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Deviation Inequalities for Stochastic Differential Equations with Jumps

Wu Ye

(*Department of Electronic Information Science, Fujian Jiang Xia University, Fuzhou, 350108*)

Abstract

By adopting the martingale technique, we derive deviation inequalities for Lipschitz functions of general jump-diffusion processes. Our results extend related works for pure jump Lévy processes under Cramér's like assumption, while our approach is considerably efficient for the situation that Lévy measure does not have finite exponential moments.

Keywords: Deviation inequalities, jump-diffusion processes, Lipschitz property.

AMS Subject Classification: 60F10, 60G44, 60G51, 60J25, 60J75.

§1. Introduction and Main Results

The object of this paper is a jump-diffusion process on $\mathbb{R}^d$, which is determined by the following SDE:

$$dX_t = b(X_{t-})dt + \sigma(X_{t-})dW_t + \int c(X_{t-}, z)\widetilde{\mu}(dz, dt), \tag{1.1}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\otimes d}$, $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, W_t is a d-dimensional Brownian motion, and $\widetilde{\mu}(dz, dt) = \mu(dz, dt) - \nu(dz)dt$ is a compensated Poisson measure on $\mathbb{R}^d\backslash\{0\}\times \mathbb{R}_+$; namely, $\mu(dz, dt)$ is a Poisson point measure on $\mathbb{R}^d\backslash\{0\} \times \mathbb{R}_+$ with intensity measure $\nu(dz)dt$. In the literature, e.g. see [1], ν is referred to a Lévy measure.

Deviation inequalities are of great importance in the probability theory and its applications. During recent years deviation inequalities for jump processes have received a lot of attention. For instance, based on the covariance representation of Lévy processes, deviation inequalities for pure Lévy jump processes were obtained in [4, Theorem 1] under exponential moments condition on the corresponding Lévy measure; while [12, Proposition 3.4] used the martingale technique to get similar conclusions, and this approach further was employed to study Poisson-type deviation inequalities for continuous-time Markov chains with finite range in [6]. Deviation inequalities for jump-diffusion processes have

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been investigated in [8, Section 4.2], where coefficients in the associated SDE (1.1) are continuously differentiable and satisfy the dissipative condition, while the Lévy measure also fulfills a Cramér's like assumption.

The purpose of this paper is to study general deviation inequalities for jump-diffusion processes given by (1.1). Our results extend these of pure jump Lévy processes in [4, Theorem 1] and [12, Proposition 3.4], and also cover [8, Theorem 2.2 (2)] for jump-diffusion processes. The main idea, different from the approach in [8], is quite elementary. It is based on the martingale technique developed in [12] and the Itô formula for Lévy type stochastic integral, e.g. see [1, Theorem 4.4.7]. Note that, the semigroup corresponding to Lévy processes is a convolution operator, so it preserves the set of bounded Lipschitz continuous functions. However, in general such characterization does not hold for jump-diffusion processes. Here, we adopt the coupling approach in recent paper [13] to study the Lipschitz property of semigroups associated with the SDE (1.1).

We always assume the following conditions on the coefficients in the SDE (1.1):

(H1) The coefficients σ_{ij}, b_i, and $c(\cdot,\xi)$ for any fixed $\xi\in\mathbb{R}^d$ are continuous on $\mathbb{R}^d$, and for all $x\in\mathbb{R}^d$, $\displaystyle\int c(x,z)^2\nu(\mathrm{d}z)<\infty$.

(H2) There exists a constant $C\in\mathbb{R}$ such that for all x, $y\in\mathbb{R}^d$,

$$\|\sigma(x)-\sigma(y)\|^2+2\langle b(x)-b(y),x-y\rangle+\int|c(x,z)-c(y,z)|^2\nu(\mathrm{d}z)\le 2C|x-y|^2,\quad(1.2)$$

where for matrix $A\in\mathbb{R}^{d\otimes d}$, $\|A\|=\mathrm{tr}(AA^*)$, and A^* is the transpose of A.

Theorem 1.1 Let $(X_t)_{t\ge0}$ be the jump-diffusion process given by (1.1), and define $a(x)=\sigma(x)\sigma(x)^*$ for $x\in\mathbb{R}^d$. Then, we have the following two statements.

(1) Suppose that for every $x\in\mathbb{R}^d$

$$A(x):=\sup_{\theta:|\theta_{ij}|\le1\text{ and }\theta_{ij}=\theta_{ji},\,1\le i,j\le d}\ \sum_{i,j=1}^d a_{ij}(x)\theta_{ij}\le A_0,$$

and there exist a non-negative Borel-measurable function $c(z)$ and a positive constant λ such that $|c(x,z)|\le c(z)$ for every x, $z\in\mathbb{R}^d$,

$$\beta(\lambda):=\int\big(e^{\lambda c(z)}-1-\lambda c(z)\big)\nu(\mathrm{d}z)<\infty.\quad(1.3)$$

Then for any $f\in\mathrm{Lip}_b(\mathbb{R}^d)$ with $C_{\mathrm{Lip}}(f)=1$, $0<\kappa\le\lambda(1\wedge e^{-Ct})$ and $t>0$,

$$\mathsf{E}^x\exp[\kappa(f(X_t)-\mathsf{E}^xf(X_t))]\le\exp\Big[\int_0^t h(t,s,\kappa)\mathrm{d}s\Big],\quad(1.4)$$

where

$$h(t,s,\kappa)=\frac{A_0}{2}\kappa^2e^{2C(t-s)}+\beta\big(\kappa e^{C(t-s)}\big).$$

In particular, for any $f \in \mathrm{Lip}(\mathbb{R}^d)$ with $C_{\mathrm{Lip}}(f)=1$ and r, $t>0$,

$$\mathsf{P}^x(|f(X_t)-\mathsf{E}^x f(X_t)|\geq r)\leq \inf_{0<\kappa\leq\lambda(1\wedge e^{-Ct})}\exp\Big[-\kappa r+\int_0^t h(t,s,\kappa)\mathrm{d}s\Big]. \tag{1.5}$$

(2) Assume that for every $x\in\mathbb{R}^d$,

$$A(x):=\sup_{\theta:|\theta_{ij}|\leq 1 \text{ and } \theta_{ij}=\theta_{ji},\,1\leq i,j\leq d}\sum_{i,j=1}^d a_{ij}(x)\theta_{ij}\leq A_0,$$

and

$$\int c(x,z)^2\nu(\mathrm{d}z)\leq C_0. \tag{1.6}$$

Then for any $f\in\mathrm{Lip}_b(\mathbb{R}^d)$ with $C_{\mathrm{Lip}}(f)=1$ and $t>0$,

$$\mathsf{E}^x(f(X_t)-\mathsf{E}^x f(X_t))^2\leq \frac{(A_0+C_0)(e^{2Ct}-1)}{2C}.$$

In particular, for any $f\in\mathrm{Lip}(\mathbb{R}^d)$ with $C_{\mathrm{Lip}}(f)=1$ and r, $t>0$,

$$\mathsf{P}^x(|f(X_t)-\mathsf{E}^x f(X_t)|\geq r)\leq \frac{(A_0+C_0)(e^{2Ct}-1)}{2Cr^2}.$$

Let us make some comments on Theorem 1.1. First, according to Proposition 2.1 below, condition (1.2) ensures that the semigroup $(P_t)_{t\geq 0}$ of the SDE (1.1) maps $\mathrm{Lip}_b(\mathbb{R}^d)$ into itself; moveover, for every $t>0$ and $f\in\mathrm{Lip}_b(\mathbb{R}^d)$, $C_{\mathrm{Lip}}(P_tf)\leq e^{Ct}C_{\mathrm{Lip}}(f)$, where $C_{\mathrm{Lip}}(f)$ is denoted by the Lipschitz constant of the Lipschitz continuous function f. Second, for pure Lévy jump process; that is, in the SDE (1.1), $b(x)=b_0$, $\sigma(x)=0$ and $c(x,z)=z$ for every x, $z\in\mathbb{R}^d$, Theorem 1.1 (1) is reduced into [12, Proposition 3.4]. The readers can refer to [4, Pages 1224–1225] for the optimality of Theorem 1.1 (1) for Poisson processes. Third, to derive the deviation inequality (1.5) the essential point is Cramér's like condition (1.3). It follows, from the condition $|c(x,z)|\leq c(z)$ for every x, $z\in\mathbb{R}^d$ and the inequality $e^s-1-s\geq s^2/2$ for $s\geq 0$, that (1.3) implies (1.6). The statement (2) shows that our technique yields the deviation inequality for jump-diffusion processes with power low form, freeing us from the exponential moments condition (1.3).

The remainder of this paper is organized as follows. In Section 2, we point out the statement that, (1.2) implies the semigroup corresponding to the SDE (1.1) maps $\mathrm{Lip}_b(\mathbb{R}^d)$ into itself. Section 3 is devoted to the proof of Theorem 1.1, which is based on the martingale technique.

§2. Preliminaries and Lipschitz Property

First, it is easy to check that under assumptions (H1) and (H2), there exists a constant $C_1 > 0$ such that for any $x \in \mathbb{R}^d$,

$$\|\sigma(x)\|^2 + \langle b(x), x\rangle + \int |c(x,z)|^2 \nu(\mathrm{d}z) \leq C_1(1+|x|^2).$$

Therefore, according to the arguments of the existence and the pathwise uniqueness of strong solution to equation (1.1), cf. [1, Chapter 6, Theorems 6.2.3 and 6.2.11] or [11, Chapter 3, Theorems 117 and 118], we know the SDE (1.1) has the pathwise uniquely and nonexplosive strong solution. We denote this solution by $X = (X_t)_{t\geq 0}$.

On the other hand, let $(P_t)_{t\geq 0}$ be a Markov semigroup, which maps $\mathrm{Lip}_b(\mathbb{R}^d)$ into itself; that is, for any $f \in \mathrm{Lip}_b(\mathbb{R}^d)$ and $t \geq 0$, $P_t f \in \mathrm{Lip}_b(\mathbb{R}^d)$. We are interested in the rate function defined by

$$g(t) := \sup_{f\in \mathrm{Lip}_b(\mathbb{R}^d)} \frac{C_{\mathrm{Lip}}(P_t f)}{C_{\mathrm{Lip}}(f)}, \qquad t \geq 0.$$

By the property of Markov semigroups, $g(t+s) \leq g(t)g(s)$ for any $s,t \geq 0$. Noting that $g(0) = 1$, it is easy to claim, cf. see the proof of [2, Theorem 1.4], that $g(t) \leq e^{c_0 t}$ for $t \geq 0$, where

$$c_0 = \lim_{t\to 0} \frac{\log g(t)}{t} = \sup_{t>0} \frac{\log g(t)}{t}.$$

The next assertion follows from the proofs of [13, Theorem 2.2] and [8, Lemma 3.3].

Proposition 2.1 Let $(X_t)_{t\geq 0}$ be the unique jump-diffusion process given by (1.1), and $(P_t)_{t\geq 0}$ be its Markov semigroup. If (1.2) holds, then for any $f \in \mathrm{Lip}_b(\mathbb{R}^d)$ and $t \geq 0$, $P_t f \in \mathrm{Lip}_b(\mathbb{R}^d)$ and $\sup\limits_{f\in \mathrm{Lip}_b(\mathbb{R}^d)} C_{\mathrm{Lip}}(P_t f)/C_{\mathrm{Lip}}(f) \leq e^{Ct}$.

Condition (1.2) indicates that the coefficients in the SDE (1.1) are Lipschitz continuous. In particular, when $C < 0$, it is nothing else but the dissipative condition, see [8, (1.2)]. Therefore, Proposition 2.1 shows that the Lipschitz continuity of the coefficients in the SDE (1.1) implies the exponential growth of the Lipschitz constant corresponding to its semigroups on $\mathrm{Lip}_b(\mathbb{R}^d)$. According to [6, Definition 2.1], (1.2) is equivalent to saying that the Wasserstein curvature of $(X_t)_{t\geq 0}$ is bounded below by $-C$. The Wasserstein curvature is related to the ergodicity of the process $(X_t)_{t\geq 0}$. The readers are urged to see [2, Theorem 5.23] and [7, Theorem 2.4] for more details. It is well known that the semigroup $(P_t)_{t\geq 0}$ corresponding to Lévy processes is a convolution operator, and it preserves the Lipschitz constant, i.e. $g(t) = 1$ for all $t \geq 0$. In this case, since the coefficients in Lévy processes are constants, (1.2) holds with $C = 0$, which indicates that Proposition 2.1 is optimal.

We close this section with the sketch of the proof of Proposition 2.1.

Sketch of the Proof of Proposition 2.1 Let $(X_t^x)_{t\ge 0}$ and $(X_t^y)_{t\ge 0}$ be jump-diffusion processes given by (1.1) with starting points x, $y \in \mathbb{R}^d$, respectively. Then, by the Itô formula and the assumption (1.2), we have

$$
\begin{aligned}
\mathrm{d}|X_t^x - X_t^y|^2 &= 2\langle X_{t-}^x - X_{t-}^y, b(X_{t-}^x) - b(X_{t-}^y)\rangle \mathrm{d}t \\
&\quad + 2\langle X_{t-}^x - X_{t-}^y, \sigma(X_{t-}^x) - \sigma(X_{t-}^y)\rangle \mathrm{d}W_t + \|\sigma(X_{t-}^x) - \sigma(X_{t-}^y)\|^2 \mathrm{d}t \\
&\quad + 2\int \langle X_{t-}^x - X_{t-}^y, c(X_{t-}^x, z) - c(X_{t-}^y, z)\rangle \widetilde{\mu}(\mathrm{d}z, \mathrm{d}t) \\
&\quad + \int |c(X_{t-}^x, z) - c(X_{t-}^y, z)|^2 \, \mu(\mathrm{d}z, \mathrm{d}t) \\
&= 2\langle X_{t-}^x - X_{t-}^y, b(X_{t-}^x) - b(X_{t-}^y)\rangle \mathrm{d}t + \|\sigma(X_{t-}^x) - \sigma(X_{t-}^y)\|^2 \mathrm{d}t \\
&\quad + \int |c(X_{t-}^x, z) - c(X_{t-}^y, z)|^2 \nu(\mathrm{d}z)\mathrm{d}t \\
&\quad + 2\langle X_{t-}^x - X_{t-}^y, \sigma(X_{t-}^x) - \sigma(X_{t-}^y)\rangle \mathrm{d}W_t \\
&\quad + \int \big(2\langle X_{t-}^x - X_{t-}^y, c(X_{t-}^x, z) - c(X_{t-}^y, z)\rangle \\
&\quad + |c(X_{t-}^x, z) - c(X_{t-}^y, z)|^2\big) \widetilde{\mu}(\mathrm{d}z, \mathrm{d}t) \\
&\le 2C|X_{t-}^x - X_{t-}^y|^2 \mathrm{d}t + \mathrm{d}M_t,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{d}M_t :=\ & 2\langle X_{t-}^x - X_{t-}^y, \sigma(X_{t-}^x) - \sigma(X_{t-}^y)\rangle \mathrm{d}W_t \\
& + \int \big(2\langle X_{t-}^x - X_{t-}^y, c(X_{t-}^x, z) - c(X_{t-}^y, z)\rangle \\
& + |c(X_{t-}^x, z) - c(X_{t-}^y, z)|^2\big) \widetilde{\mu}(\mathrm{d}z, \mathrm{d}t)
\end{aligned}
$$

is a local martingale. It follows from the integral and differential inequality in [3, Lemma A1 and Corollary A.2] that

$$
\mathsf{E}|X_t^x - X_t^y|^2 \le e^{2Ct}|x - y|^2, \qquad t \ge 0.
$$

In particular,

$$
\mathsf{E}|X_t^x - X_t^y| \le e^{Ct}|x - y|, \qquad t \ge 0. \tag{2.1}
$$

Recall that, for any two probability measures P_1 and P_2, the Wasserstein metric between P_1 and P_2 with respect to the metric function ρ is given by

$$
W_\rho(P_1, P_2) := \inf \int \rho(x, y) P(\mathrm{d}x, \mathrm{d}y),
$$

where the infimum is taken over all probability measures P on $\mathbb{R}^{2d}$ with marginal distributions P_1 and P_2. It is known that, cf. see [2, Theorem 5.10],

$$W_\rho(P_1,P_2)=\sup\Big\{\int f\mathrm{d}P_1-\int f\mathrm{d}P_2 : f\in \mathrm{Lip}_{b,\rho}(\mathbb{R}^d), C_{\mathrm{Lip},\rho}(f)\le 1\Big\}, \tag{2.2}$$

where $\mathrm{Lip}_{b,\rho}(\mathbb{R}^d)$ is denoted by the set of bounded Lipschitz continuous functions with respect to the metric ρ, and $C_{\mathrm{Lip},\rho}(f)$ is the corresponding Lipschitz constant of $f\in \mathrm{Lip}_{b,\rho}(\mathbb{R}^d)$.

Taking $\rho(x,y)=|x-y|$, (2.1) implies that

$$W(P_t(x,\cdot),P_t(y,\cdot))\le e^{Ct}|x-y|,$$

which along with (2.2) yield the required assertion. □

§3. Proof of Theorem 1.1

We will derive deviation inequalities for the jump diffusion process (1.1). According to different integrable conditions (1.3) and (1.6) on the Lévy measure ν, we will split the proof of Theorem 1.1 into two parts.

Proof of Theorem 1.1 (1) The proof is based on the martingale technique developed in [12, Section 3]. The deviation inequality (1.5) restricted on $\mathrm{Lip}_b(\mathbb{R}^d)$ follows from (1.4) and the Chebyshev inequality. To remove the boundedness assumption, we follow the standard approximation, cf. see the proof of [4, Theorem 1]. For any $n\ge 1$, let f_n where $f_n=f$ if $|f|\le n$, $f_n=n$ if $f>n$ and $f_n=-n$ if $f\le -n$. Then, $f_n\in \mathrm{Lip}_b(\mathbb{R}^d)$ such that its Lipschitz constant at most $C_{\mathrm{Lip}}(f)$. The desired assertion follows by applying Fatou's lemma. Thus, to complete the proof it suffices to prove (1.4), and its argument is divided into the following five steps.

(a) For $t>0$ and $f\in \mathrm{Lip}_b(\mathbb{R}^d)$, set

$$F(s,x)=P_{t-s}f(x),\qquad x\in\mathbb{R}^d,\ 0\le s\le t.$$

Then, by Proposition 2.1, for any fixed $s\in[0,t]$, $F(s,\cdot)\in \mathrm{Lip}_b(\mathbb{R}^d)$. Before moving further, we apply to $F(s,x)$ the mollification argument; that is, convolve it with p_ε, $\varepsilon>0$, where for $x\in\mathbb{R}^d$, $p_\varepsilon(x)=\varepsilon^{-d}p(\varepsilon^{-1}x)$ and $p(x)=c\exp[1/(|x|^2-1)]\mathbb{1}_{\{|x|<1\}}$ (c is a normalizing constant). For any $\varepsilon>0$, define $F_\varepsilon(s,x)=(p_\varepsilon * F(s,\cdot))(x)$ for all $s\in[0,t]$ and $x\in\mathbb{R}^d$. Then, $F_\varepsilon\in C^{1,\infty}(\mathbb{R}_+\times\mathbb{R}^d)$, and $F_\varepsilon(s,x)$ pointwise converges to $F(s,x)$ as ε tends to zero.

(b) By using the Itô formula of Lévy type stochastic integral, cf. see [1, Theorem 4.4.7], for every $t > 0$,

$$\begin{aligned} F_\varepsilon(t, X_t) &= F_\varepsilon(0, X_0) + \int_0^t \partial_s F_\varepsilon(s, X_{s-})\mathrm{d}s + \int_0^t \langle \nabla_x F_\varepsilon(s, X_{s-}), b(X_{s-})\rangle \mathrm{d}s \\ &\quad + \int_0^t \langle \nabla_x F_\varepsilon(s, X_{s-}), \sigma(X_{s-})\mathrm{d}W_s\rangle \\ &\quad + \frac{1}{2}\int_0^t |\sigma(X_{s-})^* \nabla_x F_\varepsilon(s, X_{s-})|^2 \mathrm{d}s \\ &\quad + \int_0^t \int \big(F_\varepsilon(s, X_{s-} + c(X_{s-}, z)) - F_\varepsilon(s, X_{s-})\big)\widetilde{\mu}(\mathrm{d}z, \mathrm{d}s) \\ &\quad + \int_0^t \int \big(F_\varepsilon(s, X_{s-} + c(X_{s-}, z)) - F_\varepsilon(s, X_{s-}) \\ &\quad - \langle \nabla_x F_\varepsilon(s, X_{s-}), c(X_{s-}, z)\rangle\big)\nu(\mathrm{d}z)\mathrm{d}s. \end{aligned}$$

Noticing that, for any $s \in (0, t)$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} \partial_s F_\varepsilon(s, x) &= -\Big[\langle \nabla_x F_\varepsilon(s, x), b(x)\rangle + \frac{1}{2}|\sigma(x)^* \nabla_x F_\varepsilon(s, x)|^2 \\ &\quad + \int \big(F_\varepsilon(s, x + c(x, z)) - F_\varepsilon(s, x) - \langle \nabla_x F_\varepsilon(s, x), c(x, z)\rangle\big)\nu(\mathrm{d}z)\Big], \end{aligned}$$

we arrive at

$$\begin{aligned} F_\varepsilon(t, X_t) &= F_\varepsilon(0, X_0) + \int_0^t \langle \nabla_x F_\varepsilon(s, X_{s-}), \sigma(X_{s-})\mathrm{d}W_s\rangle \\ &\quad + \int_0^t \int \big(F_\varepsilon(s, X_{s-} + c(X_{s-}, z)) - F_\varepsilon(s, X_{s-})\big)\widetilde{\mu}(\mathrm{d}z, \mathrm{d}s). \end{aligned} \tag{3.1}$$

That is, the process $(Z_s^{f,\varepsilon})_{0 \le s \le t}$ defined by

$$Z_s^{f,\varepsilon} := F_\varepsilon(s, X_s) - F_\varepsilon(0, X_0)$$

is a real P^x-martingale with respect to the truncated filtration $(\mathcal{F}_s)_{0 \le s \le t}$.

(c) For any $\kappa > 0$, also according to the Itô formula of Lévy type stochastic integral, e.g. see [1, Chapter 5.2.2, (5.4), Page 288],

$$\begin{aligned} \mathrm{d}e^{\kappa Z_t^{f,\varepsilon}} &= e^{\kappa Z_{t-}^{f,\varepsilon}}\Big[\kappa\langle \nabla_x F_\varepsilon(t, X_{t-}), \sigma(X_{t-})\mathrm{d}W_t\rangle \\ &\quad + \int \big(e^{\kappa(F_\varepsilon(t, X_{t-} + c(X_{t-}, z)) - F_\varepsilon(t, X_{t-}))} - 1\big)\widetilde{\mu}(\mathrm{d}z, \mathrm{d}t) + C_t^{f,\varepsilon}\mathrm{d}t\Big] \\ &=: \mathrm{d}M_t^{f,\varepsilon} + e^{\kappa Z_{t-}^{f,\varepsilon}} C_t^{f,\varepsilon}\mathrm{d}t. \end{aligned}$$

Here,

$$\begin{aligned} \mathrm{d}M_t^{f,\varepsilon} &= e^{\kappa Z_{t-}^{f,\varepsilon}}\Big[\kappa\langle \nabla_x F_\varepsilon(t, X_{t-}), \sigma(X_{t-})\mathrm{d}W_t\rangle \\ &\quad + \int \big(e^{\kappa(F_\varepsilon(t, X_{t-} + c(X_{t-}, z)) - F_\varepsilon(t, X_{t-}))} - 1\big)\widetilde{\mu}(\mathrm{d}z, \mathrm{d}t)\Big] \end{aligned}$$

and

$$\begin{aligned} C_t^{f,\varepsilon} &= \frac{\kappa^2}{2}|\sigma(X_{t-})^*\nabla_x F_\varepsilon(t,X_{t-})|^2 \\ &+\int \Big(e^{\kappa(F_\varepsilon(t,X_{t-}+c(X_{t-},z))-F_\varepsilon(t,X_{t-}))}-1 \\ &-\kappa(F_\varepsilon(t,X_{t-}+c(X_{t-},z))-F_\varepsilon(t,X_{t-}))\Big)\nu(\mathrm{d}z). \end{aligned}$$

Note that, $(M_s^{f,\varepsilon})_{0\le s\le t}$ is a martingale with $\mathsf{E}^x(M_s^{f,\varepsilon})=0$ for $0\le s\le t$. Furthermore, due to the Itô product formula, e.g. see [1, Theorem 4.4.13] (also called the integration-by-parts formula in the literature, e.g. see [9, Proposition 2.28, Page 129] or [10, Chapter II, Corollary 2]), one has

$$\begin{aligned} &\exp\Big(\kappa Z_t^{f,\varepsilon}-\int_0^t C_s^{f,\varepsilon}\mathrm{d}s\Big) \\ =\ &1+\int_0^t \exp\Big(-\int_0^s C_u^{f,\varepsilon}\mathrm{d}u\Big)\mathrm{d}e^{\kappa Z_s^{f,\varepsilon}}+\int_0^t e^{\kappa Z_s^{f,\varepsilon}}\mathrm{d}\Big(\exp\Big(-\int_0^s C_u^{f,\varepsilon}\mathrm{d}u\Big)\Big) \\ &+\Big[e^{\kappa Z_\cdot^{f,\varepsilon}},\exp\Big(-\int_0^\cdot C_u^{f,\varepsilon}\mathrm{d}u\Big)\Big]_0^t \\ =\ &1+\int_0^t \exp\Big(-\int_0^s C_u^{f,\varepsilon}\mathrm{d}u\Big)\mathrm{d}M_s^{f,\varepsilon}+\int_0^t C_s^{f,\varepsilon}\exp\Big(\kappa Z_s^{f,\varepsilon}-\int_0^s C_u^{f,\varepsilon}\mathrm{d}u\Big)\mathrm{d}s \\ &-C_s^{f,\varepsilon}\int_0^t \exp\Big(\kappa Z_s^{f,\varepsilon}-\int_0^s C_u^{f,\varepsilon}\mathrm{d}u\Big)\mathrm{d}s+0 \\ =\ &1+\int_0^t \exp\Big(-\int_0^s C_u^{f,\varepsilon}\mathrm{d}u\Big)\mathrm{d}M_s^{f,\varepsilon} \end{aligned}$$

for all $t\ge 0$. Here, $\Big[e^{\kappa Z_\cdot^{f,\varepsilon}},\exp\Big(-\int_0^\cdot C_u^{f,\varepsilon}\mathrm{d}u\Big)\Big]$ is the quadratic variation process associated with the processes $e^{\kappa Z_\cdot^{f,\varepsilon}}$ and $\exp\Big(-\int_0^\cdot C_u^{f,\varepsilon}\mathrm{d}u\Big)$, which is equal to zero, cf. see [5, Chapter II, Proposition 4.49 (d)] (also according to the Kunita-Watanabe inequality, e.g. see [9, Proposition 2.34, Page 134], since the quadratic variation process corresponding to the process $\exp\Big(-\int_0^\cdot C_u^{f,\varepsilon}\mathrm{d}u\Big)$ is zero, e.g. see remark before [10, Chapter II, Theorem 25]). Hence, the process $(K_s^{f,\varepsilon})_{0\le s\le t}$ defined by

$$K_s^{f,\varepsilon}:=\exp\Big(\kappa Z_s^{f,\varepsilon}-\int_0^s C_u^{f,\varepsilon}\mathrm{d}u\Big)$$

is also a real P^x-martingale with respect to the truncated filtration $(\mathcal{F}_s)_{0\le s\le t}$. In particular, for any $t>0$ and $f\in \mathrm{Lip}_b(\mathbb{R}^d)$,

$$\mathsf{E}^x\Big[\exp\Big(\kappa Z_t^{f,\varepsilon}-\int_0^t C_s^{f,\varepsilon}\mathrm{d}s\Big)\Big]=1.$$

(d) Now, we turn to estimate the term $C_s^{f,\varepsilon}$ for $s \in (0,t)$. Let $0 < \kappa \le \lambda(1 \wedge e^{-ct})$. First, according to the definition of $F_\varepsilon(s,x)$ and Proposition 2.1, $F_\varepsilon(s,\cdot) \in \mathrm{Lip}_b(\mathbb{R}^d)$ and

$$C_{\mathrm{Lip}}(F_\varepsilon(s,\cdot)) \le C_{\mathrm{Lip}}(F(s,\cdot)) \le e^{C(t-s)} C_{\mathrm{Lip}}(f).$$

Therefore, thanks to Lemma 3.1 below, we get that

$$\begin{aligned}
&\int \Big(e^{\kappa(F_\varepsilon(s,X_{s-}+c(X_{s-},z))-F_\varepsilon(s,X_{s-}))} - 1 \\
&\quad - \kappa(F_\varepsilon(s,X_{s-}+c(X_{s-},z)) - F_\varepsilon(s,X_{s-}))\Big)\nu(\mathrm{d}z) \\
\le\ &\int \Big(e^{\kappa C_{\mathrm{Lip}}(F(s,\cdot))|c(X_{s-},z)|} - 1 - \kappa C_{\mathrm{Lip}}(F(s,\cdot))|c(X_{s-},z)|\Big)\nu(\mathrm{d}z) \\
\le\ &\int \Big(e^{\kappa e^{C(t-s)} C_{\mathrm{Lip}}(f)|c(X_{s-},z)|} - 1 - \kappa e^{C(t-s)} C_{\mathrm{Lip}}(f)|c(X_{s-},z)|\Big)\nu(\mathrm{d}z).
\end{aligned}$$

On the other hand, also due to the definition of $F_\varepsilon(s,x)$, for every $1 \le i \le d$,

$$|\partial_i F(s,\cdot)| \le e^{C(t-s)} C_{\mathrm{Lip}}(f),$$

and so

$$\begin{aligned}
|\sigma(X_{s-})^* \nabla_x F_\varepsilon(s,X_{s-})|^2 &= \sum_{i,j=1}^d \partial_i F_\varepsilon(s,X_{s-}) a_{ij}(X_{s-}) \partial_j F_\varepsilon(s,X_{s-}) \\
&\le e^{2C(t-s)} C_{\mathrm{Lip}}(f)^2 A(X_{s-}),
\end{aligned}$$

where

$$A(x) = \sup_{\theta:|\theta_{ij}|\le 1 \text{ and } \theta_{ij}=\theta_{ji},\, 1\le i,j\le d} \ \sum_{i,j=1}^d a_{ij}(x)\theta_{ij}.$$

Combining all the estimates, for $0 \le s \le t$, we have

$$\begin{aligned}
C_s^{f,\varepsilon} &\le \frac{A_0}{2}\kappa^2 e^{2C(t-s)} C_{\mathrm{Lip}}(f)^2 \\
&\quad + \int \Big(e^{\kappa e^{C(t-s)} C_{\mathrm{Lip}}(f) c(z)} - 1 - \kappa e^{C(t-s)} C_{\mathrm{Lip}}(f) c(z)\Big)\nu(\mathrm{d}z) \\
&=: C_s^{f,0}.
\end{aligned}$$

We note here that the term $C_s^{f,0}$ does not depend on the process X_t and the parameter ε.

(e) For any $t > 0$ and $\lambda > 0$,

$$\begin{aligned}
\mathsf{E}^x\big[\exp(\kappa Z_t^{f,\varepsilon})\big] &= \mathsf{E}^x\Big[\exp\Big(\kappa Z_t^{f,\varepsilon} - \int_0^t C_s^{f,\varepsilon}\mathrm{d}s\Big)\exp\Big(\int_0^t C_s^{f,\varepsilon}\mathrm{d}s\Big)\Big] \\
&\le \mathsf{E}^x\Big[\exp\Big(\kappa Z_t^{f,\varepsilon} - \int_0^t C_s^{f,\varepsilon}\mathrm{d}s\Big)\Big]\exp\Big(\int_0^t C_s^{f,0}\mathrm{d}s\Big) \\
&= \exp\Big(\int_0^t C_s^{f,0}\mathrm{d}s\Big).
\end{aligned}$$

Letting $\varepsilon \to 0$ yields the first required assertion. The proof is complete. □

Remark 1 In the step (a) of the proof above, the construction $F_\varepsilon(s,x)$ by using the standard mollification argument to $F(s,x)$ is necessary, since the property that $F_\varepsilon(s,\cdot)\in C^2(\mathbb{R}^d)$ is required by applying the Itô formula. This point fills out a small gap in the proof of [12, Proposition 3.4]. In the step (b), the conclusion that $(Z_s^{f,\varepsilon})_{0\le s\le t}$ is a martingale could be easily deduced from its definition. Indeed, for any $0\le s_1<s_2\le t$, by the Markov property,

$$\begin{aligned}
\mathsf{E}^x(Z_{s_2}^{f,\varepsilon}|\mathcal{F}_{s_1}) &= \mathsf{E}^x(F_\varepsilon(s_2,X_{s_2})-F_\varepsilon(0,X_0)|\mathcal{F}_{s_1})\\
&= \mathsf{E}^x((p_\varepsilon * P_{t-s_2})f(X_{s_2})|\mathcal{F}_{s_1})-F_\varepsilon(0,x)\\
&= \mathsf{E}^x[\mathsf{E}^x((p_\varepsilon * P_t)f|\mathcal{F}_{s_2})|\mathcal{F}_{s_1}]-F_\varepsilon(0,x)\\
&= \mathsf{E}^x((p_\varepsilon * P_t)f|\mathcal{F}_{s_1})-F_\varepsilon(0,x)\\
&= (p_\varepsilon * P_t)f(X_{s_1})-F_\varepsilon(0,x)=Z_{s_1}^{f,\varepsilon}.
\end{aligned}$$

Note that, the expression (3.1) of $Z_t^{f,\varepsilon}$ is key to steps (c) and (d) of the proof.

The lemma below has been used in the proof above.

Lemma 3.1 For any $x\in\mathbb{R}$, it holds that $e^x-1-x\le e^{|x|}-1-|x|$; moreover, the function $f(x)=e^x-1-x$ is increasing on $[0,\infty)$.

Proof The second assertion is easily proved. For the first one, we only need to consider the case that $x\le 0$. It suffices to prove that $e^x-1-x\le e^{-x}-1+x$. That is, $e^x-e^{-x}\le 2x$ for $x\le 0$. Set $h(x)=e^x-e^{-x}-2x$. Then, the facts that $h(0)=0$ and $h'(x)=e^x+e^{-x}-2\ge 0$ for all $x\le 0$ will immediately give us the required assertion. □

We now turn to the second part of the proof of Theorem 1.1.

Proof of Theorem 1.1 (2) We will adopt same notations in the proof of Theorem 1.1 (1). According to steps (a) and (b) in its proof, we know that

$$(Z_s^{f,\varepsilon})_{0\le s\le t}:=(F_\varepsilon(s,X_s)-F_\varepsilon(0,X_0))_{0\le s\le t}$$

is a real P^x-martingale with respect to the truncated filtration $(\mathcal{F}_s)_{0\le s\le t}$, where

$$\begin{aligned}
F_\varepsilon(t,X_t) &= F_\varepsilon(0,X_0)+\int_0^t\langle\nabla_x F_\varepsilon(s,X_{s-}),\sigma(X_{s-})\mathrm{d}W_s\rangle\\
&\quad+\int_0^t\int(F_\varepsilon(s,X_{s-}+c(X_{s-},z))-F_\varepsilon(s,X_{s-}))\widetilde{\mu}(\mathrm{d}z,\mathrm{d}s).
\end{aligned}$$

Applying the Itô formula of Lévy type stochastic integral to the function $g(x)=x^2$

yields that

$$\begin{aligned}(Z_t^{f,\varepsilon})^2 &= 2\int_0^t \langle \nabla_x F_\varepsilon(s,X_{s-}),\sigma(X_{s-})\mathrm{d}W_s\rangle \\ &\quad+\int_0^t\int \big[(Z_{s-}^{f,\varepsilon}+F_\varepsilon(s,X_{s-}+c(X_{s-},z))-F_\varepsilon(s,X_{s-}))^2-(Z_{s-}^{f,\varepsilon})^2\big]\widetilde{\mu}(\mathrm{d}z,\mathrm{d}s) \\ &\quad+\int_0^t |\sigma(X_{s-})^*\nabla_x F_\varepsilon(s,X_{s-})|^2\mathrm{d}s \\ &\quad+\int_0^t\int \big[(Z_{s-}^{f,\varepsilon}+F_\varepsilon(s,X_{s-}+c(X_{s-},z))-F_\varepsilon(s,X_{s-}))^2-(Z_{s-}^{f,\varepsilon})^2 \\ &\quad-2Z_{s-}^{f,\varepsilon}(F_\varepsilon(s,X_{s-}+c(X_{s-},z))-F_\varepsilon(s,X_{s-}))\big]\nu(\mathrm{d}z)\mathrm{d}s \\ &=: M_t^{f,\varepsilon}+\int_0^t C_s^{f,\varepsilon}\mathrm{d}s,\end{aligned}$$

where $(M_s^{f,\varepsilon})_{0\le s\le t}$ is a martingale, and

$$\begin{aligned}C_s^{f,\varepsilon} &= |\sigma(X_{s-})^*\nabla_x F_\varepsilon(s,X_{s-})|^2 \\ &\quad+\int (F_\varepsilon(s,X_{s-}+c(X_{s-},z))-F_\varepsilon(s,X_{s-}))^2\nu(\mathrm{d}z).\end{aligned}$$

Therefore, for any $t>0$,

$$\mathsf{E}^x(Z_t^{f,\varepsilon})^2=\mathsf{E}^x\Big(\int_0^t C_s^{f,\varepsilon}\mathrm{d}s\Big). \tag{3.2}$$

Furthermore, due to part (d) in the proof of Theorem 1.1 (1), for any $s\in(0,t)$,

$$|\sigma(X_{s-})^*\nabla_x F_\varepsilon(s,X_{s-})|^2\le e^{2C(t-s)}C_{\mathrm{Lip}}(f)^2A_0.$$

On the other hand,

$$\begin{aligned}&\int (F_\varepsilon(s,X_{s-}+c(X_{s-},z))-F_\varepsilon(s,X_{s-}))^2\nu(\mathrm{d}z) \\ \le\ & C_{\mathrm{Lip}}(F_\varepsilon(s,\cdot))^2\int c(X_{s-},z)^2\nu(\mathrm{d}z)\le e^{2C(t-s)}C_0C_{\mathrm{Lip}}(f)^2.\end{aligned}$$

Thus, for any $s\in(0,t)$,

$$C_s^{f,\varepsilon}\le e^{2C(t-s)}C_{\mathrm{Lip}}(f)^2(A_0+C_0),$$

which combining with (3.2) yields that

$$\mathsf{E}^x(Z_t^{f,\varepsilon})^2\le\frac{C_{\mathrm{Lip}}(f)^2(A_0+C_0)(e^{2Ct}-1)}{2C}.$$

The first desired assertion follows from the inequality above by letting $\varepsilon\to0$. The second conclusion is a consequence of the first one, by applying the Chebyshev inequality and the approximation procedure. □

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带跳随机微分方程的偏差不等式

吴 晔

(福建江夏学院电子信息科学系, 福州, 350108)

利用鞅方法, 我们给出跳扩散过程的偏差不等式, 推广了之前关于纯Lévy跳过程在类Cramér条件下的结论, 同时我们的方法对于Lévy测度不具有指数矩的情形也是适用的.

关键词: 偏差不等式, 跳扩散过程, Lipschitz性质.

学科分类号: O211.62.