从SPVII分布生成的新的多元偏态t分布的有关性质 *

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摘　要

随机向量的t分布属于椭球等高分布族, 然而, 它是对称分布. 在许多诸如经济学、生理学、社会学等领域中, 有时回归模型中的随机误差不再满足对称性, 通常表现出高度的偏态性(skewness). 于是就有了偏态椭球等高分布族. 本文在已有的多元偏态t分布的基础上, 着重研究它的分布性质, 包括线性组合分布、边缘分布、条件分布及各阶矩.

关键词: 偏态Pearson VII型分布, 偏态t分布, 偏态正态分布, 密度生成函数, 矩生成函数.

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§1.　从SPVII分布生成的新的多元偏态t分布

t分布属于Pearson VII型分布, 本节在多元偏态Pearson VII型分布的基础上给出新的多元偏态t分布的定义及随机表示.

1.1　多元偏态PVII分布的定义

设随机向量$X_{p\times1}$服从Pearson VII型分布([1] P93), 记为$X\sim\mathrm{PVII}_p(\mathbf{0},\Omega,M,v)$, $M>p/2$, $v>0$, $\Omega_{p\times p}>0$, 密度函数为$g^{(p)}(u)=C_pg(u;p)$, 其中, $u=x'\Omega x>0$为中间变量, $g(u;p)=(1+u/v)^{-M}$为密度生成函数, $C_p=\Gamma(M)/[(\pi v)^{p/2}\Gamma(M-p/2)]$为标准化常数.

将X分块为$X=(X_1',X_2')'$, 其中, $X_1:p_1\times1$, $X_2:p_2\times1$, $p=p_1+p_2$, 且X_1满足线性约束条件: $X_1>\mathbf{0}_{p_1\times1}$, 记$p_c$表示约束条件的概率, 即$p_c=\mathsf{P}(X_1>\mathbf{0})$. Ω分块为

$$\begin{pmatrix}\Omega_{11}&\Omega_{12}\\\Omega_{21}&\Omega_{22}\end{pmatrix}\triangleq\begin{pmatrix}I_{p_1}&\Omega_{12}\\\Omega_{21}&\Omega_{22}\end{pmatrix},$$

本文以下部分我们都令$\Omega_{11}=I_{p_1}$, $\Omega_{11.2}=I_{p_1}-\Omega_{12}\Omega_{22}^{-1}\Omega_{21}$.

定义 1.1 ([2] P4)　若随机向量$Z_{p_2\times1}$有如下分布密度:

$$\begin{aligned}f_Z(z)&=\frac{F(\alpha'z(v+Q_2)^{-1/2}|\mathbf{0},I_{p_1};M,1)}{F(\mathbf{0}|\mathbf{0},I_{p_1};M-p_2/2,v)}\cdot f\Big(z|\mathbf{0},\Omega_{22};M-\frac{p_1}{2},v\Big)\\&=2^{p_1}\cdot F(\alpha'z(v+Q_2)^{-1/2}|\mathbf{0},I_{p_1};M,1)\cdot f\Big(z|\mathbf{0},\Omega_{22};M-\frac{p_1}{2},v\Big).\qquad(1.1)\end{aligned}$$

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这里, $\alpha=\Omega_{22}^{-1}\Omega_{21}\Omega_{11.2}^{-1/2}$, $Q_2=z'\Omega_{22}^{-1}z$, $F(\alpha'z(v+Q_2)^{-1/2}|\mathbf{0},I_{p_1};M,1)$表示$\mathrm{PVII}_{p_1}(\mathbf{0},I_{p_1};M,1)$分布在$\alpha'z(v+Q_2)^{-1/2}$处的值, $F(\mathbf{0}|\mathbf{0},I_{p_1};M-p_2/2,v)$表示$\mathrm{PVII}_{p_1}(\mathbf{0},I_{p_1},M-p_2/2,v)$分布在$\mathbf{0}$处的值, 等于$2^{-p_1}$, $f(z|\mathbf{0},\Omega_{22};M-p_1/2,v)$表示$\mathrm{PVII}_{p_2}(\mathbf{0},\Omega_{22},M-p_1/2,v)$的分布密度, 称$Z$服从多元偏态$\mathrm{PVII}_{p_2}(\cdot)$分布, 记为$\mathrm{SPVII}_{p_2}(\mathbf{0},\Omega,M,v,\alpha)$, 称$\alpha$为偏态系数.

注记 1 这里, $\mathrm{SPVII}_{p_2}(\mathbf{0},\Omega,M,v,\alpha)$中, $\Omega:p\times p$, 与随机向量Z的维数$p_2\times 1$是不同的, $\mathbf{0}:p_2\times 1$, 下同.

若$M=(v+p)/2$, 称PVII_p为自由度为v的多元t分布([1] P93), 记为$\mathrm{t}_p(\mu,\Omega;v)$, 其中

$$g^{(p)}(u)=\frac{\Gamma[(v+p)/2]}{\Gamma(v/2)(\pi v)^{p/2}}\Big(1+\frac{u}{v}\Big)^{-(v+p)/2},\qquad u>0.$$

定义 1.2 ([2] P5) 若随机向量$Z_{p_2\times 1}$有如下的分布密度:

$$\begin{aligned} f_Z(z) &= \frac{F(\alpha'z(v+Q_2)^{-1/2}(v+p_2)^{1/2}|\mathbf{0},I_{p_1};v+p_2)}{F(\mathbf{0}|\mathbf{0},I_{p_1};v)}\cdot f(z|\mathbf{0},\Omega_{22};v)\\ &= 2^{p_1}\cdot F(\alpha'z(v+Q_2)^{-1/2}(v+p_2)^{1/2}|\mathbf{0},I_{p_1};v+p_2)\cdot f(z|\mathbf{0},\Omega_{22};v). \end{aligned}\tag{1.2}$$

这里, $\alpha=\Omega_{22}^{-1}\Omega_{21}\Omega_{11.2}^{-1/2}$, $Q_2=z'\Omega_{22}^{-1}z$, $F(\alpha'z(v+Q_2)^{-1/2}(v+p_2)^{1/2}|\mathbf{0},I_{p_1};v+p_2)$表示自由度为$v+p_2$的$\mathrm{t}_{p_1}(\mathbf{0},I_{p_1};v+p_2)$分布在$\alpha'z(v+Q_2)^{-1/2}(v+p_2)^{1/2}$处的值, $F(\mathbf{0}|\mathbf{0},I_{p_1};v)$表示自由度为$v$的$\mathrm{t}_{p_1}(\mathbf{0},I_{p_1};v)$分布在$\mathbf{0}$处的值, 等于$2^{-p_1}$, $f(z|\mathbf{0},\Omega_{22};v)$表示自由度为$v$的$\mathrm{t}_{p_2}(\mathbf{0}_{p_2\times 1},\Omega_{22};v)$的分布密度, 称$Z$服从多元偏态$\mathrm{t}_{p_2}(\cdot)$分布, 记为$\mathrm{St}_{p_2}(\mathbf{0},\Omega,v,\alpha)$, 称$\alpha$为偏态系数.

1.2 两种随机表示方法

上述定义是通过线性约束的条件方法([2] P4)给出的, 也可以通过变换的方法得到.

定理 1.1 ([2] P8) (变换方法) 设$U^*=(U_1^{*\prime},U_2^{*\prime})'\sim\mathrm{PVII}_p(\mathbf{0},\Psi^*,M,v)$, $M>p/2$, $v>0$, $\Psi^*_{p\times p}=\begin{pmatrix} I_{p_1} & \mathbf{0}_{p_1\times p_2}\\ \mathbf{0}_{p_2\times p_1} & \Psi_{p_2\times p_2}\end{pmatrix}>0$, 其中$U_1^*:p_1\times 1$, $U_2^*:p_2\times 1$, $p=p_1+p_2$, $U_1^*=(U_1,\cdots,U_{p_1})'$, $U_2^*=(U_{p_1+1},\cdots,U_p)'$, 定义

$$Z_j=\delta_j(|U_1|+\cdots+|U_{p_1}|)+(1-\delta_j^2)^{1/2}U_j,\qquad -1<\delta_j<1,\ j=p_1+1,\cdots,p,\tag{1.3}$$

则$Z\triangleq(Z_{p_1+1},\cdots,Z_p)'$有(1.1)式的分布密度. 这里,

$$\lambda_i=\delta_i(1-\delta_i^2)^{-1/2},\quad i=p_1+1,\cdots,p,\quad \Delta_{p_2\times p_2}=\mathrm{diag}\{(1+\lambda_{p_1+1}^2)^{-1/2},\cdots,(1+\lambda_p^2)^{-1/2}\},$$

$$\lambda_{p_2\times p_1}=\begin{pmatrix}\lambda_{p_1+1} & \cdots & \lambda_{p_1+1}\\ \vdots & & \vdots\\ \lambda_p & \cdots & \lambda_p\end{pmatrix},\qquad \Omega_{p\times p}=\begin{pmatrix} I_{p_1} & \lambda'\Delta\\ \Delta\lambda & \Delta(\Psi+\lambda\lambda')\Delta\end{pmatrix},$$

$$\alpha=\Delta^{-1}(\Psi+\lambda\lambda')^{-1}\lambda[I_{p_1}-\lambda'(\Psi+\lambda\lambda')^{-1}\lambda]^{-1/2}.\tag{1.4}$$

§2. 多元SPVII分布的有关性质

2.1 线性组合分布

定理 2.1 设$Z_{p_2\times 1}\sim \text{SPVII}_{p_2}(\mathbf{0},\Omega,M,v,\alpha)$, 参数同定义1.1. 已知矩阵$A$可逆, b为$p_2\times 1$维向量, 则

$$\widetilde{Z}=AZ+b\sim \text{SPVII}_{p_2}(b,\widetilde{\Omega},M,v,\widetilde{\alpha}), \tag{2.1}$$

其中$\widetilde{\Omega}_{p\times p}=\begin{pmatrix} I_{p_1} & \Omega_{12}A' \\ A\Omega_{21} & A\Omega_{22}A' \end{pmatrix}$, $\widetilde{\alpha}=(A')^{-1}\alpha$.

证明: 见附录. □

推论 2.1 设$Z_{p_2\times 1}\sim \text{St}_{p_2}(\mathbf{0},\Omega,v,\alpha)$, 已知矩阵$A$可逆, b为$p_2\times 1$维向量, 则

$$\widetilde{Z}=AZ+b\sim \text{St}_{p_2}(b,\widetilde{\Omega},v,\widetilde{\alpha}), \tag{2.2}$$

其中, $\widetilde{\Omega}_{p\times p}=\begin{pmatrix} I_{p_1} & \Omega_{12}A' \\ A\Omega_{21} & A\Omega_{22}A' \end{pmatrix}$, $\widetilde{\alpha}=(A')^{-1}\alpha$.

证明: 见附录. □

2.2 边缘分布

设$Z_{p_2\times 1}\sim \text{SPVII}_{p_2}(\mathbf{0},\Omega,M,v,\alpha)$, 参数同定义1.1. 将$Z$进行分块: $Z=(Z_1',Z_2')'$, 其中, $Z_1: p_{21}\times 1$, $Z_2: p_{22}\times 1$, $p_2=p_{21}+p_{22}$, 引入如下记号:

$$\Omega_{p\times p}=\begin{pmatrix} I_{p_1} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{pmatrix}=\begin{pmatrix} \widetilde{\Omega}_{11} & \widetilde{\Omega}_{12} \\ \widetilde{\Omega}_{21} & \Omega_{33} \end{pmatrix}=\begin{pmatrix} I_{p_1} & \Omega_{12}^* \\ \Omega_{21}^* & \Omega_{22}^* \end{pmatrix}, \tag{2.3}$$

$$\alpha_{p_2\times p_1}=\Omega_{22}^{*-1}\Omega_{21}^{*}\Omega_{11.2}^{*-1/2}=\Omega_{22}^{*-1}\Omega_{21}^{*}(I_{p_1}-\Omega_{12}^{*}\Omega_{22}^{*-1}\Omega_{21}^{*})^{-1/2}. \tag{2.4}$$

定理 2.2 设$Z_{p_2\times 1}\sim \text{SPVII}_{p_2}(\mathbf{0},\Omega,M,v,\alpha)$, α同(2.4)式, 其他条件同上. 则Z_1的边缘分布为

$$f_c(z_1)=2^{p_1}\cdot F\Big(\widetilde{\alpha}'z_1(v+Q_5)^{-1/2}|\mathbf{0},I_{p_1};M-\frac{p_{22}}{2},1\Big)\cdot f\Big(z_1|\mathbf{0},\Omega_{22};M-\frac{p_{22}+p_1}{2},v\Big), \tag{2.5}$$

其中, $\widetilde{\alpha}=\Omega_{22}^{-1}\Omega_{21}\Omega_{11.2}^{-1/2}$, $Q_5=z_1'\Omega_{22}^{-1}z_1$.

证明: 见附录. □

推论 2.2 设$Z_{p_2\times 1}\sim \text{St}_{p_2}(\mathbf{0},\Omega,v,\alpha)$, α同(2.4)式, 其他条件同上. 则Z_1的边缘分布为

$$f_c(z_1)=2^{p_1}\cdot F(\widetilde{\alpha}'z_1(v+Q_5)^{-1/2}(v+p_{21})^{1/2}|\mathbf{0},I_{p_1};v+p_{21})\cdot f(z_1|\mathbf{0},\Omega_{22};v), \tag{2.6}$$

其中, $\widetilde{\alpha}=\Omega_{22}^{-1}\Omega_{21}\Omega_{11.2}^{-1/2}$, $Q_5=z_1'\Omega_{22}^{-1}z_1$.

证明: 见附录. □

2.3 条件分布

定理 2.3 设$Z_{p_2\times 1}\sim \mathrm{SPVII}_{p_2}(\mathbf{0},\Omega,M,v,\alpha)$, 条件同上. 则$Z_1|Z_2=z_2$的条件分布密度为:

$$\frac{F(\psi_{11.2}^{-1/2}m_{1.2}(v+Q_2+Q_4)^{-1/2}|\mathbf{0},I_{p_1};M,1)}{F(\psi_{11}^{-1/2}m_1(v+Q_2)^{-1/2}|\mathbf{0},I_{p_1};M-p_{21}/2,1)}\cdot f\Big(z_1|m_2,\psi_{22};M-\frac{p_1}{2},v+Q_2\Big), \tag{2.7}$$

其中, $Q_2=z_2'\Omega_{33}^{-1}z_2$, $\begin{pmatrix}\psi_{11} & \psi_{12}\\ \psi_{21} & \psi_{22}\end{pmatrix}=\begin{pmatrix}I_{p_1}-\Omega_{13}\Omega_{33}^{-1}\Omega_{31} & \Omega_{12}-\Omega_{13}\Omega_{33}^{-1}\Omega_{32}\\ \Omega_{21}-\Omega_{23}\Omega_{33}^{-1}\Omega_{31} & \Omega_{22}-\Omega_{23}\Omega_{33}^{-1}\Omega_{32}\end{pmatrix}$, $\psi_{11.2}=\psi_{11}-\psi_{12}\psi_{22}^{-1}\psi_{21}$, $\begin{pmatrix}m_1\\ m_2\end{pmatrix}=\begin{pmatrix}\Omega_{13}\Omega_{33}^{-1}z_2\\ \Omega_{23}\Omega_{33}^{-1}z_2\end{pmatrix}$, $m_{1.2}=m_1+\psi_{12}\psi_{22}^{-1}(z_1-m_2)$, $Q_4=(z_1-m_2)'\cdot\psi_{22}^{-1}(z_1-m_2)$, $F(\psi_{11.2}^{-1/2}m_{1.2}(v+Q_2+Q_4)^{-1/2}|\mathbf{0},I_{p_1};M,1)$表示$\mathrm{PVII}_{p_1}(\mathbf{0},I_{p_1};M,1)$累计分布函数在$\psi_{11.2}^{-1/2}m_{1.2}(v+Q_2+Q_4)^{-1/2}$处的值, $F(\psi_{11}^{-1/2}m_1(v+Q_2)^{-1/2}|\mathbf{0},I_{p_1};M-p_{21}/2,1)$表示$\mathrm{PVII}_{p_1}(\mathbf{0},I_{p_1};M-p_{21}/2,1)$累计分布函数在$\psi_{11}^{-1/2}m_1(v+Q_2)^{-1/2}$处的值, $f(z_1|m_2,\psi_{22};M-p_1/2,v+Q_2)$表示$\mathrm{PVII}_{p_{21}}(m_2,\psi_{22},M-p_1/2,v+Q_2)$的分布密度函数.

证明: 见附录. □

推论 2.3 设$Z_{p_2\times 1}\sim \mathrm{St}_{p_2}(\mathbf{0},\Omega,M,v,\alpha)$, 条件同上. 则$Z_1|Z_2=z_2$的条件分布密度为:

$$\frac{F(\psi_{11.2}^{-1/2}m_{1.2}(v+Q_2+Q_4)^{-1/2}(v+p_2)^{1/2}|\mathbf{0},I_{p_1};v+p_2)}{F(\psi_{11}^{-1/2}m_1(v+Q_2)^{-1/2}(v+p_{22})^{1/2}|\mathbf{0},I_{p_1};v+p_{22})}\cdot\widetilde{f}(z_1|m_2,\psi_{22},v), \tag{2.8}$$

其中, $Q_2=z_2'\Omega_{33}^{-1}z_2$, $\begin{pmatrix}\psi_{11} & \psi_{12}\\ \psi_{21} & \psi_{22}\end{pmatrix}=\begin{pmatrix}I_{p_1}-\Omega_{13}\Omega_{33}^{-1}\Omega_{31} & \Omega_{12}-\Omega_{13}\Omega_{33}^{-1}\Omega_{32}\\ \Omega_{21}-\Omega_{23}\Omega_{33}^{-1}\Omega_{31} & \Omega_{22}-\Omega_{23}\Omega_{33}^{-1}\Omega_{32}\end{pmatrix}$, $\psi_{11.2}=\psi_{11}-\psi_{12}\psi_{22}^{-1}\psi_{21}$, $\begin{pmatrix}m_1\\ m_2\end{pmatrix}=\begin{pmatrix}\Omega_{13}\Omega_{33}^{-1}z_2\\ \Omega_{23}\Omega_{33}^{-1}z_2\end{pmatrix}$, $m_{1.2}=m_1+\psi_{12}\psi_{22}^{-1}(z_1-m_2)$, $Q_4=(z_1-m_2)'\cdot\psi_{22}^{-1}(z_1-m_2)$, $F(\psi_{11.2}^{-1/2}m_{1.2}(v+Q_2+Q_4)^{-1/2}(v+p_2)^{1/2}|\mathbf{0},I_{p_1};v+p_2)$表示$\mathrm{t}_{p_1}(\mathbf{0},I_{p_1};v+p_2)$累计分布函数在$\psi_{11.2}^{-1/2}m_{1.2}(v+Q_2+Q_4)^{-1/2}(v+p_2)^{1/2}$处的值, $F(\psi_{11}^{-1/2}m_1(v+Q_2)^{-1/2}(v+p_{22})^{1/2}|\mathbf{0},I_{p_1};v+p_{22})$表示$\mathrm{t}_{p_1}(\mathbf{0},I_{p_1};v+p_{22})$累计分布函数在$\psi_{11}^{-1/2}m_1(v+Q_2)^{-1/2}(v+p_{22})^{1/2}$处的值,

$$\widetilde{f}(z_1|m_2,\psi_{22},v)=\frac{\Gamma[(v+p_{22}+p_{21})/2]}{\Gamma[(v+p_{22})/2]\pi^{p_{21}/2}}(v+Q_2)^{-p_{21}/2}|\psi_{22}|^{-1/2}\Big(1+\frac{Q_4}{v+Q_2}\Big)^{-(v+p_{22}+p_{21})/2}. \tag{2.9}$$

§3. 多元SPVII分布的各阶矩

3.1 多元偏态正态分布(SN)各阶矩与多元SPVII分布各阶矩的关系

由(1.1)式知, 直接求SPVII分布的各阶矩比较困难, 但多元偏态正态分布(SN)的各阶矩比较容易求出. 本小节将通过多元偏态正态分布(SN)各阶矩与SPVII分布各阶矩的关系来确定SPVII分布的各阶矩. 由定理1.1知,

$$Z \triangleq (Z_{p_1+1},\cdots,Z_p)' \overset{d}{=} \delta|U_1^*| + \Delta U_2^*. \tag{3.1}$$

$$U^* = (U_1^{*\prime}, U_2^{*\prime})' \sim \mathrm{PVII}_p(\mathbf{0}, \Psi^*, M, v), \tag{3.2}$$

其中, $\delta = \begin{pmatrix} \delta_{p_1+1} & \cdots & \delta_{p_1+1} \\ \vdots & & \vdots \\ \delta_p & \cdots & \delta_p \end{pmatrix}_{p_2\times p_1}$, $|U_1^*| = (|U_1|,|U_2|,\cdots,|U_{p_1}|)'$. 这里, 对于$U^*$服从任意椭球等高分布($U^* \sim \mathrm{EC}_p(\mathbf{0},\Psi^*; g^{(p)})$), 都有如下的随机表示:

$$\begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} \overset{d}{=} R \begin{pmatrix} I_{p_1} & \mathbf{0} \\ \mathbf{0} & A' \end{pmatrix} \begin{pmatrix} \mathcal{U}^{(p_1)} \\ \mathcal{U}^{(p_2)} \end{pmatrix}, \tag{3.3}$$

其中, R与$(\mathcal{U}^{(p_1)\prime},\mathcal{U}^{(p_2)\prime})'$独立, $A: p_2\times p_2$, 且$A'A = \Psi$, 可令$A = (\Omega_{22} - \delta\delta')^{1/2}\Delta^{-1}$, R有如下分布密度([1] P77):

$$h_R(r) = \frac{2\pi^{p/2}}{\Gamma(p/2)} r^{p-1} g^{(p)}(r^2), \tag{3.4}$$

这里, $g^{(p)}(\cdot)$可以是任意椭球等高分布的密度. 所以

$$Z \overset{d}{=} R[\delta|\mathcal{U}^{(p_1)}| + (\Omega_{22}-\delta\delta')^{1/2}\mathcal{U}^{(p_2)}] \triangleq R[B'|\mathcal{U}^{(p_1)}| + C'\mathcal{U}^{(p_2)}]. \tag{3.5}$$

所以有

$$\mathsf{E}\prod_{j=1}^{p_2} Z_j^{s_j} = \mathsf{E}\Big(R^{\sum_{j=1}^{p_2} s_j}\Big)\cdot \mathsf{E}\Big\{\prod_{j=1}^{p_2}[B_j'|\mathcal{U}^{(p_1)}| + C_j'\mathcal{U}^{(p_2)}]^{s_j}\Big\}, \tag{3.6}$$

其中, B_j'表示B'的第j行, C_j'表示C'的第j行, s_j为非负整数, Z_j为Z_2^*的分量, $j = 1,\cdots,p_2$.

令随机向量$X_{p_2\times 1}$服从偏态正态分布, 记为$X \sim \mathrm{SN}_{p_2}(\mathbf{0},\Omega,\alpha)$, 则由(3.5)式

$$X \overset{d}{=} R_0[B'|\mathcal{U}^{(p_1)}| + C'\mathcal{U}^{(p_2)}], \tag{3.7}$$

其中, $R_0^2 \sim \chi_p^2$. 再由(3.6)式得

$$\mathsf{E}\prod_{j=1}^{p_2} X_j^{s_j} = \mathsf{E}\Big(R_0^{\sum_{j=1}^{p_2} s_j}\Big)\cdot \mathsf{E}\Big\{\prod_{j=1}^{p_2}[B_j'|\mathcal{U}^{(p_1)}| + C_j'\mathcal{U}^{(p_2)}]^{s_j}\Big\}, \tag{3.8}$$

其中, X_j为X的分量, $j=1,\cdots,p_2$. 所以由(3.6)、(3.8)式得

$$\mathsf{E}\prod_{j=1}^{p_2}Z_j^{s_j}=\Big[\mathsf{E}\Big(R^{\sum\limits_{j=1}^{p_2}s_j}\Big)\Big/\mathsf{E}\Big(R_0^{\sum\limits_{j=1}^{p_2}s_j}\Big)\Big]\cdot\mathsf{E}\prod_{j=1}^{p_2}X_j^{s_j}. \tag{3.9}$$

若$Z\sim\mathrm{SPVII}_{p_2}(\mathbf{0},\Omega,M,v,\alpha)$, 则$R$的分布可由(3.4)式求出, $R_0\sim\chi_p$, 关键求$\mathsf{E}\prod\limits_{j=1}^{p_2}X_j^{s_j}$.

3.2 多元SN分布及各阶矩

对于多元正态分布, 有([2] P2, 例1)

$$g^{(p)}(u)=(2\pi)^{-p/2}e^{-u/2}, \tag{3.10}$$

仍按线性约束的条件方法可求得$\mathrm{SN}_{p_2}(\mathbf{0},\Omega,\alpha)$的分布密度.

定理 3.1 设$X_p=(X_1',X_2')'\sim\mathrm{N}_p(\mathbf{0},\Omega)$, 在线性约束$X_1>\mathbf{0}_{p_1\times1}$条件下, 随机向量$Z=(X_2|X_1>\mathbf{0})$有如下的分布密度:

$$f_Z(z)=\frac{\Phi_{p_1}(\alpha'z|\mathbf{0},I_{p_1})}{\Phi_{p_1}(\mathbf{0}|\mathbf{0},I_{p_1})}\cdot\phi_{p_2}(z|\mathbf{0},\Omega_{22})=2^{p_1}\cdot\Phi_{p_1}(\alpha'z|\mathbf{0})\cdot\phi_{p_2}(z|\mathbf{0},\Omega_{22}), \tag{3.11}$$

这里, $\alpha=\Omega_{22}^{-1}\Omega_{21}\Omega_{11.2}^{-1/2}$, $\Phi_{p_1}(\alpha'z|\mathbf{0},I_{p_1})$表示$N_{p_1}(\mathbf{0},I_{p_1})$分布在$\alpha'z$处的值, $\Phi_{p_1}(\mathbf{0}|\mathbf{0},I_{p_1})$表示$N_{p_1}(\mathbf{0},I_{p_1})$分布在$\mathbf{0}$处的值, 等于$2^{-p_1}$, $\phi_{p_2}(z|\mathbf{0},\Omega_{22})$表示$N_{p_2}(\mathbf{0},\Omega_{22})$的分布密度. 称$Z$服从多元偏态正态分布, 记为$\mathrm{SN}_{p_2}(\mathbf{0},\Omega,\alpha)$, 称$\alpha$为偏态系数.

证明: 证明过程与构造多元偏态PVII分布类似, 从略. □

其次, 求$\mathrm{SN}_{p_2}(\mathbf{0},\Omega,\alpha)$的矩生成函数(Moment generating function), 记为$M(t)$.

定理 3.2 ([3] P340, Theorem 4.1) 设随机向量$X_{p_2\times1}\sim\mathrm{SN}_{p_2}(\mathbf{0},\Omega,\alpha)$, 则它的矩生成函数:

$$\begin{aligned}M(t)&=\Phi_{p_1}^{-1}(\mathbf{0}|\mathbf{0},I_{p_1})\cdot\exp\Big\{\frac{1}{2}(t'\Omega_{22}t)\Big\}\cdot\Phi_{p_1}((\alpha'\Omega_{22}\alpha+I_{p_1})^{-1/2}\alpha'\Omega_{22}t|\mathbf{0},I_{p_1})\\&=2^{p_1}\cdot\exp\Big\{\frac{1}{2}(t'\Omega_{22}t)\Big\}\cdot\Phi_{p_1}((\alpha'\Omega_{22}\alpha+I_{p_1})^{-1/2}\alpha'\Omega_{22}t|\mathbf{0},I_{p_1}).\end{aligned} \tag{3.12}$$

最后, 求$\mathrm{SN}_{p_2}(\mathbf{0},\Omega,\alpha)$的前二阶矩.

引理 3.1 ([4] P51, 引理1) (对向量求导的连锁法则) 设$X_{n\times1},Y_{k\times1},Z_{p\times1}$, 则

$$\frac{\partial Z'}{\partial X}=\frac{\partial Y'}{\partial X}\cdot\frac{\partial Z'}{\partial Y},\qquad\frac{\partial Z}{\partial X'}=\frac{\partial Z}{\partial Y'}\cdot\frac{\partial Y}{\partial X'}. \tag{3.13}$$

引理 3.2 ([5] P324, Appendix)

$$\frac{\partial t'At}{\partial t}=2At,\quad\frac{\partial t'At}{\partial t'}=(2At)',\quad\frac{\partial Dt}{\partial t'}=D,\quad\frac{\partial(Dt)'}{\partial t}=D',\quad\frac{\partial Dt}{\partial t}=\mathrm{vec}(D'), \tag{3.14}$$

这里, $A_{p\times p}$为对称阵, $D_{k\times p}$为任意矩阵, $t_{p\times1}$为向量.

定理 3.3 设随机向量$X_{p_2\times 1}\sim \mathrm{SN}_{p_2}(\mathbf{0},\Omega,\alpha)$, 其中, $\alpha=\Omega_{22}^{-1}\Omega_{21}\Omega_{11.2}^{-1/2}$, 则

$$\begin{aligned}\mathsf{E}X &= \frac{\Phi_{p_1-1}(\mathbf{0}|\mathbf{0},I_{p_1-1})}{\Phi_{p_1}(\mathbf{0}|\mathbf{0},I_{p_1})\sqrt{2\pi}}\cdot\Omega_{22}\alpha(\alpha'\Omega_{22}\alpha+I_{p_1})^{-1/2}\cdot\mathbf{1}\\ &= \sqrt{\frac{2}{\pi}}\cdot\Omega_{22}\alpha(\alpha'\Omega_{22}\alpha+I_{p_1})^{-1/2}\cdot\mathbf{1},\end{aligned}\tag{3.15}$$

其中, $\mathbf{1}_{p_1\times 1}=(1,\cdots,1)'$, $\Phi_{p_1-1}(\mathbf{0}|\mathbf{0},I_{p_1-1})$表示$\mathrm{N}_{p_1-1}(\mathbf{0},I_{p_1-1})$分布在$\mathbf{0}$处的值, 等于$2^{-(p_1-1)}$.

证明: 令$D_{p_1\times p_2}\triangleq(\alpha'\Omega_{22}\alpha+I_{p_1})^{-1/2}\alpha'\Omega_{22}$, 则(3.12)式化为

$$M(t)=2^{p_1}\exp\left\{\frac{1}{2}(t'\Omega_{22}t)\right\}\cdot\Phi_{p_1}(Dt|\mathbf{0},I_{p_1}).\tag{3.16}$$

利用引理3.1和引理3.2, 令$\mathbf{m}_{p_1\times 1}=(m_1,\cdots,m_{p_1})'\triangleq Dt$, 得

$$\frac{\partial M(t)}{\partial t}=2^{p_1}\exp\left\{\frac{1}{2}(t'\Omega_{22}t)\right\}\cdot\left[\Phi_{p_1}(Dt|\mathbf{0},I_{p_1})\cdot\Omega_{22}t+D'\cdot\frac{\partial\Phi_{p_1}(\mathbf{m}|\mathbf{0},I_{p_1})}{\partial\mathbf{m}}\right].$$

因为

$$\frac{\partial\Phi_{p_1}(\mathbf{m}|\mathbf{0},I_{p_1})}{\partial m_j}=(2\pi)^{-1/2}e^{-m_j^2/2}\cdot\Phi_{p_1-1}(\mathbf{m}^j|\mathbf{0},I_{p_1-1}),\tag{3.17}$$

这里, $j=1,\cdots,p_1$, $\mathbf{m}^j$表示向量$\mathbf{m}$去掉第j个分量后重新生成的p_1-1维向量. 所以,

$$\frac{\partial\Phi_{p_1}(\mathbf{m}|\mathbf{0},I_{p_1})}{\partial\mathbf{m}}|_{\mathbf{m}=\mathbf{0}}=(2\pi)^{-1/2}\cdot\Phi_{p_1-1}(\mathbf{0}|\mathbf{0},I_{p_1-1})\cdot\mathbf{1},$$

其中, $\mathbf{1}=(1,\cdots,1)'$. 代入$\mathsf{E}X=\partial M(t)/\partial t|_{t=\mathbf{0}}$化简即得(3.15)式. □

定理 3.4 设随机向量$X_{p_2\times 1}\sim \mathrm{SN}_{p_2}(\mathbf{0},\Omega,\alpha)$, 其中, $\alpha=\Omega_{22}^{-1}\Omega_{21}\Omega_{11.2}^{-1/2}$, 则

$$\begin{aligned}\mathsf{E}XX' &= \Phi_{p_1}^{-1}(\mathbf{0}|\mathbf{0},I_{p_1})\Phi_{p_1}(\mathbf{0}|\mathbf{0},I_{p_1})\Omega_{22}\\ &\quad+\Phi_{p_1}^{-1}(\mathbf{0}|\mathbf{0},I_{p_1})(2\pi)^{-1}\Phi_{p_1-2}(\mathbf{0}|\mathbf{0},I_{p_1-2})\cdot D'AD\\ &= \Omega_{22}+\frac{2}{\pi}\cdot D'AD,\end{aligned}\tag{3.18}$$

其中, $D_{p_1\times p_2}=(\alpha'\Omega_{22}\alpha+I_{p_1})^{-1/2}\alpha'\Omega_{22}$, $\Phi_{p_1-2}(\mathbf{0}|\mathbf{0},I_{p_1-2})$表示$N_{p_1-2}(\mathbf{0},I_{p_1-2})$分布在$\mathbf{0}$处的值, 等于$2^{-(p_1-2)}$.

$$A_{p_1\times p_1}=\begin{pmatrix}0&1&\cdots&1\\1&0&\ddots&\vdots\\\vdots&\ddots&\ddots&1\\1&\cdots&1&0\end{pmatrix}.$$

证明: 利用引理3.1、引理3.2和定理3.3即可, 见附录. □

3.3 多元SPVII分布的各阶矩

仅给出多元SPVII分布的前二阶矩. 由(3.9)式可知,

$$\mathsf{E}Z=\frac{\mathsf{E}R}{\mathsf{E}R_0}\cdot\mathsf{E}X,\qquad \mathsf{E}ZZ'=\frac{\mathsf{E}R^2}{\mathsf{E}R_0^2}\cdot\mathsf{E}XX'. \tag{3.19}$$

定理 3.5 设随机向量$Z\sim \mathrm{SPVII}_{p_2}(\mathbf{0},\Omega,M,v,\alpha)$, 则

$$\mathsf{E}Z=\frac{\Gamma[M-(p+1)/2]}{\Gamma(M-p/2)}\sqrt{\frac{v}{2}}\cdot\mathsf{E}X,\qquad \mathsf{E}ZZ'=\frac{v}{2M-p-2}\cdot\mathsf{E}XX', \tag{3.20}$$

其中, $\mathsf{E}X$等于(3.15)式, $\mathsf{E}XX'$等于(3.18)式.

证明: 由(3.19)式只需求$\mathsf{E}R$、$\mathsf{E}R_0$、$\mathsf{E}R^2$、$\mathsf{E}R_0^2$即可, 见附录. □

推论 3.1 设随机向量$Z\sim \mathrm{St}_{p_2}(\mathbf{0},\Omega,v,\alpha)$, 则

$$\mathsf{E}Z=\frac{\Gamma[(v-1)/2]}{\Gamma(v/2)}\sqrt{\frac{v}{2}}\cdot\mathsf{E}X,\qquad \mathsf{E}ZZ'=\frac{v}{v-2}\cdot\mathsf{E}XX', \tag{3.21}$$

其中, $\mathsf{E}X$等于(3.15)式, $\mathsf{E}XX'$等于(3.18)式.

附　　录

定理2.1的证明.

证明: 由$Z\sim \mathrm{SPVII}_{p_2}(\mathbf{0},\Omega,M,v,\alpha)$, 可令$X_{p\times1}=\begin{pmatrix}X_1\\X_2\end{pmatrix}\sim \mathrm{PVII}_p(\mathbf{0},\Omega,M,v)$, 其中$X_1:p_1\times1$, $X_2:p_2\times1$, $p=p_1+p_2$. 则由[2] P3, 引理2.3得

$$Z\stackrel{d}{=}(X_2|X_1>\mathbf{0}),\qquad \widetilde{Z}=AZ+b\stackrel{d}{=}(AX_2+b|X_1>\mathbf{0})=(\widetilde{X}_2|\widetilde{X}_1>\mathbf{0}). \tag{A.1}$$

其中

$$\widetilde{X}=\begin{pmatrix}\widetilde{X}_1\\\widetilde{X}_2\end{pmatrix}=\begin{pmatrix}I_{p_1}&\mathbf{0}\\\mathbf{0}&A\end{pmatrix}\begin{pmatrix}X_1\\X_2\end{pmatrix}+\begin{pmatrix}\mathbf{0}\\b\end{pmatrix}\triangleq\widetilde{A}X+\widetilde{b}\sim \mathrm{PVII}_p(\widetilde{b},\widetilde{\Omega},M,v). \tag{A.2}$$

$$\widetilde{\Omega}=\widetilde{A}\Omega\widetilde{A}'=\begin{pmatrix}I_{p_1}&\Omega_{12}A'\\A\Omega_{21}&A\Omega_{22}A'\end{pmatrix}. \tag{A.3}$$

$\widetilde{Z}$的密度为([2] P5, (3.7)式):

$$\begin{aligned}f_c(\widetilde{x}_2)&=\frac{1}{p_c}\int_{\widetilde{x}_1>\mathbf{0}}f(\widetilde{x}_1,\widetilde{x}_2|\widetilde{b},\widetilde{\Omega};g^{(p)})\mathrm{d}\widetilde{x}_1\\&=\frac{1}{p_c}f(\widetilde{x}_2|b,\widetilde{\Omega}_{22};\widetilde{g}^{(p_2)})\int_{\widetilde{x}_1>\mathbf{0}}f(\widetilde{x}_1|\widetilde{x}_2,\widetilde{\mu}_{1.2},\widetilde{\Omega}_{11.2};g^{(p_1)}_{q(\widetilde{x}_2)})\mathrm{d}\widetilde{x}_1.\end{aligned} \tag{A.4}$$

化简(A.4)式:

(i)

$$
\begin{aligned}
f(\widetilde{x}|\widetilde{b},\widetilde{\Omega};g^{(p)}) &= |\widetilde{\Omega}|^{-1/2}g^{(p)}((\widetilde{x}-\widetilde{b})'\widetilde{\Omega}^{-1}(\widetilde{x}-\widetilde{b}))\\
&= \frac{\Gamma(M)}{(\pi v)^{p/2}\Gamma(M-p/2)}|\widetilde{\Omega}|^{-1/2}\Big(1+\frac{(\widetilde{x}-\widetilde{b})'\widetilde{\Omega}^{-1}(\widetilde{x}-\widetilde{b})}{v}\Big)^{-M}. \quad (A.5)
\end{aligned}
$$

(ii) 由[2] P3, 引理2.3得

$$
\widetilde{X}_2 = AX_2+b\sim \mathrm{PVII}_{p_2}\Big(b,A\Omega_{22}A',M-\frac{p_1}{2},v\Big), \quad (A.6)
$$

$$
\begin{aligned}
f(\widetilde{x}_2|b,\widetilde{\Omega}_{22};\widetilde{g}^{(p_2)}) &= |A\Omega_{22}A'|^{-1/2}\widetilde{g}^{(p_2)}(\widetilde{Q}_2)\\
&= \frac{\Gamma(M-p_1/2)}{(\pi v)^{p_2/2}\Gamma(M-p/2)}|A\Omega_{22}A'|^{-1/2}\Big(1+\frac{\widetilde{Q}_2}{v}\Big)^{-(M-p_1/2)}. \quad (A.7)
\end{aligned}
$$

这里, $\widetilde{Q}_2=(\widetilde{x}_2-b)'(A\Omega_{22}A')^{-1}(\widetilde{x}_2-b)$.

(iii) 由[2] P3, 引理2.3得

$$
\widetilde{X}_1 \sim \mathrm{PVII}_{p_1}\Big(\mathbf{0},I_{p_1},M-\frac{p_2}{2},v\Big), \quad (A.8)
$$

$$
\begin{aligned}
f(\widetilde{x}_1|\mathbf{0},I_{p_1};\widetilde{g}^{(p_1)}) &= \widetilde{g}^{(p_1)}(\widetilde{Q}_1)\\
&= \frac{\Gamma(M-p_2/2)}{(\pi v)^{p_1/2}\Gamma(M-p/2)}\Big(1+\frac{\widetilde{Q}_1}{v}\Big)^{-(M-p_2/2)}. \quad (A.9)
\end{aligned}
$$

这里, $\widetilde{Q}_1=\widetilde{x}_1'\widetilde{x}_1$. 则

$$
\begin{aligned}
p_c &= \mathsf{P}(\widetilde{x}_1>\mathbf{0})=\int_{\widetilde{x}_1>\mathbf{0}}f(\widetilde{x}_1|\mathbf{0},I_{p_1};\widetilde{g}^{(p_1)})\mathrm{d}\widetilde{x}_1\\
&= F\Big(\mathbf{0}|\mathbf{0},I_{p_1},M-\frac{p_2}{2},v\Big)=2^{-p_1}. \quad (A.10)
\end{aligned}
$$

(iv) 由[2] P4, 定理3.1的证明得

$$
\begin{aligned}
q(\widetilde{x}_2) &= \widetilde{Q}_2=(\widetilde{x}_2-b)'(A\Omega_{22}A')^{-1}(\widetilde{x}_2-b),\\
\widetilde{\mu}_{1.2} &= \Omega_{12}A'(A\Omega_{22}A')^{-1}(\widetilde{x}_2-b)=\Omega_{12}\Omega_{22}^{-1}A^{-1}(\widetilde{x}_2-b),\\
\widetilde{\Omega}_{11.2} &= I_{p_1}-\Omega_{12}A'(A\Omega_{22}A')^{-1}A\Omega_{21}=\Omega_{11.2},\\
\widetilde{\Omega}_{11.2}^{-1/2}\widetilde{\mu}_{1.2}(v+\widetilde{Q}_2)^{-1/2} &= \Omega_{11.2}^{-1/2}\Omega_{12}\Omega_{22}^{-1}A^{-1}(\widetilde{x}_2-b)(v+\widetilde{Q}_2)^{-1/2}\\
&= \alpha'A^{-1}(\widetilde{x}_2-b)(v+\widetilde{Q}_2)^{-1/2}\\
&\triangleq \widetilde{\alpha}'(\widetilde{x}_2-b)(v+\widetilde{Q}_2)^{-1/2},
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\widetilde{x}_1>\mathbf{0}}f(\widetilde{x}_1|\widetilde{x}_2,\widetilde{\mu}_{1.2},\widetilde{\Omega}_{11.2};g_{q(\widetilde{x}_2)}^{(p_1)})\mathrm{d}\widetilde{x}_1\\
&= \frac{\Gamma(M)}{\Gamma(M-p_1/2)\pi^{p_1/2}}\int_{y<\widetilde{\Omega}_{11.2}^{-1/2}\widetilde{\mu}_{1.2}(v+\widetilde{Q}_2)^{-1/2}}(1+y'y)^{-M}\mathrm{d}y\\
&= F(\widetilde{\alpha}'(\widetilde{x}_2-b)(v+\widetilde{Q}_2)^{-1/2}|\mathbf{0},I_{p_1};M,1). \quad (A.11)
\end{aligned}
$$

由(ii) (iii) (iv), 得$\widetilde{Z}$的分布密度为:

$$\begin{aligned} f_{\widetilde{Z}}(\widetilde{x}_2) &= 2^{p_1}\cdot F(\widetilde{\alpha}'(\widetilde{x}_2-b)(v+\widetilde{Q}_2)^{-1/2}|\mathbf{0},I_{p_1};M,1) \\ &\quad \cdot f\Big(\widetilde{x}_2|b,A\Omega_{22}A';M-\frac{p_1}{2},v\Big). \end{aligned} \tag{A.12}$$

其中, $\widetilde{\alpha}=(A')^{-1}\alpha$. 根据定义1.1, (2.1)式得证. □

推论2.1的证明.

证明: 证明过程同定理2.1. 注意到$\widetilde{Z}$的分布密度为:

$$\begin{aligned} f_{\widetilde{Z}}(\widetilde{x}_2) &= 2^{p_1}\cdot F(\widetilde{\alpha}'(\widetilde{x}_2-b)(v+\widetilde{Q}_2)^{-1/2}(v+p_2)^{1/2}|\mathbf{0},I_{p_1};v+p_2) \\ &\quad \cdot f(\widetilde{x}_2|b,A\Omega_{22}A';v). \end{aligned}$$

根据定义1.2, (2.2)式得证. □

定理2.2的证明.

证明: 由$Z_{p_2\times 1}\sim \mathrm{SPVII}_{p_2}(\mathbf{0},\Omega,M,v,\alpha)$, 得

$$X_{p\times 1}\triangleq\begin{pmatrix} X_1\\ Z_1\\ Z_2\end{pmatrix}\sim \mathrm{PVII}_p(\mathbf{0},\Omega,M,v),$$

其中$X_1:p_1\times 1$. 则在线性约束$X_1>\mathbf{0}$的条件下, X的分布密度为:

$$f_c(x)=f_c(x_1,z_1,z_2)=\frac{1}{p_c}f(x_1,z_1,z_2|\mathbf{0},\Omega;M,v). \tag{A.13}$$

其中, $p_c=\mathsf{P}(X_1>\mathbf{0})=F(\mathbf{0}|\mathbf{0},I_{p_1};M-p_2/2,v)=2^{-p_1}$.

由(A.13)式得到在线性约束$X_1>\mathbf{0}$的条件下, $(X_1',Z_1')'$的分布密度为([2] P2, 引理2.2):

$$\begin{aligned} f_c(x_1,z_1) &= \frac{1}{p_c}\int_{z_2} f(x_1,z_1,z_2|\mathbf{0},\Omega;M,v)\mathrm{d}z_2 \\ &= \frac{1}{p_c}f(x_1,z_1|\mathbf{0},\widetilde{\Omega}_{11};\widetilde{g}^{(p_1+p_{21})}) \\ &= \frac{1}{p_c}f\Big(x_1,z_1|\mathbf{0},\widetilde{\Omega}_{11};M-\frac{p_{22}}{2},v\Big). \end{aligned} \tag{A.14}$$

其中,

$$\begin{aligned} & f(x_1,z_1|\mathbf{0},\widetilde{\Omega}_{11};\widetilde{g}^{(p_1+p_{21})}) \\ =\ & |\widetilde{\Omega}_{11}|^{-1/2}\widetilde{g}^{(p_1+p_{21})}(Q_4) \\ =\ & \frac{\Gamma(M-p_{22}/2)}{(\pi v)^{(p_1+p_{21})/2}\Gamma(M-p_{22}/2-p_1+p_{21}/2)}|\widetilde{\Omega}_{11}|^{-1/2}\Big(1+\frac{Q_4}{v}\Big)^{-(M-p_{22}/2)} \\ =\ & \frac{\Gamma(M-p_{22}/2)}{(\pi v)^{(p_1+p_{21})/2}\Gamma(M-p/2)}|\widetilde{\Omega}_{11}|^{-1/2}\Big(1+\frac{Q_4}{v}\Big)^{-(M-p_{22}/2)}. \end{aligned} \tag{A.15}$$

这里, $Q_4=\begin{pmatrix} x_1', & z_1'\end{pmatrix}\widetilde{\Omega}_{11}^{-1}\begin{pmatrix} x_1\\ z_1\end{pmatrix}=\begin{pmatrix} x_1', & z_1'\end{pmatrix}\begin{pmatrix} I_{p_1} & \Omega_{12}\\ \Omega_{21} & \Omega_{22}\end{pmatrix}^{-1}\begin{pmatrix} x_1\\ z_1\end{pmatrix}$.

由(A.14)式, 对X_1积分, 即得到Z_1的边缘密度:

$$\begin{aligned} f_c(z_1) &= \frac{1}{p_c}\int_{x_1>0} f\Big(x_1,z_1|\mathbf{0},\widetilde{\Omega}_{11};M-\frac{p_{22}}{2},v\Big)\mathrm{d}x_1\\ &= \frac{1}{p_c}f(z_1|\mathbf{0},\Omega_{22};\widetilde{g}^{(p_{21})})\int_{x_1>\mathbf{0}} f(x_1|z_1,\mu_{1.2},\Omega_{11.2};g_{q(z_1)}^{(p_1)})\mathrm{d}x_1. \end{aligned}\tag{A.16}$$

其中,

$$\begin{aligned} \mu_{1.2} &= \Omega_{12}\Omega_{22}^{-1}z_1,\\ \Omega_{11.2} &= I_{p_1}-\Omega_{12}\Omega_{22}^{-1}\Omega_{21},\\ f(z_1|\mathbf{0},\Omega_{22};\widetilde{g}^{(p_{21})}) &= |\Omega_{22}|^{-1/2}\widetilde{g}^{(p_{21})}(Q_5)\\ &= \frac{\Gamma(M-p_{22}/2-p_1/2)}{(\pi v)^{p_{21}/2}\Gamma(M-p_{22}/2-p_1/2-p_{21}/2)}\\ &\quad\cdot|\Omega_{22}|^{-1/2}\Big(1+\frac{Q_5}{v}\Big)^{-(M-p_{22}/2-p_1/2)}\\ &= \frac{\Gamma[M-(p_{22}+p_1)/2]}{(\pi v)^{p_{21}/2}\Gamma(M-p/2)}|\Omega_{22}|^{-1/2}\Big(1+\frac{Q_5}{v}\Big)^{-(M-(p_{22}+p_1)/2)}, \quad (A.17)\\ Q_5 &= q(z_1)=z_1'\Omega_{22}^{-1}z_1,\\ \Omega_{11.2}^{-1/2}\mu_{1.2} &= \Omega_{11.2}^{-1/2}\Omega_{12}\Omega_{22}^{-1}z_1\triangleq\widetilde{\alpha}'z_1. \end{aligned}$$

由(A.17)式得$Z_1\sim \mathrm{PVII}_{p_{21}}(\mathbf{0},\Omega_{22},M-(p_{22}+p_1)/2,v)$.

类似(A.11)式得

$$\begin{aligned} &\int_{x_1>\mathbf{0}} f(x_1|z_1,\mu_{1.2},\Omega_{11.2};g_{q(z_1)}^{(p_1)})\mathrm{d}x_1\\ =&\ \frac{\Gamma(M-p_{22}/2)}{\Gamma(M-p_{22}/2-p_1/2)\pi^{p_1/2}}\int_{y<\Omega_{11.2}^{-1/2}\mu_{1.2}(v+Q_5)^{-1/2}}(1+y'y)^{-(M-p_{22}/2)}\mathrm{d}y\\ =&\ F\Big(\widetilde{\alpha}'z_1(v+Q_5)^{-1/2}|\mathbf{0},I_{p_1};M-\frac{p_{22}}{2},1\Big). \end{aligned}\tag{A.18}$$

所以由(A.17)式、(A.18)式, (A.16)式即可化为(2.5)式. □

推论2.2的证明.

证明: 计算边缘分布时只需令定理2.2中的$M=(v+p)/2$即可. 计算条件分布$X_1|Z_1=z_1$时, 由(A.18)式

$$\begin{aligned} &\int_{x_1>\mathbf{0}} f(x_1|z_1,\mu_{1.2},\Omega_{11.2};g_{q(z_1)}^{(p_1)})\mathrm{d}x_1\\ =&\ \frac{\Gamma[(v+p)/2-p_{22}/2]}{\Gamma[(v+p)/2-p_{22}/2-p_1/2]\pi^{p_1/2}}\int_{y<\Omega_{11.2}^{-1/2}\mu_{1.2}(v+Q_5)^{-1/2}}(1+y'y)^{-((v+p)/2-p_{22}/2)}\mathrm{d}y\\ =&\ \frac{\Gamma[(v+p_{21}+p_1)/2]}{\Gamma[(v+p_{21})/2]\pi^{p_1/2}}\int_{y<\widetilde{\alpha}'z_1(v+Q_5)^{-1/2}}(1+y'y)^{-(v+p_{21}+p_1)/2}\mathrm{d}y. \end{aligned}\tag{A.19}$$

令$y_{p_1\times 1}=t_{p_1\times 1}(v+p_{21})^{-1/2}$, $J(y\to t)=(v+p_{21})^{-p_1/2}$, 则(A.19)式得

$$\begin{aligned}&\int_{x_1>\mathbf{0}} f(x_1|z_1,\mu_{1.2},\Omega_{11.2};g_{q(z_1)}^{(p_1)})\mathrm{d}x_1\\ =\ &\frac{\Gamma[(v+p_{21}+p_1)/2]}{\Gamma[(v+p_{21})/2]\pi^{p_1/2}(v+p_{21})^{p_1/2}}\int_{t<\widetilde{\alpha}'z_1(v+Q_5)^{-1/2}(v+p_{21})^{1/2}}\Big(1+\frac{t't}{v+p_{21}}\Big)^{-(v+p_{21}+p_1)/2}\mathrm{d}t\\ =\ &F(\widetilde{\alpha}'z_1(v+Q_5)^{-1/2}(v+p_{21})^{1/2}|\mathbf{0},I_{p_1};v+p_{21}).\end{aligned}\tag{A.20}$$

(2.6)式得证. □

定理2.3的证明.

证明: 由$Z_{p_2\times 1}\sim \mathrm{SPVII}_{p_2}(\mathbf{0},\Omega,M,v,\alpha)$, 得

$$X_{p\times 1}\triangleq\begin{pmatrix}X_1\\Z_1\\Z_2\end{pmatrix}\triangleq\begin{pmatrix}Y\\Z_2\end{pmatrix}\sim \mathrm{PVII}_p(\mathbf{0},\Omega,M,v),$$

其中$X_1:p_1\times 1$, $Y:(p_1+p_{21})\times 1$. 则由[2] P4, 定理3.1的证明得条件分布$Y|Z_2=z_2$的密度为:

$$\begin{aligned}&f(x_1,z_1|\widetilde{\mu}_{1.2},\widetilde{\Omega}_{11.2};g_{q(z_2)}^{(p_1+p_{21})})=|\widetilde{\Omega}_{11.2}|^{-1/2}g_{q(z_2)}^{(p_1+p_{21})}(Q_3)\\ =\ &\frac{\Gamma(M)}{\Gamma[M-(p_1+p_{21})/2]\pi^{(p_1+p_{21})/2}}(v+Q_2)^{-(p_1+p_{21})/2}|\widetilde{\Omega}_{11.2}|^{-1/2}\Big(1+\frac{Q_3}{v+Q_2}\Big)^{-M}.\end{aligned}\tag{A.21}$$

其中,

$$\begin{aligned}Q_2&=q(z_2)=z_2'\Omega_{33}^{-1}z_2,\\ Q_3&=(y-\widetilde{\mu}_{1.2})'\widetilde{\Omega}_{11.2}^{-1}(y-\widetilde{\mu}_{1.2}),\\ \widetilde{\mu}_{1.2}&=\widetilde{\Omega}_{12}\Omega_{33}^{-1}z_2=\begin{pmatrix}\Omega_{13}\\ \Omega_{23}\end{pmatrix}\Omega_{33}^{-1}z_2\triangleq\begin{pmatrix}m_1\\ m_2\end{pmatrix},\\ \widetilde{\Omega}_{11.2}&=\widetilde{\Omega}_{11}-\widetilde{\Omega}_{12}\Omega_{33}^{-1}\widetilde{\Omega}_{21}\\ &=\begin{pmatrix}I_{p_1}&\Omega_{12}\\ \Omega_{21}&\Omega_{22}\end{pmatrix}-\begin{pmatrix}\Omega_{13}\\ \Omega_{23}\end{pmatrix}\Omega_{33}^{-1}\begin{pmatrix}\Omega_{31}&\Omega_{32}\end{pmatrix}\\ &\triangleq\begin{pmatrix}\psi_{11}&\psi_{12}\\ \psi_{21}&\psi_{22}\end{pmatrix}.\end{aligned}\tag{A.22}$$

所以$(Y|Z_2=z_2)\sim \mathrm{PVII}_{p_1+p_{21}}(\widetilde{\mu}_{1.2},\widetilde{\Omega}_{11.2},M,v+Q_2)$. 在此基础上, 再求$Z_1|X_1>\mathbf{0}$的分布密度为([2] P4, 定理3.1的证明):

$$\begin{aligned}f_c(z_1)&=\frac{1}{p_c}\int_{x_1>\mathbf{0}}f(x_1,z_1|\widetilde{\mu}_{1.2},\widetilde{\Omega}_{11.2};M,v+Q_2)\mathrm{d}x_1\\ &=\frac{1}{p_c}f(z_1|m_2,\psi_{22};\widetilde{g}^{(p_{21})})\int_{x_1>\mathbf{0}}f(x_1|z_1,m_{1.2},\psi_{11.2};g_{q(z_1)}^{(p_1)})\mathrm{d}x_1.\end{aligned}\tag{A.23}$$

其中,

$$\begin{aligned} m_{1.2} &= m_1+\psi_{12}\psi_{22}^{-1}(z_1-m_2), \\ \psi_{11.2} &= \psi_{11}-\psi_{12}\psi_{22}^{-1}\psi_{21}, \\ q(z_1) &= (z_1-m_2)'\psi_{22}^{-1}(z_1-m_2)\triangleq Q_4. \end{aligned} \tag{A.24}$$

下面分别计算(A.23)式中的三项.

(i) 由[2] P4, 定理3.1的证明得

$$g_{Q_4}^{(p_1)}(u)=\frac{\Gamma(M)}{\Gamma(M-p_1/2)\pi^{p_1/2}}(v+Q_2+Q_4)^{-p_1/2}\Big(1+\frac{u}{v+Q_2+Q_4}\Big)^{-M}, \qquad u>0. \tag{A.25}$$

所以,

$$\begin{aligned} &\int_{x_1>\mathbf{0}} f(x_1|z_1,m_{1.2},\psi_{11.2};g_{q(z_1)}^{(p_1)})\mathrm{d}x_1 \\ =&\int_{x_1>\mathbf{0}} |\psi_{11.2}|^{-1/2}g_{Q_4}^{(p_1)}(Q_5)\mathrm{d}x_1 \\ =&\int_{x_1>\mathbf{0}} \frac{\Gamma(M)}{\Gamma(M-p_1/2)\pi^{p_1/2}}|\psi_{11.2}|^{-1/2}(v+Q_2+Q_4)^{-p_1/2}\Big(1+\frac{Q_5}{v+Q_2+Q_4}\Big)^{-M}\mathrm{d}x_1. \end{aligned} \tag{A.26}$$

其中, $Q_5=(x_1-m_{1.2})'\psi_{11.2}^{-1}(x_1-m_{1.2})$. 令$t_{p_1\times 1}=\psi_{11.2}^{-1/2}(x_1-m_{1.2})(v+Q_2+Q_4)^{-1/2}$, $J(x_1\to t)=|\psi_{11.2}|^{1/2}(v+Q_2+Q_4)^{p_1/2}$, 代入(A.26)式:

$$\begin{aligned} &\int_{x_1>\mathbf{0}} f(x_1|z_1,m_{1.2},\psi_{11.2};g_{q(z_1)}^{(p_1)})\mathrm{d}x_1 \\ =&\ \frac{\Gamma(M)}{\Gamma(M-p_1/2)\pi^{p_1/2}}\int_{t<\psi_{11.2}^{-1/2}m_{1.2}(v+Q_2+Q_4)^{-1/2}}(1+t't)^{-M}\mathrm{d}t \\ =&\ F(\psi_{11.2}^{-1/2}m_{1.2}(v+Q_2+Q_4)^{-1/2}|\mathbf{0},I_{P_1};M,1). \end{aligned} \tag{A.27}$$

(A.27)式表示$\mathrm{PVII}_{p_1}(\mathbf{0},I_{P_1};M,1)$累计分布函数在$\psi_{11.2}^{-1/2}m_{1.2}(v+Q_2+Q_4)^{-1/2}$处的值.

(ii) 由[2] P3, 引理2.3得

$$\widetilde{g}^{(p_{21})}(u)=\frac{\Gamma(M-p_1/2)}{\Gamma(M-p_1/2-p_{21}/2)\pi^{p_{21}/2}}(v+Q_2)^{-p_{21}/2}\Big(1+\frac{u}{v+Q_2}\Big)^{-(M-p_1/2)}, \qquad u>0. \tag{A.28}$$

所以,

$$\begin{aligned} &f(z_1|m_2,\psi_{22};\widetilde{g}^{(p_{21})}) \\ =&\ |\psi_{22}|^{-1/2}\widetilde{g}^{(p_{21})}(Q_4) \\ =&\ \frac{\Gamma(M-p_1/2)}{\Gamma(M-p_1/2-p_{21}/2)\pi^{p_{21}/2}}|\psi_{22}|^{-1/2}(v+Q_2)^{-p_{21}/2}\Big(1+\frac{Q_4}{v+Q_2}\Big)^{-(M-p_1/2)} \\ =&\ f\Big(z_1|m_2,\psi_{22};M-\frac{p_1}{2},v+Q_2\Big) \\ \sim&\ \mathrm{PVII}_{p_{21}}\Big(m_2,\psi_{22},M-\frac{p_1}{2},v+Q_2\Big). \end{aligned} \tag{A.29}$$

(iii) 同(ii)证明过程,

$$\begin{aligned} p_c &= \mathsf{P}(X_1 > \mathbf{0}_{p_1\times 1}) \\ &= \int_{X_1>\mathbf{0}} f(x_1|m_1,\psi_{11};\widetilde{g}^{(p_1)})\mathrm{d}x_1 \\ &= \int_{X_1>\mathbf{0}} |\psi_{11}|^{-1/2}\widetilde{g}^{(p_1)}(Q_6)\mathrm{d}x_1 \\ &= \int_{X_1>\mathbf{0}} \frac{\Gamma(M-p_{21}/2)}{\Gamma(M-p_{21}/2-p_1/2)\pi^{p_1/2}}|\psi_{11}|^{-1/2}(v+Q_2)^{-p_1/2} \\ &\quad \cdot\Big(1+\frac{Q_6}{v+Q_2}\Big)^{-(M-p_{21}/2)}\mathrm{d}x_1. \end{aligned} \tag{A.30}$$

其中, $Q_6=(x_1-m_1)'\psi_{11}^{-1}(x_1-m_1)$. 令$t_{p_1\times 1}=\psi_{11}^{-1/2}(x_1-m_1)(v+Q_2)^{-1/2}$, $J(x_1\to t)=|\psi_{11}|^{1/2}(v+Q_2)^{p_1/2}$代入(A.30)式, 得

$$\begin{aligned} p_c &= \frac{\Gamma(M-p_{21}/2)}{\Gamma(M-p_{21}/2-p_1/2)\pi^{p_1/2}}\int_{t<\psi_{11}^{-1/2}m_1(v+Q_2)^{-1/2}}(1+t't)^{-(M-p_{21}/2)}\mathrm{d}t \\ &= F\Big(\psi_{11}^{-1/2}m_1(v+Q_2)^{-1/2}|\mathbf{0},I_{p_1};M-\frac{p_{21}}{2},1\Big). \end{aligned} \tag{A.31}$$

(A.31)式表示$\mathrm{PVII}_{p_1}(\mathbf{0},I_{p_1};M-p_{21}/2,1)$累计分布函数在$\psi_{11}^{-1/2}m_1(v+Q_2)^{-1/2}$处的值.
由(A.27)、(A.29)、(A.31)知(A.23)式即可得到(2.7)式. □

定理3.4的证明.

证明: 令$p_c=\Phi_{p_1}(\mathbf{0}|\mathbf{0},I_{p_1})$, $\mathbf{F}_{p_1\times 1}\triangleq\partial\Phi_{p_1}(\mathbf{m}|\mathbf{0},I_{p_1})/\partial\mathbf{m}$, 由引理3.2得

$$\begin{aligned} \frac{\partial^2 M(t)}{\partial t\partial t'} &= p_c^{-1}\exp\Big\{\frac{1}{2}(t'\Omega_{22}t)\Big\} \\ &\quad\cdot\Big[\Phi_{p_1}(Dt|\mathbf{0},I_{p_1})\cdot\Omega_{22}t\otimes(\Omega_{22}t)'+\Phi_{p_1}(Dt|\mathbf{0},I_{p_1})\cdot\Omega_{22} \\ &\quad+\Omega_{22}t\otimes(D'\cdot\mathbf{F})'+(D'\cdot\mathbf{F})\otimes(\Omega_{22}t)'+D'\cdot\frac{\partial\mathbf{F}}{\partial t'}\Big]. \end{aligned} \tag{A.32}$$

由引理3.1知

$$\frac{\partial\mathbf{F}}{\partial t'}=\frac{\partial\mathbf{F}}{\partial(Dt)'}\cdot\frac{\partial(Dt)}{\partial t'}=\frac{\partial\mathbf{F}}{\partial\mathbf{m}'}\cdot D. \tag{A.33}$$

关键求$\partial\mathbf{F}/\partial\mathbf{m}'$. 由(3.17)式, 得

$$\begin{aligned} F_j &= \frac{\partial\Phi_{p_1}(\mathbf{m}|\mathbf{0},I_{p_1})}{\partial m_j} \\ &= (2\pi)^{-1/2}e^{-m_j^2/2}\cdot\int_{y_1<m_1}\cdots\int_{y_{j-1}<m_{j-1}}\int_{y_{j+1}<m_{j+1}} \\ &\quad\cdots\int_{y_{p_1}<m_{p_1}}(2\pi)^{-(p_1-1)/2}e^{-(1/2)\sum\limits_{i=1,i\neq j}^{p_1}y_i^2}dy_1\cdots\mathrm{d}y_{j-1}\mathrm{d}y_{j+1}\cdots\mathrm{d}y_{p_1}. \end{aligned} \tag{A.34}$$

1. 当$k=j$时, $k,j=1,\cdots,p_1$,

$$\begin{aligned}\frac{\partial F_j}{\partial m_k} &= \frac{\partial F_j}{\partial m_j}\\ &= (2\pi)^{-1/2}e^{-m_j^2/2}\cdot(-m_j)\cdot\int_{y_1<m_1}\cdots\int_{y_{j-1}<m_{j-1}}\int_{y_{j+1}<m_{j+1}}\\ &\quad\cdots\int_{y_{p_1}<m_{p_1}}(2\pi)^{-(p_1-1)/2}e^{-(1/2)\sum\limits_{i=1,i\neq j}^{p_1}y_i^2}\mathrm{d}y_1\cdots\mathrm{d}y_{j-1}\mathrm{d}y_{j+1}\cdots\mathrm{d}y_{p_1}. \quad (A.35)\end{aligned}$$

所以$\partial F_j/\partial m_j|_{\mathbf{m}=\mathbf{0}}=0$.

2. 当$k\neq j$时, $k,j=1,\cdots,p_1$,

$$\begin{aligned}&\frac{\partial F_j}{\partial m_k}\\ =\ & (2\pi)^{-1}e^{-(m_j^2+m_k^2)/2}\cdot\int_{y_1<m_1}\cdots\int_{y_{k-1}<m_{k-1}}\int_{y_{k+1}<m_{k+1}}\cdots\int_{y_{j-1}<m_{j-1}}\int_{y_{j+1}<m_{j+1}}\\ &\cdots\int_{y_{p_1}<m_{p_1}}(2\pi)^{-(p_1-2)/2}e^{-(1/2)\sum\limits_{i=1,i\neq j,k}^{p_1}y_i^2}\mathrm{d}y_1\cdots\mathrm{d}y_{k-1}\mathrm{d}y_{k+1}\cdots\mathrm{d}y_{j-1}\mathrm{d}y_{j+1}\cdots\mathrm{d}y_{p_1}\\ =\ & (2\pi)^{-1}e^{-(m_j^2+m_k^2)/2}\cdot\Phi_{p_1-2}(\mathbf{m}^{j,k}|\mathbf{0},I_{p_1-2}). \quad (A.36)\end{aligned}$$

这里, $\mathbf{m}^{k,j}$表示向量$\mathbf{m}$去掉第k,j个分量后重新生成的p_1-2维向量. 所以

$$\frac{\partial F_j}{\partial m_k}\Big|_{\mathbf{m}=\mathbf{0}}=(2\pi)^{-1}\cdot\Phi_{p_1-2}(\mathbf{0}|\mathbf{0},I_{p_1-2}). \quad (A.37)$$

综上所述,

$$\frac{\partial \mathbf{F}}{\partial \mathbf{m}'}\Big|_{\mathbf{m}=\mathbf{0}}=(2\pi)^{-1}\cdot\Phi_{p_1-2}(\mathbf{0}|\mathbf{0},I_{p_1-2})\cdot\begin{pmatrix}0&1&\cdots&1\\1&0&\ddots&\vdots\\\vdots&\ddots&\ddots&1\\1&\cdots&1&0\end{pmatrix}. \quad (A.38)$$

再由(A.32)、(A.33)、(A.38)式得到

$$\begin{aligned}\mathsf{E}XX' &= \frac{\partial^2 M(t)}{\partial t\partial t'}\Big|_{t=\mathbf{0}}\\ &= p_c^{-1}[\Phi_{p_1}(\mathbf{0}|\mathbf{0},I_{p_1})\cdot\Omega_{22}+(2\pi)^{-1}\cdot\Phi_{p_1-2}(\mathbf{0}|\mathbf{0},I_{p_1-2})\cdot D'AD].\end{aligned}$$

即得(3.18)式.　　□

定理3.5的证明.

证明: $Z\sim \mathrm{SPVII}_{p_2}(\mathbf{0},\Omega,M,v,\alpha)$, 所以$R$有如下分布密度((3.4)式):

$$\begin{aligned} h_R(r) &= \frac{2\pi^{p/2}}{\Gamma(p/2)}r^{p-1}g^{(p)}(r^2) \\ &= \frac{2\pi^{p/2}}{\Gamma(p/2)}r^{p-1}\cdot\frac{\Gamma(M)}{(\pi v)^{p/2}\Gamma(M-p/2)}\Big(1+\frac{r^2}{v}\Big)^{-M}, \end{aligned} \tag{A.39}$$

$$\begin{aligned} \mathsf{E}R &= \int_0^{+\infty} r h_R(r)\mathrm{d}r \\ &= \frac{2\Gamma(M)}{v^{p/2}\Gamma(p/2)\Gamma(M-p/2)}\int_0^{+\infty} r^p\Big(1+\frac{r^2}{v}\Big)^{-M}\mathrm{d}r. \end{aligned} \tag{A.40}$$

令$t\triangleq r^2/v$, $J(r\to t)=(1/2)v^{1/2}t^{-1/2}$, 则化简上式得

$$\mathsf{E}R=\frac{\Gamma[M-(p+1)/2]\Gamma[(p+1)/2]}{\Gamma(M-p/2)\Gamma(p/2)}v^{1/2}. \tag{A.41}$$

因为$R_0^2\sim\chi_p^2$, 所以

$$\mathsf{E}R_0=\mathsf{E}(R_0^2)^{1/2}=\mathsf{E}T^{1/2}=\int_0^{+\infty}\frac{(1/2)^{p/2}}{\Gamma(p/2)}e^{-t/2}t^{p/2-1+1/2}\mathrm{d}t=\frac{\Gamma[(p+1)/2]}{\Gamma(p/2)}2^{1/2}. \tag{A.42}$$

由(3.19)、(A.41)、(A.42)得

$$\mathsf{E}Z=\frac{\Gamma[M-(p+1)/2]}{\Gamma(M-p/2)}\sqrt{\frac{v}{2}}\cdot\mathsf{E}X.$$

由(A.39)式知

$$\mathsf{E}R^2=\frac{p}{2M-p-2}v. \tag{A.43}$$

因为$R_0^2\sim\chi_p^2$, 所以$\mathsf{E}R_0^2=\mathsf{E}T=p$. □

参 考 文 献

[1] 方开泰, 张尧庭, 广义多元分析, 科学出版社, 1993.

[2] 温阳俊, 朱道元, 从偏态Pearson VII分布生成的新的多元偏态t分布, 应用概率统计, **25(5)**(2009), 485–498.

[3] Dey, D.K. and Liu, J.F., A new construction for skew multivariate distributions, *J. Multivar. Anal.*, **95(2)**(2005), 323–344.

[4] 朱道元, 吴诚鸥, 秦伟良, 多元统计分析与软件SAS, 东南大学出版社, 1999.

[5] Genton, M.G., He, L. and Liu, X., Moments of skew-normal random vectors and their quadratic forms, *Statist. Probab. Lett.*, **51(4)**(2001), 319–325.

[6] Kim, H.M. and Mallick, B.K., Moments of random vectiors with skew t distribution and their quadratic forms, *Statist. Prob. Lett.*, **63(4)**(2003), 417–423.

[7] Arellano-Valle, R.B., del Pino, G. and San Martín, E., Definition and probabilistic properties of skew-distributions, *Statist. Probab. Lett.*, **58(2)**(2002), 111–121.

[8] Gupta, A.K., Gonzáles-Farías, G. and Domínguez-Molina, J.A., A multivariate skew normal distribution , *J. Multivar. Anal.*, **89(1)**(2004), 181–190.

[9] Zocks, S., *Patametric Statistical Inference*, Oxford: Pergamon Press, 1981.

Properties of New Multivariate Skew t Distributions Generated from Skew Pearson VII Distributions

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Multivariate t distributions belong to elliptically contoured distributions. However, they are symmetric. In many fields such as economics, psychology and sociology, sometimes error structures in a regression type models no longer satisfy symmetric property. Generally there is a presence of high skewness. Therefore, multivariate skew elliptical distributions have been developed. In this paper, properties about known family of multivariate skew t distributions are stressed. Linear transformations, marginal and conditional density are given. Also moments are derived.

Keywords: Skew Pearson type VII distributions, skew t distributions, skew normal distributions, density generator, moment generating function.

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