S_\alpha(\sigma,\beta,\mu),$$

where $S_\alpha(\sigma_i,\beta_i,\mu_i)$ denotes the stable distribution with the parameter α, $\sigma=(\sigma_1^\alpha+\sigma_2^\alpha)^{1/\alpha}$, $\beta=(\beta_1\sigma_1^\alpha+\beta_2\sigma_2^\alpha)/(\sigma_1^\alpha+\sigma_2^\alpha)$ and $\mu=\mu_1+\mu_2$.

Lemma 2 $X\sim S_\alpha(\sigma,\beta,\mu)$, $0<\alpha<2$, then

$$\begin{cases}\lim\limits_{\lambda\to\infty}\lambda^\alpha\mathsf{P}(X>\lambda)=C_\alpha\dfrac{1+\beta}{2}\sigma^\alpha;\\ \lim\limits_{\lambda\to\infty}\lambda^\alpha\mathsf{P}(X<-\lambda)=C_\alpha\dfrac{1-\beta}{2}\sigma^\alpha,\end{cases}$$

where

$$C_\alpha=\begin{cases}\dfrac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}, & \text{when}\ \ \alpha\neq 0,\\ 2/\pi, & \text{when}\ \ \alpha=1.\end{cases} \tag{7}$$

To compare the performance of the three charts for detecting the mean change $\{\mu_i\}$ in the stable process, we need some of corresponding notation. Let $\mathsf{P}(\cdot)$ and $\mathsf{E}(\cdot)$ denote the probability and expectation operators when there is no change. Denote $\mathsf{P}_{\tau\mu_i}(\cdot)$ and $\mathsf{E}_{\tau\mu_i}(\cdot)$ as the probability and expectation when the change point is at τ and the mean change value is $\{\mu_i\}$. When $\mu_i\equiv\mu$, μ is usually called the mean shift value. The two most commonly used operating measures in SPC are the in-control average run length (ARL_0) and the out-of-control average run length (ARL_μ), defined by

$$\text{ARL}_0(T)=\mathsf{E}(T),\qquad \text{ARL}_{\mu_i}(T)=\mathsf{E}_{\mu_i}(T),$$

where T is a stopping time (or the alarming time) outside a control limit with a detecting procedure. When $\mu_i\equiv\mu$ and $\tau=1$, we denote $\mathsf{E}_{\mu_i}(T)$ by $\text{ARL}_\mu(T)$.

Usually, comparisons of control chart performance are made by designing the charts to have a common ARL_0 and then comparing the ARL_{μ_i}'s of the control charts for a given change μ_i and a change point τ. The chart with the smaller ARL_{μ_i} is considered to have better performance.

Now we mention and prove our main theorems. To simplify the proof of the theorems we assume that $\tau=1$.

Theorem 1 For i.i.d. (independent identity distribution) random variables $X_1, X_2, \cdots, X_j \sim S_\alpha(1, 0, \mu_j)$, let $\underline{\mu} = \inf[\mu]$, $\overline{\mu} = \sup[\mu]$, where $1 < \alpha \leq 2$, $\mu_j \geq 0$, then for EWMA control charts, we have

$$\mathrm{ARL}_{\mu_i}(T_{\mathrm{E}}) \geq \frac{2[c\sqrt{r} - \overline{\mu}\sqrt{2-r}]^\alpha}{C_\alpha(\sqrt{2-r})^\alpha} - 1.$$

Furthermore, if X_i is positive random variable, then we have

$$\mathrm{ARL}_{\mu_i}(T_{\mathrm{E}}) \leq \frac{2[c - \underline{\mu}\sqrt{r(2-r)}]^\alpha}{C_\alpha(\sqrt{r(2-r)})^\alpha} - 1,$$

where $C_\alpha = (1-\alpha)/[\Gamma(2-\alpha)\cos(\pi\alpha/2)]$ and c big enough.

Proof For

$$T_{\mathrm{E}}(c) = \inf\Big\{n \geq 1 : \frac{\sqrt{2-r}}{\sqrt{r[1-(1-r)^{2n}]}} \sum_{i=0}^{n-1} r(1-r)^i X_{n-i} \geq c\Big\}, \tag{8}$$

we have

$$\begin{aligned}
\mathrm{ARL}_{\mu_i}(T_{\mathrm{E}}) &= \sum_{m=1}^{\infty} \mathsf{P}(T_{\mathrm{E}} > m) \\
&= \sum_{m=1}^{\infty} \mathsf{P}\Big(\frac{\sqrt{2-r}}{\sqrt{r[1-(1-r)^{2n}]}} \sum_{i=0}^{n-1} r(1-r)^i X_{n-i} < c,\ 1 \leq n \leq m\Big) \\
&= \sum_{m=1}^{\infty} \mathsf{P}\Big(\sum_{i=0}^{n-1} r(1-r)^i X_{n-i} < \frac{c\sqrt{r[1-(1-r)^{2n}]}}{\sqrt{2-r}},\ 1 \leq n \leq m\Big) \\
&\geq \sum_{m=1}^{\infty} \mathsf{P}\Big(X_{n-i} < \frac{c\sqrt{r}}{\sqrt{2-r}}\sqrt{-1 + \frac{2}{1-(1-r)^n}},\ 0 \leq i \leq n-1,\ 1 \leq n \leq m\Big),
\end{aligned}$$

for $0 \leq n \leq m$, we get

$$\sqrt{-1 + \frac{2}{1-(1-r)^n}} \geq \sqrt{-1 + \frac{2}{1-(1-r)^m}},$$

so

$$\begin{aligned}
\mathrm{ARL}_{\mu_i}(T_{\mathrm{E}}) &= \sum_{m=1}^{\infty} \mathsf{P}\Big(X_n < \frac{c\sqrt{r}}{\sqrt{2-r}}\sqrt{-1 + \frac{2}{1-(1-r)^m}},\ 1 \leq n \leq m\Big) \\
&= \sum_{m=1}^{\infty} \mathsf{P}\Big(X_n - \mu_n < \frac{c\sqrt{r}}{\sqrt{2-r}}\sqrt{-1 + \frac{2}{1-(1-r)^m}} - \mu_n,\ 1 \leq n \leq m\Big) \\
&\geq \sum_{m=1}^{\infty} \mathsf{P}\Big(X_n - \mu_n < \frac{c\sqrt{r}}{\sqrt{2-r}} - \overline{\mu},\ 1 \leq n \leq m\Big) \\
&= \sum_{m=1}^{\infty} \Big[1 - \frac{C_\alpha}{2}\Big(\frac{\sqrt{2-r}}{c\sqrt{r} - \overline{\mu}\sqrt{2-r}}\Big)^\alpha\Big]^m \\
&= \frac{2[c\sqrt{r} - \overline{\mu}\sqrt{2-r}]^\alpha}{C_\alpha(\sqrt{2-r})^\alpha} - 1.
\end{aligned}$$

If X_i is positive random variable, then

$$\begin{aligned}\mathrm{ARL}_{\mu_i}(T_{\mathrm{E}}) &= \sum_{m=1}^{\infty}\mathsf{P}(T_{\mathrm{E}}>m)\\ &= \sum_{m=1}^{\infty}\mathsf{P}\Big(\frac{\sqrt{2-r}}{\sqrt{r[1-(1-r)^{2n}]}}\sum_{i=0}^{n-1}r(1-r)^iX_{n-i}<c,\ 1\le n\le m\Big)\\ &\le \sum_{m=1}^{\infty}\mathsf{P}\Big(X_n<\frac{c\sqrt{1-(1-r)^{2n}}}{\sqrt{r(2-r)}},\ 1\le n\le m\Big).\end{aligned}$$

Note that X_i $(i=1,2,\cdots)$ is i.i.d. random variables. It follows from Lemma 2 that

$$\begin{aligned}&\sum_{m=1}^{\infty}\mathsf{P}\Big(X_n<\frac{c\sqrt{1-(1-r)^{2n}}}{\sqrt{r(2-r)}},\ 1\le n\le m\Big)\\ =&\sum_{m=1}^{\infty}\mathsf{P}\Big(X_n-\mu_n<\frac{c\sqrt{1-(1-r)^{2n}}}{\sqrt{r(2-r)}}-\mu_n,\ 1\le n\le m\Big)\\ \le&\sum_{m=1}^{\infty}\mathsf{P}\Big(X_n-\mu_n<\frac{c}{\sqrt{r(2-r)}}-\underline{\mu},\ 1\le n\le m\Big)\\ \approx&\sum_{m=1}^{\infty}\Big[1-\frac{C_\alpha}{2}\Big(\frac{\sqrt{r(2-r)}}{c-\underline{\mu}\sqrt{r(2-r)}}\Big)^{\alpha}\Big]^m\\ =&\frac{2(c-\underline{\mu}\sqrt{r(2-r)})^{\alpha}}{C_\alpha(\sqrt{r(2-r)})^{\alpha}}-1.\end{aligned}$$

So the second inequality is proved. #

Theorem 2 For i.i.d. random variables $X_1,X_2,\cdots,X_j\sim S_\alpha(1,0,\mu_j)$, let $\underline{\mu}=\inf[\mu]$, $\overline{\mu}=\sup[\mu]$, where $1<\alpha\le 2$, $\mu_j\ge 0$, then for GEWMA control charts, we have

$$\mathrm{ARL}_{\mu_i}(T_{\mathrm{GE}})\ge c^{2\alpha/(\alpha+4)}e^{-C_\alpha 2^{\alpha/2-1}[(e-1)/(e+1)]^{\alpha/2}},\tag{9}$$

where c is big enough.

Proof It follows that

$$T_{\mathrm{GE}}(c)=\inf\Big\{n\ge 1:\max_{1\le k\le n}\Big[\frac{\sqrt{2-1/k}}{\sqrt{(1/k)\cdot[1-(1-1/k)^{2n}]}}\sum_{i=0}^{n-1}\frac{1}{k}\Big(1-\frac{1}{k}\Big)^iX_{n-i}\Big]\ge c\Big\}.\tag{10}$$

Then

$$\begin{aligned}&\mathrm{ARL}_{\mu_i}(T_{\mathrm{GE}})\\ =&\sum_{m=1}^{\infty}\mathsf{P}(T_{\mathrm{GE}}>m)\\ =&\sum_{m=1}^{\infty}\mathsf{P}\Big(\frac{\sqrt{2-1/k}}{\sqrt{(1/k)\cdot[1-(1-1/k)^{2n}]}}\sum_{i=0}^{n-1}\frac{1}{k}\Big(1-\frac{1}{k}\Big)^iX_{n-i}<c,\ 1\le k\le n,\ 1\le n\le m\Big)\\ =&\sum_{m=1}^{\infty}\mathsf{P}\Big(\sum_{i=0}^{n-1}\frac{1}{k}\Big(1-\frac{1}{k}\Big)^iX_{n-i}<\frac{\sqrt{1/k}}{\sqrt{2-1/k}}\frac{\sqrt{1+(1-1/k)^n}}{\sqrt{1-(1-1/k)^n}}\sum_{i=0}^{n-1}\frac{1}{k}\Big(1-\frac{1}{k}\Big)^ic,\\ &1\le k\le n,\ 1\le n\le m\Big)\end{aligned}$$

$$\begin{aligned}
&\ge \sum_{m=1}^{\infty} \mathsf{P}\Big(X_i < \frac{\sqrt{1/k}}{\sqrt{2-1/k}}\frac{\sqrt{1+(1-1/k)^n}}{\sqrt{1-(1-1/k)^n}}c,\ 1\le i\le n,\ 1\le k\le n,\ 1\le n\le m\Big)\\
&= \lim_{N_1\to\infty}\sum_{m=1}^{N_1} \mathsf{P}\Big(X_i < \frac{\sqrt{1/k}}{\sqrt{2-1/k}}\frac{\sqrt{1+(1-1/k)^n}}{\sqrt{1-(1-1/k)^n}}c,\ 1\le i\le n,\ 1\le k\le n,\ 1\le n\le m\Big)\\
&\ge \lim_{N_1\to\infty}\sum_{m=1}^{N_1} \mathsf{P}\Big(X_i < \frac{\sqrt{1/k}}{\sqrt{2-1/k}}\frac{\sqrt{1+(1-1/k)^{N_1}}}{\sqrt{1-(1-1/k)^{N_1}}}c,\ 1\le i\le n,\ 1\le k\le n,\ 1\le n\le m\Big).
\end{aligned}$$

Let

$$g(a)=\frac{(1-a)(1+a^n)}{(1+a)(1-a^n)}.$$

So

$$g'(a)=\frac{(1-a^2)(1+a^2+\cdots+a^{2(n-1)}-na^{n-1})}{(1+a)(1-a^n)^2},$$

$$\sqrt{g\Big(1-\frac{1}{k}\Big)}=\frac{\sqrt{1/k}}{\sqrt{2-1/k}}\frac{\sqrt{1+(1-1/k)^n}}{\sqrt{1-(1-1/k)^n}}.$$

When $0\le a<1$, we have $g'(a)\le 0$. If $k\le N_1$, then

$$\frac{\sqrt{1/k}}{\sqrt{2-1/k}}\frac{\sqrt{1+(1-1/k)^{N_1}}}{\sqrt{1-(1-1/k)^{N_1}}}\ge\frac{\sqrt{1/N_1}}{\sqrt{2-1/N_1}}\frac{\sqrt{1+(1-1/N_1)^{N_1}}}{\sqrt{1-(1-1/N_1)^{N_1}}}.$$

Let $N_1=c^{2\alpha/(\alpha+4)}$. We have

$$\begin{aligned}
\mathrm{ARL}_{\mu_i}(T_{\mathrm{GE}}) &\ge \sum_{m=1}^{N_1}\mathsf{P}\Big(X_i<\frac{\sqrt{1/N_1}}{\sqrt{2-1/N_1}}\frac{\sqrt{1+(1-1/N_1)^{N_1}}}{\sqrt{1-(1-1/N_1)^{N_1}}}c,\ 1\le i\le m\Big)\\
&= \sum_{m=1}^{N_1}\mathsf{P}\Big(X_i-\mu_j<\frac{\sqrt{1/N_1}}{\sqrt{2-1/N_1}}\frac{\sqrt{1+(1-1/N_1)^{N_1}}}{\sqrt{1-(1-1/N_1)^{N_1}}}c-\mu_j,\ 1\le i\le m\Big)\\
&\ge \sum_{m=1}^{N_1}\mathsf{P}\Big(X_i-\mu_j<\frac{\sqrt{1/N_1}}{\sqrt{2-1/N_1}}\frac{\sqrt{1+(1-1/N_1)^{N_1}}}{\sqrt{1-(1-1/N_1)^{N_1}}}c-\overline{\mu},\ 1\le i\le m\Big).
\end{aligned}$$

Let

$$u=\frac{\sqrt{1/N_1}}{\sqrt{2-1/N_1}}\frac{\sqrt{1+(1-1/N_1)^{N_1}}}{\sqrt{1-(1-1/N_1)^{N_1}}}c-\overline{\mu}.$$

Then

$$\mathsf{P}\Big(X_j-\mu_j<\frac{\sqrt{1/N_1}}{\sqrt{2-1/N_1}}\frac{\sqrt{1+(1-1/N_1)^{N_1}}}{\sqrt{1-(1-1/N_1)^{N_1}}}c-\overline{\mu}\Big)\approx 1-\frac{C_\alpha}{2u^\alpha}.$$

So

$$\begin{aligned}
\mathrm{ARL}_{\mu_i}(T_{\mathrm{GE}}) &\ge N_1\Big(1-\frac{C_\alpha}{2u^\alpha}\Big)^{N_1}\\
&\approx N_1\Big(1-C_\alpha\Big/\Big[2\Big(\frac{\sqrt{1/N_1}}{\sqrt{2-1/N_1}}\frac{\sqrt{1+(1-1/N_1)^{N_1}}}{\sqrt{1-(1-1/N_1)^{N_1}}}c-\overline{\mu}\Big)^\alpha\Big]\Big)^{N_1}.
\end{aligned}$$

Note that c is big enough, so dose N_1. We have

$$N_1\Big(1-C_\alpha\Big/\Big[2\Big(\frac{\sqrt{1/N_1}}{\sqrt{2-1/N_1}}\frac{\sqrt{1+(1-1/N_1)^{N_1}}}{\sqrt{1-(1-1/N_1)^{N_1}}}c-\overline{\mu}\Big)^\alpha\Big]\Big)^{N_1}\approx c^{2\alpha/(\alpha+4)}e^{-C_\alpha 2^{\alpha/2-1}[(e-1)/(e+1)]^{\alpha/2}}.$$

This completes the proof of Theorem 2. #

Theorem 3 For i.i.d. random variables $X_1, X_2, \cdots, X_j \sim S_\alpha(1,0,\mu_j)$, let $\underline{\mu} = \inf[\mu] > 0$, $\overline{\mu} = \sup[\mu]$ and $2/\alpha < \beta < 2$, where $1 < \alpha \le 2$, $\mu_j \ge 0$, then for GLR control charts, we have

$$c^{2\alpha/(6-\alpha)} e^{-C_\alpha/4} \le \mathrm{ARL}_{\mu_i}(T_{\mathrm{GL}}) \le \frac{c^2}{\underline{\mu}^2} + \frac{c^\beta}{\underline{\mu}} + o(1) \tag{11}$$

for large c.

Proof Since

$$T_{\mathrm{GL}}(c) = \inf\Big\{n \ge 1 : \max_{1\le k\le n}\Big(\sum_{i=n-k+1}^{n} X_i/k^{1/2}\Big) \ge c\Big\},$$

it follows that

$$\begin{aligned}
& \mathrm{ARL}_{\mu_i}(T_{\mathrm{GL}}) \\
= & \sum_{m=1}^{\infty} \mathsf{P}(T_{\mathrm{GL}} > m) \\
= & \sum_{m=1}^{\infty} \mathsf{P}\Big(\Big(\sum_{i=n-k+1}^{n} X_i/k^{1/2}\Big) < c,\ 1 \le k \le n,\ 1 \le n \le m\Big) \\
= & \sum_{m=1}^{\infty} \mathsf{P}\Big(\Big(\sum_{i=n-k+1}^{n} (X_i-\mu_i)/k^{1/\alpha}\Big) < \Big(ck^{1/2} - \sum_{i=n-k+1}^{n}\mu_i\Big)\Big/k^{1/\alpha},\ 1 \le k \le n,\ 1 \le n \le m\Big) \\
\ge & \sum_{m=1}^{\infty} \mathsf{P}\Big(\Big(\sum_{i=n-k+1}^{n} (X_i-\mu_i)/k^{1/\alpha}\Big) < \frac{ck^{1/2}-k\overline{\mu}}{k^{1/\alpha}},\ 1 \le k \le n,\ 1 \le n \le m\Big).
\end{aligned}$$

Let $Y_{n,k} = \sum\limits_{i=n-k+1}^{n} (X_i-\mu_i)/k^{1/\alpha}$, $1 \le k \le n$, According to Lemma 1 and 2, we get $Y_{n,k} \sim S_\alpha(1,0,0)$. By Theorem 5.1 in Esary (1967), we have

$$\begin{aligned}
& \mathsf{P}\Big(\Big(\sum_{i=n-k+1}^{n} (X_i-\mu_i)/k^{1/\alpha}\Big) < \frac{ck^{1/2}-k\overline{\mu}}{k^{1/\alpha}},\ 1 \le k \le n,\ 1 \le n \le m\Big) \\
\ge\ & \mathsf{P}\Big(Y_{n,k} < \frac{ck^{1/2}-k\overline{\mu}}{k^{1/\alpha}},\ 1 \le k \le n,\ 1 \le n \le m\Big).
\end{aligned}$$

Now let

$$f(k) = \frac{ck^{1/2}-k\mu_i}{k^{1/\alpha}} = ck^{1/2-1/\alpha} - \mu_i k^{1-1/\alpha}.$$

Then

$$f'(k) = \Big(\frac{1}{2}-\frac{1}{\alpha}\Big)ck^{-(1/2+1/\alpha)} - \mu_i\Big(1-\frac{1}{\alpha}\Big)k^{1-1/\alpha},$$

when $1 \le \alpha \le 2$, we have

$$\frac{1}{2}-\frac{1}{\alpha} \le 0, \qquad 1-\frac{1}{\alpha} \ge 0.$$

So $f'(k) \le 0$ when $1 \le \alpha \le 2$. Thus $f(k)$ can arrive its minimum at the point N_1. Let $N_1 =$

$c^{2\alpha/(6-\alpha)}$, then

$$\begin{aligned}\mathrm{ARL}_{\mu_i}(T_{\mathrm{GL}}) &\geq \sum_{m=1}^{\infty}\mathsf{P}\Big(Y_{n,k}<\frac{ck^{1/2}-k\overline{\mu}}{k^{1/\alpha}},\ 1\leq k\leq n,\ 1\leq n\leq m\Big)\\ &\geq \sum_{m=1}^{N_1}\Big[\mathsf{P}\Big(Y_{n,k}<\frac{cN_1^{1/2}-N_1\overline{\mu}}{N_1^{1/\alpha}}\Big)\Big]^{m(m+1)/2}\\ &\geq \sum_{m=1}^{N_1}\Big[\mathsf{P}\Big(Y_{n,k}<\frac{cN_1^{1/2}-N_1\overline{\mu}}{N_1^{1/\alpha}}\Big)\Big]^{N_1(N_1+1)/2}\\ &\approx N_1\Big[1-\frac{C_\alpha N_1}{2(cN_1^{1/2}-N_1\overline{\mu})^{\alpha}}\Big]^{N_1(N_1+1)/2}.\end{aligned}$$

Because

$$N_1\Big[1-\frac{C_\alpha N_1}{2(cN_1^{1/2}-N_1\overline{\mu})^{\alpha}}\Big]^{N_1(N_1+1)/2}\approx c^{2\alpha/(6-\alpha)}e^{-C_\alpha/4},$$

we get the left inequality.

We next prove the upper inequality. Let $m(c)=c^2/\underline{\mu}^2+c^{\beta}/\underline{\mu}$ and $m_j=m(c)j$, $j\geq 0$. Since $\Big\{\sum\limits_{i=m_{j-1}+1}^{m_j}(X_i-\mu_i)\Big\}$, $1\leq j\leq k$, are mutually independent and have a identity distribution, it follows that

$$\begin{aligned}&\mathsf{P}\Big(\Big(\sum_{i=n-k+1}^{n}(X_i-\mu_i)/k^{1/\alpha}\Big)<\frac{ck^{1/2}-k\overline{\mu}}{k^{1/\alpha}},\ 1\leq k\leq n,\ 1\leq n\leq m\Big)\\ \leq\ &\mathsf{P}\Big(\Big(\sum_{i=m_{j-1}+1}^{m_j}(X_i-\mu_i)/(m(c)^{1/\alpha})\Big)<\frac{cm(c)^{1/2}-m(c)\overline{\mu}}{m(c)^{1/\alpha}},\ 1\leq j\leq [m/m(c)]\Big)\\ =\ &\Big[\mathsf{P}\Big(\Big(\sum_{i=0}^{m_1}(X_i-\mu_i)/(m(c)^{1/\alpha})\Big)<\frac{cm(c)^{1/2}-m(c)\overline{\mu}}{m(c)^{1/\alpha}}\Big)\Big]^{[m/m(c)]}.\end{aligned}$$

Note that

$$\mathsf{P}\Big(\Big(\sum_{i=1}^{m_1}(X_i-\mu_i)/(m(c)^{1/\alpha})\Big)<\frac{cm(c)^{1/2}-m(c)\overline{\mu}}{m(c)^{1/\alpha}}\Big)=\frac{1}{c^{\beta-2/\alpha}}+o(1)$$

for large c. Thus

$$\begin{aligned}&\mathrm{ARL}_{\mu_i}(T_{\mathrm{GL}})\\ =\ &\sum_{m=0}^{\infty}\mathsf{P}(T_{\mathrm{GL}}>m)\\ =\ &1+\sum_{j=1}^{\infty}\sum_{m=m_{j-1}+1}^{m_j}\mathsf{P}(T_{\mathrm{GL}}>m)\\ \leq\ &1+m(c)\sum_{j=1}^{\infty}\Big[\mathsf{P}\Big(\Big(\sum_{i=0}^{m_1}(X_i-\mu_i)/(m(c)^{1/\alpha})\Big)<\frac{cm(c)^{1/2}-m(c)\overline{\mu}}{m(c)^{1/\alpha}}\Big)\Big]^{j-1}\\ \sim\ &\leq 1+m(c)\sum_{j=1}^{\infty}\Big(\frac{1}{c^{\beta-2/\alpha}}\Big)^{j-1}\sim 1+m(c)\end{aligned}$$

for large c. #

§ 3. Numerical Illustration

In this section, we show some simulation results of ARL's of CUSUM, EWMA, GLR and CUSUM charts. The numerical results of ARL's were obtained based on a 10,000-repetition experiment. The following Tables compare the simulation results for various values of the mean change with change point $\tau = 1$. r_k is the reference pattern, δ is the reference value, and c denotes various values of the control limit. The mean change μ is listed respectively in the first columns.

Table 1 $X_i \sim S_{1.8}(1, 0, \mu)$

	CUSUM			EWMA					GLR	GEWMA
μ	$\delta = 0.5$	$\delta = 1$	$\delta = 1.5$	$r = 0.01$	$r = 0.05$	$r = 0.1$	$r = 0.2$	$r = 0.5$		
0	500	500	500	499	501	500.6	501.7	500.9	499	499
0.125	261	348	403	300.8	444.9	481.2	499.7	501.5	481	478
0.25	145	220	298	139.7	299.3	431.2	487.5	500.8	420	428
0.5	59.7	80.2	132	49	108.1	264.6	446.5	495.3	214	233
1	24.5	24.4	30.8	15.1	27.2	57.6	295.3	472.6	67.5	72.0
1.5	15.3	13.5	14.1	7.6	12.8	20.3	116	431.7	31.6	32.5
2	11.2	9.34	9.05	4.7	7.7	11.02	36.6	381.7	18.6	18.5
3	7.27	5.82	5.34	2.47	3.8	5.16	9.35	254.8	8.74	8.18
4	5.46	4.30	3.81	1.66	2.4	3.16	4.92	112.4	5.26	4.83
c	9.070	12.82	15.50	3.985	5.35	6.36	8.05	11.09	8.604	8.836

Table 2 $X_i \sim S_{1.5}(1, 0, \mu)$

	CUSUM			EWMA					GLR	GEWMA
μ	$\delta = 0.5$	$\delta = 1$	$\delta = 1.5$	$r = 0.01$	$r = 0.05$	$r = 0.1$	$r = 0.2$	$r = 0.5$		
0	499	501	499	499	500.5	500	500	500.8	500	499
0.125	342	418	457	442.7	498.7	499.4	500.4	501.3	495	496
0.25	233	312	413	325.7	484.2	498	497.7	501.9	492	494
0.5	115	165	259	147.4	434.6	484.1	496.3	501.7	477	482
1	48.9	53.2	75.3	47.5	256.8	434.1	484.8	496.6	311	330
1.5	30.1	29.0	33.4	23.3	99.1	352.9	462.7	493.8	173	185
2	21.9	19.8	20.9	13.8	42.9	250	433.7	488.9	105	113
3	14.2	12.3	12.1	6.7	16.9	67.7	350.3	479.2	49.5	52.2
4	10.5	8.84	8.45	4.05	9.7	20.7	246.1	458.2	28.4	29.3
c	18.40	28.62	37.92	7.38	11.99	15.85	21.2	30.15	21.19	21.45

From the tables above, we find that the result of the EWMA makes great differences if we choose different weight parameter r. In the first table ($\alpha = 1.8$), comparing the performance of the CUSUM with that of the EWMA, we find that the EWMA is better than the CUSUM when μ is large and the CUSUM is better than the EWMA when μ is small. For the GLR, the simulation results in Table 1 are quite different from what we know in the case of normal distribution (the special case of stable distribution when $\alpha = 2$). In normal distribution, GLR performs very well, and is better than the CUSUM chart except the size of the mean shift μ is nearly δ. However, the result does not hold for the case in which the observation process is subject to the stable distribution ($\alpha = 1.8$). Moreover the GEWMA has a similar performance as that of EWMA though the GEWMA does not depend the weight parameter r. From Table 2 we see that the EWMA has the better performance than CUSUM does when r equals 0.01, the CUSUM and EWMA charts perform better than the GLR and EWMA if the suitable reference value δ and weight parameter r can been chosen respectively for the CUSUM and EWMA. Moreover, the performance of the GEWMA in the case of $\alpha = 1.8$ is better than that in the case of $\alpha = 1.5$.

§ 4. Conclusion

In this paper, we focus on the estimation of the ARL for the EWMA, GEWMA and GLR control charts when the control limit c is big enough. From Theorem 1 we find that the ARL of EWMA approaches to Ac^α, where A is constant, when X_i is positive random variable. From theorem 2 it follows that $\mathrm{ARL}_{\mu_i}(T_{\mathrm{GE}}) \geq Bc^{2\alpha/(\alpha+4)}$, where B is constant. Theorem 3 tell us that $Cc^{2\alpha/(6-\alpha)} \leq \mathrm{ARL}_{\mu_i}(T_{\mathrm{GL}}) \leq Dc^2 + Ec^\beta$, where C, D and E are constants. The comparison of numerical simulation results show that the EWMA chart can perform better than the other charts if a suitable weight parameter is chosen for the EWMA. And the CUSUM has the better performs than that of the GEWMA and GLR charts in detecting any size of mean shifts for the case of $\alpha = 1.5$.

As we know that when $0 < \alpha \leq 1$, the stable random variable has infinite mean and variance. So it is an interesting problem how to monitor the change of the stable Lévy process when both the mean and variance are infinite.

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监测 Lévy 稳定过程均值变点的平均运行时间估计

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平均运行长度 (时间) (ARL) 是判断一控制图监测变点效果好坏的一个重要工具．本文主要研究 Lévy 稳定过程的均值变点监测问题．我们给出了三个控制图，即 EWMA, GEWMA 和 GLR 的 ARL 估计，并通过数值模拟比较了 4 个控制图监测均值变点的效果和差异．

关 键 词:　变点监测，平均运行长度， Lévy 稳定过程， EWMA, GLR, GEWMA, CUSUM.
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