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Some Properties of a Generalized Fractional Brownian Motion *

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Abstract

In the paper, we give a new generalization of fraction Brownian motion (gfBm). We study the existence of the local nondeterminism and the joint continuity of the local time of gfBm, and we get upper and lower bounds of Hausdorff dimensions of the level sets of a gfBm.

Keywords: Hausdorff dimension, strong local nondeterminism, level set, local time, generalized fractional Brownian motion.

AMS Subject Classification: 60F15, 60G15, 60G17.

§1. Introduction

Given a constant $H \in (0,1)$, a fractional Brownian motion in R^+ with index H is a real valued, centered Gaussian process $W_H = \{W_H(t), t \in R\}$, with the covariance function

$$\mathsf{E}[W_H(s)W_H(t)] = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s-t|^{2H}).$$

Fractional Brownian motion (fBm) was introduced by Mandelbrot and Van Ness (1968) as a moving average Gaussian process. From the covariance function, it is easy to verify that W_H is a self similar process and it has stationary increments. Pitt (1978) discovered the strong local nondeterminism of the fBm. Xiao (1997) proved the Hölder conditions for the local times and the Hausdorff measure of the level sets of the fBm. Since H is independent of the time parameter t, the regularity of the fBm is the same all along its paths. This property is undesirable when we model some phenomena that do not admit a constant Hölder exponent; for instance, the use of the fBm for synthesizing artificial mountains does not allow to take into account erosion phenomena. To relax this restriction, as a

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generalization of the fBm, Peltier and Lévy (1995) and Benassi et al. (1997) independently introduced the following definition: Let $H_t = H(t) : [0,\infty) \to (0,1)$ be a Hölder function of exponent $\beta > 0$, i.e. for any $t_1, t_2 \in [0,\infty)$ such that $|t_1 - t_2| < 1$, there exists a constant $C > 0$ such that

$$|H(t_1) - H(t_2)| \le C|t_1 - t_2|^{\beta}.$$

Then

$$\widetilde{W}(t) := W_{H_t}(t) = \frac{1}{\Gamma(H_t + 1/2)} \int_{-\infty}^{t} [(t-u)_+^{H_t - 1/2} - (-u)_+^{H_t - 1/2}] \mathrm{d}W(u) \tag{1.1}$$

is called a multifractional Brownian motion (mBm), where $W(u)$ is a Brownian motion. Peltier and Lévy (1995) showed that a mBm has continuous sample paths with probability one and studied its local Hölder properties. Lin (2002) studied large increment behavior of a mBm, and had the following result:

Let $0 \le a_T \le T$, $a_T \to \infty$ as $T \to \infty$ and $D^2(s,t) = \mathsf{E}(\widetilde{W}(t) - \widetilde{W}(s))^2$ for $0 \le s < t$, under the condition:

$$|H_t - H_s| = o((1 - s/t)^{H_t} (\log t)^{-1}) \qquad \text{as } \ 0 < t - s \to 0, \tag{1.2}$$

with $\beta(t,h) = \{2D^2(t, t+h)(\log 1/h + \log\log 1/h)\}^{-1/2}$,

$$\limsup_{h\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \beta(t,T)|\widetilde{W}(t+s) - \widetilde{W}(t)| = 1 \qquad \text{a.s..}$$

In this paper, we consider ageneralization of fBm. Suppose that $H(t)$ is a continuous non-decreasing function with $a < H(t) \le 1 - a$ for some $0 < a < 1/2$ and any $t \ge 0$. We call $\widetilde{W}$ with such $H(t)$ as generalized fractional Brownian motion (gfBm) and study some properties of a gfBm. We investigate the existence of the local non-determinism of a gfBm in Section 2. In Section 3, we study the existence and the joint continuity of the local time of a gfBm. In Section 4, the upper and lower bounds of Hausdorff dimensions of level sets of a gfBm are given under the condition (1.2).

In this paper, C always stands for a positive constant, whose value is irrelevant. $\dim_H A$ is denoted the Hausdorff dimension of set A.

§2. Thelocal Nondeterminism of a gfBm

The concept of local nondeterminism was introduced by Berman (1973). A process is locally nondeterministic if a future observation is "relatively unpredictable" on the basis of a finite set of observations from the immediate past. For a Gaussian process, the local nondeterminism is closely related to the existence of a continuous local time. As a property of Gaussian process, it is of independent interest. Pitt (1978) discovered the strong local nondeterminism of a fractional Brownian motion: there exists a constant $0 < K_1 < \infty$, depending on H only, such that for all $t \in R$ and $0 \le r \le |t|$,

$$\mathsf{Var}\left(W_H(t)|W_H(s), |s-t| \ge r\right) = K_1 r^{2H}.$$

In his proof, the self-similarity and the stationarity play crucial roles. It is obvious that a gfBm has neither the self-similarity nor the stationarity. How about the local nondeterminism for a gfBm?

Definition 2.1 Let J be an open interval on the t-axis. we assume that there exists $d > 0$ such that

$$\mathsf{E}(X(t) - X(s))^2 > 0, \tag{2.1}$$

where $s, t \in J$ and $0 < |t-s| \le d$,

$$\mathsf{E}X^2(t) > 0 \qquad \text{for all } \ t \in J. \tag{2.2}$$

For $m \ge 2$, let $t_1, \cdots, t_m$ be arbitrary points in J with $t_1 < \cdots < t_m$. Let

$$V_m = \frac{\mathsf{Var}\left\{X(t_m) - X(t_{m-1})|X(t_1), \cdots, X(t_{m-1})\right\}}{\mathsf{Var}\left\{X(t_m) - X(t_{m-1})\right\}}.$$

The process is called locally nondeterministic on J if for every integer $m \ge 2$

$$\lim_{c \downarrow 0} \inf_{t_m - t_1 \le c} V_m > 0. \tag{2.3}$$

For a gfBm, conditions (2.1) and (2.2) are obvious, we prove (2.3).

Let

$$F(t,u) = \frac{1}{\Gamma(H_t + 1/2)}\{(t-u)_+^{H_t - 1/2} - (-u)_+^{H_t - 1/2}\},$$

then

$$\widetilde{W}(t) = \int_{-\infty}^{t} F(t,u)\mathrm{d}W(u) \tag{2.4}$$

and

$$\mathsf{Var}\{\widetilde{W}(t)\} = \int_{-\infty}^{t} F^2(t,u)\mathrm{d}u.$$

It is apparent from the representation (2.4) that

$$\{W(u), u \le s\} \supset \{\widetilde{W}(u), u \le s\}.$$

Therefore, for any time set A,

$$\mathsf{Var}\,\{\widetilde{W}(t)-\widetilde{W}(s)|\widetilde{W}(u),u\in A,u\le s\}\ge \mathsf{Var}\,\{\widetilde{W}(t)-\widetilde{W}(s)|W(u),u\le s\}. \tag{2.5}$$

By the independence of increments of a Brownian motion, we have

$$\begin{aligned}&\mathsf{Var}\,\{\widetilde{W}(t)-\widetilde{W}(s)|W(u),u\le s\}\\ =\;&\mathsf{Var}\,\Big(\int_s^t F(t,u)\mathrm{d}W(u)\Big)=\int_s^t F^2(t,u)\mathrm{d}u,\qquad 0<s<t.\end{aligned} \tag{2.6}$$

Lemma 2.1 For $0\le s\le t<T$ and $t-s\to 0$,

$$\mathsf{Var}\,\{\widetilde{W}(t)-\widetilde{W}(s)\}=O((t-s)^{2H_t}+(H_t-H_s)^2t^{2H_t}\log^2 t).$$

Proof The proof is similar to Lin (2002). □

Now, we show the following theorem.

Theorem 2.1 Let $\widetilde{W}(t)$ be a gfBm, Suppose that

$$|H_t-H_s|=O((1-s/t)^{H_t}(\log t)^{-1})\qquad \text{as}\ \ 0<t-s\to 0. \tag{2.7}$$

Then $\widetilde{W}(t)$ is locally nondeterministic on open interval $(0,T)$.

Proof By (2.5) and (2.6), V_m is at least

$$\frac{\displaystyle\int_s^t F^2(t,u)\mathrm{d}u}{\mathsf{Var}\,\{\widetilde{W}(t)-\widetilde{W}(s)\}}, \tag{2.8}$$

where $t=t_m$ and $s=t_{m-1}$. Moreover

$$\int_s^t F^2(t,u)\mathrm{d}u=\frac{1}{\Gamma(H_t+1/2)^2H_t}(t-s)^{2H_t}. \tag{2.9}$$

By Lemma 2.1 and (2.9), (2.8) is at least

$$\frac{(t-s)^{2H_t}/(\Gamma(H_t+1/2)^2H_t)}{(t-s)^{2H_t}+(H_t-H_s)^2t^{2H_t}\log^2 t}>0$$

as $0\le s\le t<T$ and $t-s\to 0$. The proof is complete. □

§3. The Jointly Continuity of the Local Time of a gfBm

Berman (1973) gave the sufficient conditions for joint continuity of local time on a Gaussian process (See also German (1980)).

Lemma 3.1 Let $X(t)$, $0 \le t \le T$, be a Gaussian process with mean 0 and satisfy the following three conditions:

(1) $X(t) = 0$ almost surely;

(2) $X(t)$ is locally nondeterministic on $(0, T)$;

(3) There exists $b(t)$ such that $b^2(t) \le \mathsf{E}(X(s+t) - X(s))^2$ for all s and

$$\int_0^{\epsilon} \frac{\mathrm{d}t}{\{b(t)\}^{1+\delta}} < \infty \tag{3.1}$$

for some $\epsilon, \delta > 0$. Then the local time of $X(t)$ exists and is joint continuous in the sense that $(x, t) \to L(x, t)$ is continuous on $R \times [0, T]$.

We just verify (3.1). First we quote an well known inequality (cf., e.g., Lin et al. (1999)).

Lemma 3.2 If X and Y are independent, $\mathsf{E}X = 0$, $\mathsf{E}|X|^p < \infty$, $\mathsf{E}|Y|^p < \infty$, $1 \le p \le \infty$, then

$$\mathsf{E}|Y|^p \le \mathsf{E}|X + Y|^p. \tag{3.2}$$

Now, for $0 \le s < t$, let

$$X = W_{H_t}(s) - W_{H_s}(s), \qquad Y = \frac{1}{\Gamma(H_t + 1/2)} \int_s^t (t-u)^{H_t - 1/2} \mathrm{d}W(u).$$

Then

$$\widetilde{W}(t) - \widetilde{W}(s) = X + Y.$$

By (3.2), we have

$$\begin{aligned} \mathsf{E}|\widetilde{W}(t) - \widetilde{W}(s)|^2 &\ge \mathsf{E}|Y|^2 \\ &= \frac{1}{\Gamma^2(H_t + 1/2)} \int_s^t (t-u)^{2H_t - 1} \mathrm{d}u \\ &= \frac{1}{\Gamma^2(H_t + 1/2)(2H_t)} (t-s)^{2H_t}. \end{aligned}$$

Note that

$$\frac{1}{\Gamma^2(H_t + 1/2)(2H_t)} \ge C^2 > 0$$

for any $t \ge 0$. Let $b(t) = Ct^{H_t}$, we have (3.1) for some $\delta > 0$ small enough.

Now, Lemma (3.1) implies the following theorem

Theorem 3.1 Let $\widetilde{W}(t)$ be a gfBm and satisfy the condition (2.7). Then it has a joint continuous local time.

§4. The Hausdorff Dimensions of the Level Sets of a gfBm

The set $E(x,T)=\{t\in[0,T],\widetilde{W}(t)=x\}$ is called the level set of $\widetilde{W}(t)$ in x, where x is the interior of range of $\widetilde{W}(t)$. At first, we show the upper bound of the Hausdorff dimensions of the level sets. The following lemma is similar to Theorem 3.1 of Lin (2002).

Lemma 4.1 Suppose that the gfBm $\widetilde{W}(\cdot)$ satisfies condition (1.2). Let $D^2(s,t)=\mathsf{E}(\widetilde{W}(t)-\widetilde{W}(s))^2$ for $0\le s<t$. Then, with $\beta(t,h)=\{2D^2(t,t+h)(\log 1/h+\log\log 1/h)\}^{-1/2}$,

$$\limsup_{h\to 0}\sup_{0\le t\le 1-h}\sup_{0\le s\le h}\beta(t,h)|\widetilde{W}(t+s)-\widetilde{W}(t)|=1\qquad \text{a.s.} \tag{4.1}$$

Theorem 4.1 Suppose that the gfBm $\widetilde{W}(\cdot)$ satisfies condition of Lemma 4.1, then for any $T\in R^+$ and almost all $x\in R$, the Hausdorff dimension

$$\dim_H E(x,T)\le 1-H(0). \tag{4.2}$$

Proof By Lemma 2.1 and the condition of Lemma 4.1, $D^2(t+s,t)=O(s^{2H_{t+s}})$, so the order of Hölder exponent of $D(\cdot,\cdot)$ is at least $\min_{t\in[0,T]}H_t=H(0)$. For any $\epsilon>0$, it follows from (4.1) that for small $s>0$,

$$|\widetilde{W}(t+s)-\widetilde{W}(t)|\le CD^{1+\epsilon}(t+s,t)\qquad \text{a.s..}$$

By an argument similar to the proof of Lemma 8.2.2 of Adler (1981), we have (4.2). □

Next we consider the lower bound. First, we show the following inequality.

Lemma 4.2 Let $\widetilde{W}(t)$ be a gfBm, $0<s<t$, then

$$\int_R\int_R \mathsf{E}\exp\{i[u_1\widetilde{W}(t)+u_2\widetilde{W}(s)]\}\mathrm{d}u_1\mathrm{d}u_2\le C[(t-s)^{-H_t}s^{-H_s}]. \tag{4.3}$$

Proof Since $\widetilde{W}(t)$ is a Gaussian process, the left of (4.3) equals

$$\int_R\int_R\exp\Big\{-\frac{1}{2}\mathsf{E}[u_1\widetilde{W}(t)+u_2\widetilde{W}(s)]^2\Big\}\mathrm{d}u_1\mathrm{d}u_2. \tag{4.4}$$

Let

$$v_1=[\mathsf{E}\widetilde{W}^2(t)]^{1/2}u_1+(\mathsf{E}\widetilde{W}(s)\widetilde{W}(t)/[\mathsf{E}\widetilde{W}^2(t)]^{1/2})u_2,\qquad v_2=\Big\{\frac{\Delta}{\mathsf{E}\widetilde{W}^2(t)}\Big\}^{1/2}u_2,$$

where

$$\Delta=\mathsf{E}\widetilde{W}^2(t)\mathsf{E}\widetilde{W}^2(s)-[\mathsf{E}\widetilde{W}(t)\widetilde{W}(s)]^2.$$

Then (4.4) equals

$$\Delta^{-1/2}\int_R\int_R\exp\Big\{\frac{1}{2}(v_1^2+v_2^2)\Big\}\mathrm{d}v_1\mathrm{d}v_2=(2\pi)^2\Delta^{-1/2}. \tag{4.5}$$

Next, we calculate Δ. Let

$$W_{H_t}(s) = \int_{-\infty}^{s} F(t,u)\mathrm{d}W(u),$$

then

$$[\mathsf{E}\widetilde{W}(t)\widetilde{W}(s)]^2 = [\mathsf{E}W_{H_t}(s)W_{H_s}(s)]^2 \le \mathsf{E}W_{H_t}^2(s)\mathsf{E}W_{H_s}^2(s). \tag{4.6}$$

Let

$$W_{H_t}(s,t) = \frac{1}{\Gamma(H_t+1/2)}\int_s^t (t-u)^{H_t-1/2}\mathrm{d}W(u),$$

then

$$\mathsf{E}\widetilde{W}^2(t)\mathsf{E}\widetilde{W}^2(s) = \mathsf{E}W_{H_t}^2(s)\mathsf{E}W_{H_s}^2(s) + \mathsf{E}W_{H_t}^2(s,t)\mathsf{E}W_{H_s}^2(s). \tag{4.7}$$

Moreover

$$\begin{aligned}\mathsf{E}W_{H_s}^2(s) &= \frac{1}{\Gamma^2(H_s+1/2)}\Big\{\int_{-\infty}^{0}[(s-u)^{H_s-1/2}-(-u)^{H_s-1/2}]^2\mathrm{d}u+\int_0^s (s-u)^{2H_s-1}\mathrm{d}u\Big\}\\ &\ge Cs^{2H_s}. \end{aligned}\tag{4.8}$$

Similarly

$$\mathsf{E}W_{H_t}^2(s,t) \ge C(t-s)^{2H_t}. \tag{4.9}$$

Combining (4.6)–(4.9), we have

$$\Delta \ge \mathsf{E}W_{H_t}^2(s,t)\mathsf{E}W_{H_s}^2(s) \ge C(t-s)^{2H_t}s^{2H_s}.$$

(4.3) follows from (4.4) and (4.5). □

Lemma 4.3 Let $\widetilde{W}(t)$ be a gfBm and satisfy condition (2.7), $L(x,T)$, $T\in R$, be the local time of $\widetilde{W}(t)$, then the zero set of $L(x,T)$ $\{x: L(x,T)=0\}$ is non-dense everywhere.

Proof By Theorem 3.1, $\widetilde{W}(t)$ has jointly continuous local time, hence from Adler (1981), our conclusion is obvious. □

Now, we can show the lower bound of the Hausdorff dimension.

Theorem 4.2 Let $\widetilde{W}(t)$ be a gfBm and satisfy condition (2.7), then for any $T\in R$ and almost every x,

$$\dim_H E(x,T) \ge 1 - H_T.$$

Proof By Lemma 4.3, for any $T\in R$, we have $L(x,T)>0$ for almost every x. So we can define random measure u:

$$u(B) = \frac{L(x,[0,T]\cap B)}{L(x,T)}.$$

Then u is a probability measure on $E(x,T)$. By the energy integration formulation, we will just show for $\beta<1-H_T$, the energy integration

$$I_\beta(u)=\int_{E^2(x,T)}|s-t|^{-\beta}\mathrm{d}u(s)\mathrm{d}u(t)<\infty. \tag{4.10}$$

By an argument similar to the proof of Proposition 3.1 of Pitt (1978) (see also Theorem 8.7.4 of Adler (1981)), we have

$$\begin{aligned}\mathsf{E}I_\beta(u) &= \frac{1}{L^2(x,T)(2\pi)^2}\int_0^T\int_0^T\int_R\int_R \exp\{-i(xu_1+xu_2)\}\\ &\quad\cdot\mathsf{E}\exp\{iu_1\widetilde{W}(s)+iu_2\widetilde{W}(t)\}|t-s|^{-\beta}\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}s\mathrm{d}t. \end{aligned}\tag{4.11}$$

Therefore in order to prove (4.10), it suffices to show that the right side of (4.11) is finite. By Lemma 4.2, the right side of (4.11) is no more than

$$\begin{aligned}&\frac{1}{L^2(x,T)(2\pi)^2}\int_0^T\int_0^T\int_R\int_R|\mathsf{E}\exp\{iu_1\widetilde{W}(s)+iu_2\widetilde{W}(t)\}||t-s|^{-\beta}\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}s\mathrm{d}t\\ \le\ &\frac{C}{L^2(x,T)(2\pi)^2}\int_0^T\Big\{\int_0^t|t-s|^{-H_t}s^{-H_s}\mathrm{d}s+\int_t^T|t-s|^{-H_s}t^{-H_t}\mathrm{d}s\Big\}|t-s|^{-\beta}\mathrm{d}t\\ \le\ &\frac{C}{L^2(x,T)(2\pi)^2}\int_0^T\Big\{\int_0^T|t-s|^{-H_t}s^{-H_s}\mathrm{d}s+\int_0^T|t-s|^{-H_s}t^{-H_t}\mathrm{d}s\Big\}|t-s|^{-\beta}\mathrm{d}t\\ =\ &\frac{2C}{L^2(x,T)(2\pi)^2}\int_0^T\int_0^T|t-s|^{-H_t}s^{-H_s}|t-s|^{-\beta}\mathrm{d}s\mathrm{d}t\\ \le\ &\frac{C'}{L^2(x,T)(2\pi)^2}\int_0^T|u|^{-H_t-\beta}\mathrm{d}u<\infty\end{aligned}$$

for almost every x. Hence

$$\dim_H E(x,T)\ge 1-H_T \qquad \text{a.s..}$$

This completes the proof of Theorem 4.2. □

The lower bound of the Hausdorff dimension is not sharp, but for a fBm, we have the following corellary by combining Theorem 4.1 and Theorem 4.2.

Corollary 4.1 Let $W_H(t)$ be a fractional Browian motion, then for almost every x in the range of $W_H(t)$,

$$\dim_H E(x,T)=1-H.$$

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广义分数布朗运动的若干性质

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本文, 我们定义了一类新的分数布朗运动, 研究了它的局部非决定性和局部时的联合连续性, 最后给出了它的水平集的Hausdorff维数的上、下界.

关 键 词: Hausdorff维数, 强局部非决定性, 水平集, 局部时, 广义分数布朗运动.

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