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Statistical Analysis of Competing Failure Modes in Accelerated Life Testing Based on Assumed Copulas *

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Abstract

Among those papers discussing statistical analysis of competing failure data, most of them assume independence for failure modes. In this paper we use copula as the dependence link function to assess competing risk models in accelerated life testing. We compare via simulation the results of lifetime when the failure modes are dependent with those when the failure modes are independent, and apply our approach to a real data set in the literature.

Keywords: Competing risks, accelerating life testing (ALT), copula.
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§1. Introduction

The competing risk model assuming independent competing failure modes for each stress level has been considered by a number of authors in the engineering, medical, econometric and actuarial literature, and the list of references, scattered throughout these areas, is extensive. McCool (1978) considered a method of calculating estimate intervals for Weibull parameters of a primary failure when a secondary failure mode had the same (but unknown) Weibull shape parameter. Klein and Basu (1981, 1982) obtained maximum likelihood estimators when the lifetimes followed exponential or Weibull distribution — with common or different shape parameters under type I, type II or progressively censoring. Zhang (2002) presented analytical (maximum likelihood) methods to analyze data on competing failure modes in ALT, under two cases: ignoring the difference among failure modes or not.

The methods mentioned above are based on the use of maximum likelihood estimation which may require large sample size in order to obtain good estimators for each failure

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mode, which may be not appropriate for expensive components. Hence, a Bayesian approach is considered by some authors. DeGroot and Goel (1979) considered the partial step-stress ALT in the framework of Bayesian inference. Van Dorp et al. (1996) developed a Bayesian model for multi step-stress ALT. Bunea and Mazzuchi (2005) presented a Bayesian framework for the analysis of ALT data with possible multiple failure modes.

However, the competing failure modes are usually dependent. The literature about dependent competing failure modes is rare in the engineering, but much more in the biostatistics and econometrics. Models with copulas have become increasingly popular for modeling multivariate survival data. Carriere (1994) and Escarela and Carriere (2003) modeled dependence between two failure times by a two-dimensional copula. Carriere (1994) used a bivariate Gaussian copula to model the effect of completely eliminating of one of two competing cause of death on human mortality. In Escarela and Carriere (2003), the bivariate Frank copula was fitted to a prostate cancer data set.

In this paper, we will introduce copula into the reliability and analyze ALT data with dependent multiple failure modes. Section 2 is devoted to copulas and their properties, and provides the background material on the Archimedean Copula. Section 3 presents the constant stress ALT model and estimation of the model parameters. A simulation example is given in Section 4 to show the effectiveness of the method. The model is applied to the real data set from Klein and Basu (1981) in Section 5.

§2. Copulas and Their Properties

2.1 Definition

Copulas provide a very convenient way to model and measure the dependence among competing failure modes since they give the dependence structure which relates the known marginal distributions of failure modes to their multivariate joint distribution. In order to see this, we first provide a short introduction on copulas.

Let $u=(u_1,\ldots,u_m)'$, $u_j\in[0,1]$. An m-dimensional copula $\mathcal{C}(u)$ is conventionally defined as a multivariate cumulative distribution function with uniform margins. A probabilistic way to define the copula is provided by the theorem of Sklar (1959).

Theorem 2.1 (Sklar, 1959) Let $X_1,X_2,\ldots,X_m$ be random variables with continuous distribution functions (d.f.) $F_1(x_1),F_2(x_2),\ldots,F_m(x_m)$ respectively, and $\mathcal{H}(x_1,\ldots,x_m)$ be their joint d.f.. Then there exist a unique m-dimensional copula $\mathcal{C}$, such that for all x in $\mathcal{R}^m$

$$\mathcal{H}(x_1, \ldots, x_m) = \mathcal{C}(F_1(x_1), \ldots, F_m(x_m)). \tag{2.1}$$

Conversely, if $\mathcal{C}$ is an m-dimensional copula and $F_1(x_1), F_2(x_2), \ldots, F_m(x_m)$ are d.f.s, then $\mathcal{C}(F_1(x_1), \ldots, F_m(x_m))$ is an m-dimensional d.f. with margins $F_1(x_1), F_2(x_2), \ldots, F_m(x_m)$.

Thus, from (2.1), one can construct a dependence structure, i.e., an m-dimensional d.f. $\mathcal{H}$ by appropriately choosing a set of margins $F_1(x_1), F_2(x_2), \ldots, F_m(x_m)$ and a copula function $\mathcal{C}$. In order to construct a copula function, a corollary of Sklar's theorem can be applied, according to which a copula can be represented as an m-dimensional distribution function with continuous margins, evaluated at the inverse functions $F_1^{-1}(u_1), \ldots,$ $F_m^{-1}(u_m)$, i.e.,

$$\mathcal{C}(u_1, \ldots, u_m) = \mathcal{H}(F_1^{-1}(u_1), \ldots, F_m^{-1}(u_m)). \tag{2.2}$$

Let $S_1(x_1), S_2(x_2), \ldots, S_m(x_m)$ be survival functions of $X_1, X_2, \ldots, X_m$ respectively, and $\mathcal{S}(x_1, \ldots, x_m)$ be joint survival function. By using the probability integral transformation, $X_j \mapsto F_j(X_j) = 1 - S_j(X_j)$. It is easy to verify (see Sklar (1959)) that Sklar's theorem, given by (2.1), can be restated to express the multivariate survival function $\mathcal{S}(x_1, \ldots, x_m)$ via an appropriate copula $\overline{\mathcal{C}}$ called the survival copula of $(X_1, \ldots, X_m)$. Thus,

$$\mathcal{S}(x_1, \ldots, x_m) = \overline{\mathcal{C}}(S_1(x_1), \ldots, S_m(x_m)). \tag{2.3}$$

The survival copula, which is also a copula, relates the marginal survival functions $S_1(x_1)$, $S_2(x_2), \ldots, S_m(x_m)$ to the multivariate joint survival function $\mathcal{S}(x_1, \ldots, x_m)$ in much the same way as the copula $\mathcal{C}$ relates the marginal distribution functions to the multivariate distribution function. Note that the copula $\mathcal{C}$ and the survival copula $\overline{\mathcal{C}}$ of a random vector $(X_1, \ldots, X_m)$ are not the same in general. However, they satisfy some relationship, for example, in two-dimensional case, $\overline{\mathcal{C}}(1-u, 1-v) = 1 + u + v - \mathcal{C}(u, v)$. This will be the approach taken here in modeling the joint survival function of competing failure modes.

2.2 Measures of Association

We will consider here the standard dependence measures, Kendall's τ and Spearman's ρ_S. These measures are related to the copula since the latter is an expression of the stochastic relationship between X and Y within the entire range of values the variables can take. It is not difficult to show that

$$\rho_S(X, Y) = 12 \int_0^1 \int_0^1 \mathcal{C}(u_1, u_2) \mathrm{d}u_1 \mathrm{d}u_2 - 3, \tag{2.4}$$

and that

$$\tau(X,Y) = 4\int_0^1\int_0^1 \mathcal{C}(u_1,u_2)\mathrm{d}u_1\mathrm{d}u_2 - 1. \tag{2.5}$$

For further properties of ρ_S and τ, see Nelson (1999).

2.3 Archimedean Copula

Archimedean copulas have a wide range of applications, because

- they can be constructed easily;
- they have many nice properties;
- a lot of families belong to this class;
- they can be extended from two-dimension to m-dimension easily when satisfying some conditions.

The Archimedean copula family is given through its generator $\phi_\theta(\cdot)$ indexed by a parameter θ, i.e.

$$C_\theta(u,v) = \phi_\theta\{\phi_\theta^{-1}(u) + \phi_\theta^{-1}(v)\}, \tag{2.6}$$

where $0 \le u \le 1$, $0 \le v \le 1$, $\phi_\theta(0) = 1$, $\phi'_\theta(t) < 0$, $\phi''_\theta(t) > 0$, $0 \le \phi_\theta(t) \le 1$. In this paper Gumbel copula, which belongs to Archimedean copula family, is used to depict the dependence among competing failure modes.

Gumbel's family, with generator $\phi_\theta(t) = (-\log(t))^\theta$, $\theta \in [1,+\infty)$, can be expressed as

$$C_\theta(u,v) \equiv \exp(-[(-\log u)^\theta + (-\log v)^\theta]^{1/\theta}), \qquad \theta \in [1,+\infty). \tag{2.7}$$

The failure modes are positively associated, and independent when $\theta = 1$.

§3. The Statistical Analysis of Constant-Stress ALT

3.1 Estimation of Mean Lifetime with competing Failure Modes

First we consider a k constant stress ALT for series systems with two competing failure modes. At each stress level S_i, $i = 1,2,\ldots,k$, a number of n_i systems are tested until r_i of them fail. $(t_{i1},c_{i1}),(t_{i2},c_{i2}),\ldots,(t_{ir_i},c_{ir_i})$ are the failure data, where t_{il} denotes the failure time of l-th system under stress level S_i, $t_{i1} \le t_{i2} \le \cdots \le t_{ir_i}$ and c_{il} takes any integer in the set of $\{1,2\}$. $c_{il} = 1$ and $c_{il} = 2$ indicate the failure is caused by failure mode 1 and 2 respectively.

The approach discussed in this paper is based on the following five assumptions.

Assumption 1 The failures occur due to one of the two competing failure modes, with lifetime T_1 and T_2, and the dependence of the two competing failure modes is depicted by the bivariate Gumbel copula. Here the survival copula (2.7) is used.

Assumption 2 The lifetime of the series system is the shorter of T_j, $j = 1, 2$.

Assumption 3 The failure time of T_j under stress level S_i follows the exponential distribution with failure rate λ_{ij} with density function

$$f_{ij}(t) = \lambda_{ij} e^{-\lambda_{ij} t}, \qquad i = 1, 2, \ldots, k, \; j = 1, 2.$$

From Assumptions 1 and 3, we know that under stress level S_i $(i = 1, 2, \ldots, k)$, the survival function of the lifetime is

$$S_i(t) = \exp\{-(\lambda_{i1}^{\theta} + \lambda_{i2}^{\theta})^{1/\theta} t\}. \tag{3.1}$$

Assumption 4 Under stress level S_i, the relation between lifetime and stress (known as accelerate function, or AF for short) of the j-th failure mode satisfies the log-linear equation

$$\log \mu_{ij} = \alpha_j + \beta_j \varphi(S_i), \qquad i = 0, 1, 2, \ldots, k, \; j = 1, 2, \tag{3.2}$$

where $\mu_{ij} = 1/\lambda_{ij}$, α_j and β_j are unknown parameters, and $\varphi(S)$ is a given function of stress level S. (3.2) is a general form, which contains the Arrhenius model and the inverse power law model as two commonly used special cases.

Assumption 5 Dependence of competing failure modes maintains the same during ALT. That is, θ will not change as the stress changes.

Under stress level S_i, let g_{ij} denote the failure number of products due to failure modes j. That is,

$$g_{ij} = \sum_{l=1}^{r_i} \delta_j(c_{il}), \qquad \delta_j(c_{il}) = \begin{cases} 1, & \text{if } c_{il} = j, \\ 0, & \text{if } c_{il} \neq j. \end{cases}$$

To get the likelihood function, we notice that when $\delta_1(c_{il}) = 1$, t_{il} is the failure time due to failure mode 1, and when $\delta_1(c_{il}) = 0$, t_{il} is the failure time under failure mode 2, which can be considered as a random censoring time of failure mode 1. Thus the likelihood due to failure mode 1 under stress S_i is

$$\begin{aligned} L_{i1} = & \prod_{l=1}^{r_i} \left\{ \lim_{\Delta t \to 0} \frac{\mathsf{P}[(T_1 \le T_2) \cap (t_{il} \le T_1 \le t_{il} + \Delta t)]}{\Delta t} \right\}^{\delta_1(c_{il})} \\ & \times \left\{ \lim_{\Delta t \to 0} \frac{\mathsf{P}[(T_2 \le T_1) \cap (t_{il} \le T_2 \le t_{il} + \Delta t)]}{\Delta t} \right\}^{1-\delta_1(c_{il})} \\ & \times \{\mathsf{P}[T_1 > t_{ir_i}, T_2 > t_{ir_i}]\}^{n_i - r_i}. \end{aligned}$$

Since

$$\begin{aligned}
&\lim_{\Delta t\to 0}\frac{\mathsf{P}[(T_1\le T_2)\cap(t_{il}\le T_1\le t_{il}+\Delta t)]}{\Delta t}\\
=&\lim_{\Delta t\to 0}\frac{\mathsf{P}[T_1\le T_2|t_{il}\le T_1\le t_{il}+\Delta t]\times\mathsf{P}[t_{il}\le T_1\le t_{il}+\Delta t]}{\Delta t}\\
=&\frac{\partial C(u_1,u_2)}{\partial u_1}\Big|_{u_1=S_{i1}(t_{il}),u_2=S_{i2}(t_{il})}\times f_{i1}(t_{il})\\
=&\lambda_{i1}^{\theta}(\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta})^{1/\theta-1}\cdot\exp\{-(\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta})^{1/\theta}t_{il}\},
\end{aligned}$$

and similarly

$$\lim_{\Delta t\to 0}\frac{\mathsf{P}[(T_1\ge T_2)\cap(t_{il}\le T_1\le t_{il}+\Delta t)]}{\Delta t}=\lambda_{i2}^{\theta}(\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta})^{1/\theta-1}\cdot\exp\{-(\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta})^{1/\theta}t_{il}\},$$

we have

$$L_{i1}=\Big(\frac{\lambda_{i1}}{\lambda_{i2}}\Big)^{\theta g_{i1}}\lambda_{i2}^{\theta r_i}\Big(\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta}\Big)^{r_i(\frac{1}{\theta}-1)}\exp\Big\{-(\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta})^{1/\theta}\Big(\sum_{l=1}^{r_i}t_{il}+(n_i-r_i)t_{ir_i}\Big)\Big\}.\tag{3.3}$$

In the same manner, the likelihood due to failure mode 2 under stress S_i is

$$L_{i2}=\Big(\frac{\lambda_{i2}}{\lambda_{i1}}\Big)^{\theta g_{i2}}\lambda_{i1}^{\theta r_i}(\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta})^{r_i(1/\theta-1)}\exp\Big\{-(\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta})^{1/\theta}\Big(\sum_{l=1}^{r_i}t_{il}+(n_i-r_i)t_{ir_i}\Big)\Big\}.$$

Noticing that $g_{i1}=r_i-g_{i2}$, we know $L_{i1}=L_{i2}$. Therefore, the likelihood function under stress level S_i is $L_i=L_{i1}L_{i2}=L_{i1}^2$. Let

$$TTT_i=\sum_{l=1}^{r_i}t_{il}+(n_i-r_i)t_{ir_i}\qquad\text{and}\qquad\overline{T}_i=TTT_i/r_i,$$

the total lifetime under stress S_i. We have

$$\log L_i=2*\Big[\theta g_{i1}\log\lambda_{i1}+\theta(r_i-g_{i1})\log\lambda_{i2}+r_i\Big(\frac{1}{\theta}-1\Big)\log(\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta})-(\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta})^{1/\theta}TTT_i\Big].$$

Therefore, we obtain the system of likelihood equations

$$\frac{\mathrm{d}\log L_i}{\mathrm{d}\lambda_{i1}}=\frac{\theta g_{i1}}{\lambda_{i1}}+\frac{r_i(1-\theta)\lambda_{i1}^{\theta-1}}{\lambda_{i1}^{\theta}+\lambda_2^{\theta}}-\lambda_{i1}^{\theta-1}(\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta})^{\theta-1}TTT_i=0,\tag{3.4}$$

$$\frac{\mathrm{d}\log L_i}{\mathrm{d}\lambda_{i2}}=\frac{\theta(r_i-g_{i1})}{\lambda_{i2}}+\frac{r_i(1-\theta)\lambda_{i2}^{\theta-1}}{\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta}}-\lambda_{i2}^{\theta-1}(\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta})^{\theta-1}TTT_i=0,\tag{3.5}$$

$$\frac{\mathrm{d}\log L_i}{\mathrm{d}\theta}=0.\tag{3.6}$$

Using simple algebra calculations, formulas (3.4) and (3.5) can be transformed into the following form

$$(\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta})^{\theta-1}TTT_i=\frac{\theta g_{i1}}{\lambda_{i1}^{\theta}}+\frac{r_i(1-\theta)}{\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta}},\tag{3.7}$$

$$(\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta})^{\theta-1}TTT_i=\frac{\theta(r_i-g_{i1})}{\lambda_{i2}^{\theta}}+\frac{r_i(1-\theta)}{\lambda_{i1}^{\theta}+\lambda_{i2}^{\theta}}.\tag{3.8}$$

From equations (3.7) and (3.8), we obtain

$$\lambda_{i1}^{\theta} = \lambda_{i2}^{\theta} \cdot \frac{g_{i1}}{r_i - g_{i1}}. \tag{3.9}$$

Substituting (3.9) into (3.7) gives rise to

$$\widehat{\lambda}_{i1} = \frac{1}{\overline{T}_i} \cdot \Big(\frac{g_{i1}}{r_i}\Big)^{1/\theta}, \qquad \widehat{\lambda}_{i2} = \frac{1}{\overline{T}_i} \cdot \Big(\frac{r_i - g_{i1}}{r_i}\Big)^{1/\theta}. \tag{3.10}$$

If, in particular, $\theta = 1$, which indicates independence, our result reduces to

$$\widehat{\lambda}_{i1} = \frac{1}{\overline{T}_i} \frac{g_{i1}}{r_i}, \qquad \widehat{\lambda}_{i2} = \frac{1}{\overline{T}_i} \frac{r_i - g_{i1}}{r_i}, \tag{3.11}$$

which is the same as in Zhang (2002). Then substituting $\widehat{\lambda}_{i1}$ and $\widehat{\lambda}_{i2}$ into (3.6), we find that the equality holds true for all $\theta\,(\geq 1)$. In fact, we have from the two estimates above that $(\widehat{\lambda}_{i1}^{\theta} + \widehat{\lambda}_{i2}^{\theta})^{1/\theta} = 1/\overline{T}_i$. This means that the copula parameter θ is not identifiable when maximum likelihood method is used. This is why we assume the copula function is known. See Zheng and Klein (1994, 1995) for graphical method. Besides, for a single stress, the estimate of mean lifetime of the system is $1/\overline{T}_i$ for whether assuming independence of competing failure modes or not.

Putting $\widehat{\lambda}_{i1}$ and $\widehat{\lambda}_{i2}$ into AF (3.2), we obtain from the Markov theorem the least squares estimators (LSE) of α_j and β_j

$$\begin{cases} \widehat{\alpha}_j = \dfrac{A\sum\limits_{i=1}^{k} \ln \widehat{\mu}_{ij} - B\sum\limits_{i=1}^{k} \varphi_i \ln \widehat{\mu}_{ij}}{kA - B^2} \\ \widehat{\beta}_j = \dfrac{A\sum\limits_{i=1}^{k} \varphi_i \ln \widehat{\mu}_{ij} - B\sum\limits_{i=1}^{k} \ln \widehat{\mu}_{ij}}{kA - B^2} \end{cases}, \qquad j = 1, 2,$$

where

$$\varphi_i = \varphi(S_i), \qquad A = \sum_{i=1}^{k} \varphi_i^2, \qquad B = \sum_{i=1}^{k} \varphi_i.$$

Hence, under the use stress level S_0, the mean life time of failure mode j is

$$\widehat{\mu}_{0j} = \exp(\widehat{\alpha}_j + \widehat{\beta}_j \varphi_0).$$

Thus, under the use tress level S_0, the estimator of the mean lifetime is

$$\widehat{\mu}_0 = 1/(\widehat{\lambda}_{01}^{\theta} + \widehat{\lambda}_{02}^{\theta})^{1/\theta}, \tag{3.12}$$

where $\widehat{\lambda}_{0j} = 1/\widehat{\mu}_{0j}$, $j = 1, 2$.

3.2 The Case of m Competing Failure Modes

We consider type II censoring ALT for series systems with m competing failure modes. Let $(t_{i1}, c_{i1}), (t_{i2}, c_{i2}), \ldots, (t_{ir_i}, c_{ir_i})$ be the failure data, where t_{il} denotes the failure time of l-th product under stress level S_i, $t_{il} \le t_{i2} \le \cdots \le t_{ir_i}$, and c_{il}, takes any integer in the set of $\{1, 2, \ldots, m\}$, with j indicating the failure is caused by failure mode j.

With all the assumptions and notations in Section 3.2 except that the index j runs from 1 to m, we have

Theorem 3.1 Let $(T_1, T_2, \ldots, T_n)$ be random vector with marginal survival functions $S_1(t), S_2(t), \ldots, S_n(t)$ respectively, and $\mathcal{C}(u_1, u_2, \ldots, u_n)$ be its survival copula. Suppose $\mathcal{C}$ is an Archimedean copula with generator $\phi_\theta(\cdot)$ (shortly $\phi(\cdot)$), which is strict, i.e., $\phi(0) = \infty$. Then the generator of survival copula of T_j and $T'_j = \min\{T_1, \ldots, T_{j-1}, T_{j+1}, \ldots, T_n\}$ is also $\phi(\cdot)$.

Proof Since $\mathcal{C}$ is Archimedean copula, hence $(n-1)$-dimensional margins of $\mathcal{C}$ is also Archimedean copula (see Nelson, 1999). Therefore,

$$\begin{aligned}\mathsf{P}\{T'_j > t\} &= \mathsf{P}\{T_1 > t, \ldots, T_{j-1} > t, T_{j+1} > t, \ldots, T_n > t\} \\ &= \phi^{-1}[\phi(S_1(t)) + \cdots + \phi(S_{j-1}(t)) + \phi(S_{j+1}(t)) + \cdots + \phi(S_n(t))]. \end{aligned} \tag{3.13}$$

Thus,

$$\begin{aligned}\mathsf{P}\{T_j > t_j, T'_j > t\} &= \mathsf{P}\{T_j > t_j, T_1 > t, \ldots, T_{j-1} > t, T_{j+1} > t, \ldots, T_n > t\} \\ &= \phi^{-1}\{\phi(S_j(t_j)) + \phi(S_1(t)) + \cdots + \phi(S_{j-1}(t)) \\ &\quad + \phi(S_{j+1}(t)) + \cdots + \phi(S_n(t))\} \\ &= \phi^{-1}\{\phi(S_j(t_j)) + \phi[\phi^{-1}(\phi(S_1(t)) + \cdots + \phi(S_{j-1}(t)) \\ &\quad + \phi(S_{j+1}(t)) + \cdots + \phi(S_n(t)))]\}. \end{aligned} \tag{3.14}$$

This ends the proof. □

The generator of Gumbel copula is strict, so it satisfies the condition of Theorem 3.1. Therefore, we know from Assumption 3 and Theorem 3.1 that the survival function of T'_j is

$$S'_j = \exp\Big\{-\Big(\sum_{h\ne j}^{m} \lambda_{ih}^{\theta}\Big)^{1/\theta} t\Big\},$$

and the dependence between T_j and T'_j is depicted by bivariate Gumbel copula. Thus the likelihood function of m competing failure modes is similar to the case of two competing failure modes: just replacing λ_{i1} and λ_{i2} by λ_{ij} and $-\Big(\sum_{h\ne j}^{m} \lambda_{ih}^{\theta}\Big)^{1/\theta}$ respectively in formula (3.3).

Let L_{ij} be the likelihood function of T_j under stress S_i. Then

$$L_{ij}=\Big[\lambda_{ij}\Big(\sum_{h\neq j}^{m}\lambda_{ih}^{\theta}\Big)^{-1/\theta}\Big]^{\theta g_{ij}}\Big(\sum_{h\neq j}^{m}\lambda_{ih}^{\theta}\Big)^{r_i}\Big(\sum_{j=1}^{m}\lambda_{ij}^{\theta}\Big)^{r_i(1/\theta-1)}\exp\Big\{-\Big(\sum_{j=1}^{m}\lambda_{ij}^{\theta}\Big)^{1/\theta}TTT_i\Big\}. \tag{3.15}$$

Let L_i be the likelihood function under stress level S_i, then $L_i=\prod_{j=1}^{m}L_{ij}$. Thus

$$\begin{aligned}\log L_i &= \sum_{j=1}^{m}\log L_{ij}\\ &= \sum_{j=1}^{m}\Big[\theta g_{ij}\log\lambda_{ij}+(r_i-g_{ij})\log\Big(\sum_{h\neq j}^{m}\lambda_{ih}^{\theta}\Big)+r_i\Big(\frac{1}{\theta}-1\Big)\log\Big(\sum_{j=1}^{m}\lambda_{ij}^{\theta}\Big)\\ &\quad -\Big(\sum_{j=1}^{m}\lambda_{ij}^{\theta}\Big)^{1/\theta}TTT_i\Big].\end{aligned} \tag{3.16}$$

We can perform the same procedure as in the case of two competing failure modes and the MLEs of λ_{ij} are

$$\widehat{\lambda}_{ij}=\frac{1}{\overline{T}_i}\cdot\Big(\frac{g_{ij}}{r_i}\Big)^{1/\theta},\qquad j=1,2,\ldots,m. \tag{3.17}$$

§4. Simulation

We simulate 4 constant stress ALTs with type II censoring with temperature-accelerated stress levels $S_1=80°\mathrm{C}=353\mathrm{K}$, $S_2=100°\mathrm{C}=373\mathrm{K}$, $S_3=120°\mathrm{C}=393\mathrm{K}$, and $S_4=150°\mathrm{C}=423\mathrm{K}$. The use temperature is $S_0=25°\mathrm{C}=298\mathrm{K}$. 100 products are put on each stress level until 35, 30, 25 and 20 of them fail, respectively. Only two competing failure modes are considered and their dependence is depicted by bivariate Gumbel copula. The lifetime of the failure modes follows the exponential distribution with mean lifetime related to stress level by the Arrennius model, that is,

$$\log\theta_{ij}=\alpha_j+\beta_j/S_i,\qquad i=0,1,2,3,4,\ j=1,2.$$

We repeated 100 times for different test schemes with different dependence structures. The sample average of these 100 estimators of mean lifetime are compared.

1. Let the parameter of Gumbel copula $\theta=2$ or equivalently $\tau=1/2$. The result of simulation is presented in Table 1, where $\widehat{\mu}_0$ is calculated according to formula (3.12), $\widehat{\mu}_0'$ is the estimation under falsely-taken independent competing failure modes. We see that $\widehat{\mu}_0$ is much closer to true value than $\widehat{\mu}_0'$, though the discrepancy of the two estimates are not substantial. To a certain extent, the estimation of considering dependence is more precise.

2. Let the parameter of Gumbel copula $\theta = 4, 6$ or equivalently $\tau = 3/4, 5/6$. The result of simulation is presented in Table 2 and Table 3. We see that as the dependence of failure modes becomes stronger, $\widehat{\theta}_0$ becomes closer to the true value, while $\widehat{\theta}'_0$ departures from true value much further. This shows that the dependence structure is very important in statistical analysis.

Table 1 Estimation comparison of mean lifetime when $\tau = 1/2$

AF under failure mode 1	AF under failure mode 2	True mean lifetime	$\widehat{\mu}_0$	$\widehat{\mu}'_0$
$\ln\theta_i = -5 + 4000/S_i$	$\ln\theta_i = -14.29 + 7700/S_i$	4545	4917	3609
$\ln\theta_i = -5 + 4000/S_i$	$\ln\theta_i = -16.28 + 8876/S_i$	4550	4610	3781
$\ln\theta_i = -3.8 + 2800/S_i$	$\ln\theta_i = -15.26 + 7102/S_i$	269	300.2	244.3
$\ln\theta_i = -3.8 + 2800/S_i$	$\ln\theta_i = -11.28 + 5876/S_i$	268.9	271.9	231.5
$\ln\theta_i = -4.67 + 4637/S_i$	$\ln\theta_i = -14.26 + 8366/S_i$	53582	52773	45375
$\ln\theta_i = -4.67 + 4637/S_i$	$\ln\theta_i = -16.52 + 8163.6/S_i$	37642	36403	39565

Table 2 Estimation comparison of mean lifetime when $\tau = 3/4$

AF under failure mode 1	AF under failure mode 2	True mean lifetime	$\widehat{\mu}_0$	$\widehat{\mu}'_0$
$\ln\theta_i = -4.8 + 3235/S_i$	$\ln\theta_i = -14.37 + 6723/S_i$	426.5	486	179.4
$\ln\theta_i = -5.21 + 4216/S_i$	$\ln\theta_i = -12.26 + 7102/S_i$	7613	8549	4429
$\ln\theta_i = -2.31 + 3216/S_i$	$\ln\theta_i = -15.37 + 8121/S_i$	4827	5979	1304
$\ln\theta_i = -3.62 + 3362/S_i$	$\ln\theta_i = -12.37 + 6837/S_i$	2126	2616	991

Table 3 Estimation comparison of mean lifetime when $\tau = 5/6$

AF under failure mode 1	AF under failure mode 2	True mean lifetime	$\widehat{\mu}_0$	$\widehat{\mu}'_0$
$\ln\theta_i = -5.12 + 4356.5/S_i$	$\ln\theta_i = -16.65 + 8876/S_i$	13343	15637	2755
$\ln\theta_i = -4.89 + 4032.8/S_i$	$\ln\theta_i = -14.52 + 7872.6/S_i$	5665	6089	1559
$\ln\theta_i = -4.8 + 3235/S_i$	$\ln\theta_i = -14.37 + 6723/S_i$	426.5	469.9	74.64
$\ln\theta_i = -2.3 + 3157/S_i$	$\ln\theta_i = -18.28 + 9122/S_i$	4000	3558	203.3

§5. Real Data Application

We will apply the above method to the data set about insulated system of electromotor from Klein and Basu (1981). The original data consists of three failure modes: turn failure, phase failure and ground failure. 323K and 423K are the two use temperatures, and the four accelerated temperatures are 453K, 463K, 493K and 513K. Using the approach above, we obtain the three accelerate functions as follows

$$\log \mu_{i1} = -6.60 + 6686.55/S_i,$$
$$\log \mu_{i2} = -3.48 + 5376.72/S_i,$$
$$\log \mu_{i3} = -6.95 + 6993.09/S_i.$$

Thus the estimates of mean lifetime at temperature 323K and 423K are $\widehat{\mu}_0 = 477497.6$h and $\widehat{\mu}_0 = 6406.4$h respectively.

§6. Conclusions and Remarks

1. A very simple and elegant copula model for dependent competing risks in constant stress ALT is introduced. This model is more general and practical and simulation shows that the usual independence assumption would have a crucial effect on the reliability assessment for constant stress ALTs when the failure modes of the series system were dependent.

2. For a life test under use condition, if the failure modes of the failures were not known (masked), the mean life of the system could still be estimated as the total test time over the number of failures. Thus the reliability analysis will be the same whether the failure modes are independent or not, whether the failures are masked or not. However, as is shown in this paper, this is not the case for constant ALTs.

3. It is the first time for us to introduce copula to depict the dependent structure between/among failure modes in system reliability analysis. Though more effort need to be done in practice, advantages of it are clear, including: (1) It has the ability to capture the dependence structure between/among different failure modes; (2) it seems more effective and flexible with respect to the choice of the marginal distributions. However, as to the assumption 5, it needs to be investigated based on the physical or chemical features of the product lot from which the date set comes. The case of step stress ALT is not considered here, however, it is expected that the derivation process are the same based on the assumption of Nelson (1980). For the choice of copula, either frequentist or Bayesian method can be used . This is currently under consideration.

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基于Copulas加速寿命试验中竞争失效模型的统计分析

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在已有讨论竞争失效数据统计分析的文献中, 大多数都假设失效机理之间相互独立. 本文使用copula作为连接函数来考查加速寿命试验中的竞争失效模型. 通过模拟, 把失效机理相关时得到的结果与失效机理独立时得到的结果做了比较. 最后分析了文献中的一个实际数据.

关键词: 竞争风险, 加速寿命试验, copula.

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