Credibility Estimator of the Generalized Weighted Premium with Multitude Contracts *

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Abstract

The credibility estimator under the generalized weighted premium principle were discussed. The results were also extended to the versions of multitude contracts. By transforming the probability distribution, the inhomogeneous and homogeneous credibility estimators in the multitude models were derived, and some statistical properties of those estimators were discussed. Furthermore, the structure parameters in credibility factor were estimated by bootstrap techniques. Finally, the simulation study is presented and shows that the inhomogeneous estimator are good enough to use in practice.

Keywords: Loss function, the generalized weighted premium, credibility estimator, homogeneous estimator, bootstrap method.

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§1. Introduction

Under the framework of decision theory, it is well known that in classical credibility theory the risk premiums are derived on the basis of net premium. In practice, however, a net premium cannot meet the essential need of positive safety loading. An earlier version of credibility premium under the exponential weighted squared loss function is due to Gerber (1980) who represented the credibility premium formula regarding the Esscher premium principle as a linear combination of the collective premium and sample mean of the claim history. A recent work is due to Pan et al. (2008) who found that, as an estimator of individual premium, this credibility premium does not meet certain fundamental requirements from statistics: it does not converge to the individual premium except for

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certain special case. Due to this consideration, they suggested another type of credibility premium for Esscher premium principle that meets the requirement of convergence. Following the line of quadratic-type loss functions, the most generalized form of squared loss functions addressed in literature so far are the ones discussed by Furman and Zitikis (2008), who proposed a generalized weighted loss function, and Wen et al. (2009a) discussed the corresponding credibility estimators under generalized weighted loss functions, and prove the convergence of credibility estimators, as well as Bayes estimator under the generalized weighted premium principle.

However, the credibility estimators they derived cannot be applied to practice, since the collective premium and structure parameters in credibility factor are unknown in general. In this paper, we built the credibility models with multitude contracts, and further study credibility pricing under the generalized weighted premium principle. In this models, we derived the inhomogeneous and homogeneous credibility estimators. In addition, the structure parameters are also estimated by bootstrap techniques.

The paper is organized as follows. The formulations and preliminaries of the models are discussed in Section 2. The credibility estimators are derived in simple Bühlmann model in Section 3 and the multitude contracts models in Section 4 respectively. Furthermore, the structure parameters are estimated by bootstrap techniques in the second part of Section 4. Finally, the simulation study is presented in Section 5, which shows that the results are good enough to use in practice.

§2. Model Formulations and Preliminaries

Suppose that the loss X of a risk is a non-negative random variable with distribution function $F_X(x)$. Throughout the paper, the existence of expectations and variances of random variables are implicitly assumed when referred to. Define the generalized weighted premium of risk X as

$$H(X)=\frac{\mathsf{E}[v(X)h(X)]}{\mathsf{E}[h(X)]}, \tag{2.1}$$

where $v(x)>0$, $h(x)>0$ are assumed to be known weighted functions. In real applications, some restrictive conditions are needed to be added.

Actually, one of the important methods to construct premium calculation principles is to solve the minimization of some expected loss functions indicating the mean loss of the insurer when he adopts a certain premium calculation principle, see for example, Gerber (1980), Heilmann (1989), Wen et al. (2009a), and references therein. We can observe that the generalized weighted premium minimizes the expectation of weighted squared loss function

$$L(X,P)=(v(X)-P)^2h(X), \tag{2.2}$$

i.e.,

$$H(X)=\arg\min_P \mathsf{E}[(v(X)-P)^2h(X)].$$

Here, by taking different forms of $v(x)$ and $h(x)$, the premium defined by (2.1) subsumes many classical premium calculation principles, and some of them have positive safety loadings, see e.g. Furman and Zitikis (2008), Wen et al. (2009a):

• the expectation premium principle $H(X)=(1+\alpha)\mathsf{E}X$ for $v(x)=(1+\alpha)x$ and $h(x)=1$, where $\alpha>0$ is a positive constant;

• the Esscher premium principle $H(X)=\mathsf{E}[Xe^{\lambda X}]/\mathsf{E}[e^{\lambda X}]$ for $v(x)=x$ and $h(x)=e^{\lambda x}$, where λ is a positive constant;

• the modified variance premium principle $H(X)=\mathsf{E}X+\mathsf{Var}(X)/\mathsf{E}X$ for $v(x)=x$ and $h(x)=x$;

• Kamp's premium $H(X)=\mathsf{E}[X(1-e^{-\lambda X})]/\mathsf{E}[1-e^{-\lambda X}]$ for $v(x)=x$ and $h(x)=1-e^{-\lambda x}$, where λ is also a positive constant;

• the conditional tail expectation premium $H(X)=\mathsf{E}[XI(x>q)]/\mathsf{P}(x>q)=\mathsf{E}(X|X>q)$ for $v(x)=x$ and $h(x)=I(x>q)$.

Thus the generalized weighted premium has larger flexibility such that different insurance companies can take particular functions $v(x)$ and $h(x)$ to meet their needs and make the premium be competitive in the markets.

In order to further investigate credibility estimators of the generalized weighted premium, firstly we present a lemma as follows. Similar results can refer to Wen et al. (2009b), Zheng et al. (2012).

Lemma 2.1 Let Y be a random variable and is to be estimated/predited, and $Z=(Z_1,\ldots,Z_m)'$ is a random vector. Write $B=(b_1,b_2,\ldots,b_m)$, $b_i\in R$, and $a\in R$. Then

(1) The solutions of the minimization problem

$$\min_{a\in R,B\in R^m}\mathsf{E}[(v(Y)-a-BZ)^2h(Y)] \tag{2.3}$$

are given by

$$a=\mathsf{E}_*[v(Y)]-\mathsf{Cov}_*(v(Y),Z)[\mathsf{Cov}_*(Z)]^{-1}\mathsf{E}_*(Z),\qquad B=\mathsf{Cov}_*(v(Y),Z)[\mathsf{Cov}_*(Z)]^{-1}, \tag{2.4}$$

where P_* is a new joint distribution generated by the original joint probability of the random vector (Y, Z'), such that

$$\mathsf{P}_*(Y \in A) = \frac{\mathsf{E}[I_A(Y)h(Y)]}{\mathsf{E}[h(Y)]}, \quad \mathsf{P}_*(Z_i \in A) = \frac{\mathsf{E}[I_A(Z_i)h(Y)]}{\mathsf{E}[h(Y)]}, \qquad i = 1, 2, \ldots, m. \tag{2.5}$$

Here I_A is the indicator of set A. Accordingly, the expectation, variance, covariance or covariance matrix under P_* are denoted as E_*, Var_* and $\mathsf{Cov}_*(X)$ respectively (Note that $\mathsf{Cov}_*(X)$ denotes covariance matrix if X is a random vector, or variance if X is a random variable).

(2) The solution of conditional minimization problem

$$\min_{B\in R^m} \mathsf{E}[(v(Y) - BZ)^2 h(Y)], \qquad \text{with } \mathsf{E}_*[v(Y)] = B\mathsf{E}_*[Z] \tag{2.6}$$

is

$$B = \Big[\mathsf{Cov}_*(v(Y), Z) + \frac{\mathsf{E}_*[v(Y)] - \mathsf{Cov}_*(v(Y), Z)[\mathsf{Cov}_*(Z)]^{-1}\mathsf{E}_*(Z)}{\mathsf{E}_*(Z')[\mathsf{Cov}_*(Z)]^{-1}\mathsf{E}_*(Z)}\mathsf{E}_*(Z')\Big][\mathsf{Cov}_*(Z)]^{-1}.$$

Proof Note that

$$\mathsf{E}[(v(Y) - a - BZ)^2 h(Y)] = \mathsf{E}[h(Y)] \cdot \mathsf{E}_*[(v(Y) - a - BZ)^2 h(Y)].$$

Write $\Phi = \mathsf{E}_*[(v(Y) - a - BZ)^2]$, so we only need to minimize Φ under expected squared loss function with probability distribution "P_*". Firstly, Standard lagrange's method gives

$$B = \mathsf{Cov}_*(v(Y), Z)[\mathsf{Cov}_*(Z)]^{-1}$$

and

$$a = \mathsf{E}_*[v(Y)] - \mathsf{Cov}_*(v(Y), Z)[\mathsf{Cov}_*(Z)]^{-1}\mathsf{E}_*(Z).$$

Secondly, note that the minimization problem (2.6) is equivalent to

$$\min_{B\in R^m} \mathsf{E}_*[((v(Y) - \mathsf{E}_*(v(Y))) - BZ - \mathsf{E}_*(Z))^2], \qquad \text{with } \mathsf{E}_*[v(Y)] = B\mathsf{E}_*[Z], \tag{2.7}$$

thus we can also get

$$B = \Big[\mathsf{Cov}_*(v(Y), Z) + \frac{\mathsf{E}_*[v(Y)] - \mathsf{Cov}_*(v(Y), Z)[\mathsf{Cov}_*(Z)]^{-1}\mathsf{E}_*(Z)}{\mathsf{E}_*(Z')[\mathsf{Cov}_*(Z)]^{-1}\mathsf{E}_*(Z)}\mathsf{E}_*(Z')\Big][\mathsf{Cov}_*(Z)]^{-1},$$

with conditional Lagrange methods. □

Consequently, from the first part of the Lemma 2.1, Y can be optimally predicted under the loss function (2.2) in the class of inhomogeneous linear functions of Z by

$$\widehat{Y} = \mathsf{E}_*[v(Y)] + \mathsf{Cov}_*(v(Y), Z)[\mathsf{Cov}_*(Z)]^{-1}(Z - \mathsf{E}_*(Z)). \tag{2.8}$$

§3. The Credibility Estimators

In the credibility theory, we assumed that a risk X can be recognized by a risk parameter Θ, which is an unobservable random variable, say, with distribution density $\pi(\theta)$. Given Θ, $X_1, X_2, \ldots, X_n, X_{n+1}$ are i.i.d. copies of X, i.e., for $i = 1, 2, \ldots, n+1$, (X_i, Θ) has the same joint distribution as that of a typical representative (X, Θ). Write $\underline{X_n} = (X_1, X_2, \ldots, X_n)$. In this model, the individual premium and collective premium are defined as

$$R(\Theta) = \frac{\mathsf{E}[v(X)h(X)|\Theta]}{\mathsf{E}[h(X)|\Theta]} \tag{3.1}$$

and

$$H(X) = \frac{\mathsf{E}[v(X)h(X)]}{\mathsf{E}[h(X)]} \tag{3.2}$$

respectively.

In credibility theory, our goals are to estimate/predict the future loss X_{n+1}. Under the generalized weighted loss function, individual premium $R(\Theta)$ is the optimal predictor of X_{n+1} given Θ, i.e.,

$$R(\Theta) = \arg \min_{P=P(\Theta)} \mathsf{E}[(v(X_{n+1}) - P)^2 h(X_{n+1})|\Theta], \qquad \text{a.s..} \tag{3.3}$$

However, since the risk parameter Θ is unknown in practice, the individual premium $R(\Theta)$ is also unknown in actuarial science, and is to be estimated based on the samples $\underline{X_n}$. In credibility theory, the individual premium $R(\Theta)$ is also called as the risk premium.

Wen et al. (2009a) derived the credibility estimator for the individual premium $R(\Theta)$. The results can be presented as follows.

Theorem 3.1 (Wen et al. (2009a)) Denoting

$$\mathcal{M}_g = \{a + bg(\underline{X_n}),\ \text{with } a, b \in R \text{ and } g(\underline{X_n}) \text{ is a function of the samples } \underline{X_n}\}, \tag{3.4}$$

then the solution of

$$\mathsf{E}[L(X_{n+1}, H_g(\underline{X_n}))] = \min_{f(\cdot)\in\mathcal{M}_g} \mathsf{E}[(v(X_{n+1}) - f(\cdot))^2 h(X_{n+1})] \tag{3.5}$$

is

$$H_g(\underline{X_n}) = zg(\underline{X_n}) + \Big(1 - z\frac{\mathsf{E}_*[g_n(\Theta)]}{H(X)}\Big)H(X), \tag{3.6}$$

where

$$z = \frac{\mathsf{Cov}_*(R(\Theta), g_n(\Theta))}{\mathsf{Var}_*[g_n(\Theta)] + \mathsf{E}_*[\mathsf{Var}(g(\underline{X_n})|\Theta)]} \tag{3.7}$$

is called credibility factor, $g_n(\Theta) = \mathsf{E}[g(\underline{X_n})|\Theta]$, and the new distribution "$*$" of Θ is defined as

$$\mathsf{P}_*(\Theta \in A) = \frac{\mathsf{E}[I_A(\Theta)\mathsf{E}(h(X)|\Theta))]}{\mathsf{E}[h(X)]}. \tag{3.8}$$

§4. Multitude Contract Models

4.1 Homogeneous Credibility Estimator

In real applications, however, the structure function $\pi(\theta)$ usually cannot be completely specified so that the collective premium $H(X)$, as well as structure parameters in credibility factor z is actually unknown. Therefore, the claim experiences over a number of risks in the same portfolio have to be observed in order to estimate $\pi(\theta)$ or the collective premium by empirical Bayes techniques. More specifically, let $X_1, X_2, \ldots, X_K$ denote K risks under observation. The distribution of each X_i is characterized by its risk parameter Θ_i and contributes a sequence of claim experiences $X_i = (X_{i1}, X_{i2}, \ldots, X_{in_i})$ over n_i time periods (To simplify our exposition, the same time periods will be applied to all individuals (the so-called balanced model). With a slight variation to the model, however, it can be easily extended to the unbalanced case.). Under certain assumptions, the standard paradigms using empirical Bayes method can be applied to all the historical data to estimate the prior distribution $\pi(\theta)$ and, in turn, to predict the future loss of each X_i, $i = 1, 2, \ldots, K$, at the next period. These data are therefore structured in two dimensions with one indicating the time horizon and the other the distinct insured individuals.

We list the assumptions for multitude contracts as follows.

Assumption 4.1 Conditionally on $\Theta_i = \theta$, the random variables X_{ij} ($j = 1, 2, \ldots, n$) are independent, with the same distribution function F_θ.

Assumption 4.2 The risk parameter $\Theta_1, \Theta_2, \ldots, \Theta_K$ are independent and identically distributed as the same structure distribution function $\pi(\theta)$.

Assumption 4.3 The random vectors (Θ_i, X_i) are independent for $i=1, 2, \ldots, K$.

This section aims to estimate

$$R(\Theta_i) = \frac{\mathsf{E}[v(X_{ij})h(X_{ij})|\Theta_i]}{\mathsf{E}[h(X_{ij})|\Theta_i]}$$

(predict $X_{i,n+1}$) based on data $\underline{X} = (X_1', X_2', \ldots, X_K')$ so as to minimize the expected generalized weighted loss function $L(X_{i,n+1}, f(\cdot))$ defined as in (2.2), where $f(\cdot)$ is a function of the sample $\underline{X}$.

We constrain the function $f(\cdot)$ to be in the class

$$M_l=\Big\{a+\sum_{s=1}^{K}b_s\overline{H}_{sn},\ \text{with } a,b_s\in R \text{ and } \overline{H}_{sn}=\Big[\sum_{j=1}^{n}v(X_{sj})h(X_{sj})\Big]\Big/\Big[\sum_{j=1}^{n}h(X_{sj})\Big]\Big\},\quad(4.1)$$

i.e, to solve the following minimization problem

$$\min_{a,b_s\in R}\mathsf{E}\Big[\Big(v(X_{i,n+1})-a-\sum_{s=1}^{K}b_s\overline{H}_{sn}\Big)^2h(X_{i,n+1})\Big].\quad(4.2)$$

The solution of minimization problem (4.2), denoted by $\widehat{R(\Theta_i)}^*$, is called credibility estimator of $R(\Theta_i)$.

Firstly, we define a new joint distribution for $(\Theta,\underline{X},X_{i,n+1})$, where $\Theta=(\Theta_1,\Theta_2,\ldots,\Theta_K)$, $\underline{X}=(X_1,X_2,\ldots,X_K)$, and $X_s=(X_{s1},X_{s2},\ldots,X_{sn})$ as follows: under which the conditional independence among X_{sj}, $s=1,2,\ldots$, $j=1,2,\ldots$, and the independence among contracts retain, whereas the marginal distribution of Θ_s changes to

$$\mathsf{P}_*(\Theta_s\in A)=\frac{\mathsf{E}[I_A(\Theta_s)m_h(\Theta_i)]}{\mathsf{E}[m_h(\Theta_i)]},\qquad s=1,2,\ldots,K,\quad(4.3)$$

where I_A is the indicator of set A, and the conditional distribution of X_{st} changes to

$$\mathsf{P}_*(X_{st}\in B|\Theta)=\frac{\mathsf{E}[I_B(X_{st})h(X_{i,n+1})|\Theta]}{\mathsf{E}[h(X_{i,n+1})|\Theta]},\qquad s=1,2,\ldots,K,\ j=1,2,\ldots,n,n+1.$$

Under P_*, we can see that the marginal distribution of any X_{st} changes over different n. Note that here for $s=1,2,\ldots,K$, $t=1,2,\ldots,n$,

$$\mathsf{P}_*(X_{st}\in B|\Theta)=\frac{\mathsf{E}[I_B(X_{st})h(X_{i,n+1})|\Theta]}{\mathsf{E}[h(X_{i,n+1})|\Theta]}=\mathsf{P}(X_{st}\in B|\Theta),\quad(4.4)$$

i.e, the conditional distribution of the sample $\underline{X}$ is unchanged. Similarly, we have

$$\mathsf{P}_*(\Theta_s\in A)=\frac{\mathsf{E}[I_A(\Theta_s)m_h(\Theta_i)]}{\mathsf{E}[m_h(\Theta_i)]}=\mathsf{P}(\Theta_s\in A),\qquad s\neq i.$$

Accordingly, the expectation, variance and covariance under P_* are also denoted as, respectively, E_*, Var_* and Cov_*.

For any measurable function $l(\underline{X_n},X_{i,n+1})$, noting that $\mathsf{E}_*[l(\underline{X_n})|\Theta]=\mathsf{E}[l(\underline{X_n})|\Theta]$, then

$$\begin{aligned}\mathsf{E}_*[l(\underline{X_n},X_{i,n+1})]&=\mathsf{E}_*[\mathsf{E}_*[l(\underline{X_n},X_{i,n+1})|\Theta]]\\&=\frac{\mathsf{E}\{\mathsf{E}_*[l(\underline{X_n},X_{i,n+1})|\Theta]\mathsf{E}[h(X_{i,n+1})|\Theta]\}}{\mathsf{E}[h(X_{i,n+1})]}\\&=\frac{\mathsf{E}\{\mathsf{E}[l(\underline{X_n},X_{i,n+1})h(X_{i,n+1})|\Theta]\}}{\mathsf{E}[h(X_{i,n+1})]}\\&=\frac{\mathsf{E}[l(\underline{X_n},X_{i,n+1})h(X_{i,n+1})]}{\mathsf{E}[h(X_{i,n+1})]}.\end{aligned}\quad(4.5)$$

Let $\overline{H}_n := (\overline{H}_{1n}, \overline{H}_{2n}, \ldots, \overline{H}_{Kn})$ and $h_i(\Theta_i) = \mathsf{E}(\overline{H}_{in}|\Theta_i)$, then we derive the following theorem.

Theorem 4.1 Under the Assumptions 4.1-4.3 and new joint probability distribution of random vector $(\Theta, \underline{X}, X_{i,n+1})$ above, the credibility estimator of $R(\Theta_i)$ by solving the minimization problem (4.2) is

$$\widehat{R(\Theta_i)}^* = z_i \overline{H}_{in} + \Big(1 - z_i \frac{\mathsf{E}_*[h_i(\Theta_i)]}{H(X)}\Big) H(X), \tag{4.6}$$

where the credibility factor is

$$z_i = \frac{\mathsf{Cov}_*(R(\Theta_i), h_i(\Theta_i))}{\mathsf{Var}_*[h_i(\Theta_i)] + \mathsf{E}_*[\mathsf{Var}(\overline{H}_{in}|\Theta_i)]}. \tag{4.7}$$

Proof Denote $B = (b_1, b_2, \ldots, b_K)$, then

$$\mathsf{E}\Big[\Big(v(X_{i,n+1}) - a - \sum_{i=1}^{K} b_i \overline{H}_{in}\Big)^2 h(X_{i,n+1})\Big] = \mathsf{E}[(v(X_{i,n+1}) - a - B\overline{H}_n)^2 h(X_{i,n+1})]. \tag{4.8}$$

From (2.8), the credibility estimator $\widehat{R(\Theta_i)}^*$ can be given by

$$\begin{aligned}\widehat{R(\Theta_i)}^* &= \mathsf{E}_*[v(X_{i,n+1})] + \mathsf{Cov}_*(v(X_{i,n+1}), \overline{H}_n)[\mathsf{Cov}_*(\overline{H}_n)]^{-1}(\overline{H}_n - \mathsf{E}_*(\overline{H}_n)) \\ &= z_i \overline{H}_{in} + \Big(1 - z_i \frac{\mathsf{E}_*[h_i(\Theta_i)]}{H(X)}\Big) H(X).\end{aligned}$$

This complete the proof. □

Remark 1 From the Assumptions 4.1-4.3, it is easily to check that $\mathsf{E}_*[h_i(\Theta_i)]$, $\mathsf{Cov}_*(R(\Theta_i), h_i(\Theta_i))$, $\mathsf{Var}_*[h_i(\Theta_i)]$ and $\mathsf{E}_*[\mathsf{Var}(\overline{H}_{in}|\Theta_i)]$ are all independent of index i, so is z_i, $i = 1, 2, \ldots, K$. We denote z for z_i in the following.

Analogy to the classical credibility theory, when $H(X)$ or $\mathsf{E}_*[h_i(\Theta_i)]$ is unknown, we can solve the problem (4.2) and establish the optimal homogeneous estimator of $R(\Theta_i)$. We solve the following problem

$$\min_{b_i \in R} \mathsf{E}\Big[\Big(v(X_{i,n+1}) - \sum_{s=1}^{K} b_s \overline{H}_{sn}\Big)^2 h(X_{i,n+1})\Big], \qquad \text{with } \mathsf{E}_*[v(X_{i,n+1})] = \mathsf{E}_*\Big[\sum_{s=1}^{K} b_s \overline{H}_{sn}\Big] \tag{4.9}$$

and derive the results as follows.

Theorem 4.2 Under the Assumptions 4.1-4.3, the homogeneous credibility estimator of $R(\Theta_i)$, by solving the problem (4.9) can be derived as

$$\widehat{R(\Theta_i)}^{\text{hom}} = z\overline{H}_{in} + \Big(\frac{H(X)}{w} - z\Big)\overline{\overline{H}}, \tag{4.10}$$

where z is defined in (4.7) and $w = \mathsf{E}_*[h_i(\Theta_i)]$, $\overline{\overline{H}} = (1/K) \sum_{s=1}^{K} \overline{H}_{sn}$.

Proof From the second part of the Lemma 2.1, the homogeneous credibility estimator of $R(\Theta_i)$ is

$$\begin{aligned}\widehat{R(\Theta_i)}^{\text{hom}} &= \Big[\mathsf{Cov}_*(v(X_{i,n+1}),\overline{H}_n) \\ &\quad + \frac{\mathsf{E}_*[v(X_{i,n+1})]-\mathsf{Cov}_*(v(X_{i,n+1}),\overline{H}_n)[\mathsf{Cov}_*(\overline{H}_n)]^{-1}\mathsf{E}_*(\overline{H}_n)}{\mathsf{E}_*(\overline{H}'_n)[\mathsf{Cov}_*(\overline{H}_n)]^{-1}\mathsf{E}_*(\overline{H}_n)}\mathsf{E}_*(\overline{H}'_n)\Big] \\ &\quad \cdot[\mathsf{Cov}_*(\underline{X})]^{-1}\overline{H}_n. \end{aligned}\tag{4.11}$$

For notations convenience, we denote $u=\mathsf{Cov}_*(R(\Theta_i),h_i(\Theta_i))$, $q=\mathsf{Var}_*(h_s(\Theta_s))$, $p=\mathsf{E}_*(\mathsf{Var}(\overline{H}_{sn}|\Theta_s))$, then $z=u/(p+q)$. We obtain

$$\begin{aligned}\widehat{R(\Theta_i)}^{\text{hom}} &= \Big[ue'_i+\frac{H(X)-ue'_i[q+pI_K]^{-1}w\mathbf{1}_K}{w\mathbf{1}'_K[q+pI_K]^{-1}w\mathbf{1}_K}w\mathbf{1}'_K\Big][q+pI_K]^{-1}\overline{H}_n \\ &= \frac{u}{q+p}\overline{H}_{in}+\frac{H(X)-\dfrac{uw}{q+p}}{K\dfrac{w^2}{q+p}}\mathbf{1}'_K\frac{w}{q+p}\overline{H}_n \\ &= z\overline{H}_{in}+\frac{H(X)-zw}{w}\overline{\overline{H}} \\ &= z\overline{H}_{in}+\Big(\frac{H(X)}{w}-z\Big)\overline{\overline{H}}. \qquad \square \end{aligned}\tag{4.12}$$

Remark 2 The homogeneous credibility estimator $\widehat{R(\Theta_i)}^{\text{hom}}$ can be expressed exactly credibility form:

$$\widehat{R(\Theta_i)}^{\text{hom}}=z\overline{H}_{in}+(1-z)\overline{\overline{H}}$$

if and only if

$$\mathsf{E}_*[h_i(\Theta_i)]=H(X).\tag{4.13}$$

4.2 The Estimation of Structure Parameters

For the homogeneous credibility estimators (4.10) in multitude contract models, they cannot be applied to practice directly, because some structure parameters in credibility factor, such as $H(X)$, $\mathsf{E}_*[h_i(\Theta_i)]\mathsf{Cov}_*(R(\Theta_i),h_i(\Theta_i))$, $\mathsf{Var}_*[h_i(\Theta_i)]$ and $\mathsf{E}_*[\mathsf{Var}(\overline{H}_{in}|\Theta_i)]$ are usually unknown in practice.

From the homogeneous credibility estimator, obviously the $H(X)$ and $w=\mathsf{E}_*[h_i(\Theta_i)]$ can be estimated by

$$\widehat{H(X)}=\widehat{w}=\overline{\overline{H}}.\tag{4.14}$$

Secondly, we denote

$$u\triangleq\mathsf{Cov}_*(R(\Theta_i),h_i(\Theta_i)),\quad q\triangleq\mathsf{Var}_*[h_i(\Theta_i)]\quad\text{and}\quad p\triangleq\mathsf{E}_*[\mathsf{Var}(\overline{H}_{in}|\Theta_i)]\tag{4.15}$$

for convenience.

When the credibility estimator (4.6) are applied to practice, we first estimate these structure parameters based on the samples $\underline{X}$. By inserting the corresponding estimators for structure parameters into (4.6), we denote

$$\widetilde{R(\Theta_i)} = \widehat{z}\overline{H}_{in} + (1-\widehat{z})\overline{\overline{H}},$$

which are called the empirical Bayes credibility estimator, where $z = \widehat{u}/(\widehat{q}+\widehat{p})$.

Since

$$h_i(\Theta_i) = \mathsf{E}\Big\{\Big[\sum_{j=1}^{n} v(X_{ij})h(X_{ij})\Big]\Big/\Big[\sum_{j=1}^{n} h(X_{ij})\Big]\Big|\Theta_i\Big\}$$

is dependent on index i, so the common moment method can not be uses to estimated u and p. On the other hand, maximum likelihood method can also not be used since the distributions of random variables involved are unknown. However, a particularly effective method to estimate structure parameters in this case is to use the bootstrap techniques. For $b = 1, 2, \ldots, B$, and $j = 1, 2, \ldots, n$, let X_{ij}^{*b} be independent random variables sampled from the empirical distribution function

$$F_{in}(x) = \frac{1}{n}\sum_{j=1}^{n} I(X_{ij} \le x).$$

This resampling can be implemented by drawing nB random integers $J(j,b)$ independently from the uniform distribution on $\{1, 2, \ldots, n\}$, and setting $X_{ij}^{*b} = X_{iJ(j,b)}$. To keep the notation consistency, denote

$$\overline{H}_{in}^{b} = \Big[\sum_{j=1}^{n} v(X_{ij}^{*b})h(X_{ij}^{*b})\Big]\Big/\Big[\sum_{j=1}^{n} h(X_{ij}^{*b})\Big],$$

$$h_i = \frac{1}{B}\sum_{b=1}^{n}\overline{H}_{in}^{b} \quad \text{and} \quad s_i^2 = \frac{1}{B-1}\sum_{b=1}^{n}(\overline{H}_{in}^{b} - h_i)^2. \tag{4.16}$$

In addition, we denote

$$m_i = \frac{1}{n}\sum_{j=1}^{n} h(X_{ij}), \qquad v_i = \frac{1}{n}\sum_{j=1}^{n} h(X_{ij})v(X_{ij}). \tag{4.17}$$

Firstly, since

$$u = \mathsf{Cov}_*(R(\Theta_i), h_i(\Theta_i)) = \mathsf{E}_*[R(\Theta_i)h_i(\Theta_i)] - \mathsf{E}_*[R(\Theta_i)]\mathsf{E}_*[h_i(\Theta_i)], \tag{4.18}$$

where $R(\Theta_i) = \mathsf{E}[v(X_{ij})h(X_{ij})|\Theta_i]/\mathsf{E}[h(X_{ij})|\Theta_i]$, $h_i(\Theta_i) = \mathsf{E}[\overline{H}_{in}|\Theta_i]$ and $\mathsf{P}_*(\Theta_s \in A) = \mathsf{E}[I_A(\Theta_s)m_h(\Theta_i)]/\mathsf{E}[m_h(\Theta_i)]$, $s = 1, 2, \ldots, K$, then the bootstrap estimator of u is given by

$$\widehat{u} = \Big(\sum_{i=1}^{K}\overline{H}_{in}m_ih_i\Big)\Big/\Big(\sum_{i=1}^{K}m_i\Big) - \Big(\sum_{i=1}^{K}\overline{H}_{in}m_i\Big)\Big(\sum_{i=1}^{K}m_ih_i\Big)\Big/\Big(\sum_{i=1}^{K}m_i\Big)^2. \tag{4.19}$$

In the second, since $q = \mathsf{Var}_*[h_i(\Theta_i)] = \mathsf{E}_*[h_i^2(\Theta_i)] - (\mathsf{E}_*[h_i(\Theta_i)])^2$, then

$$\widehat{q} = \Big(\sum_{i=1}^{K} m_i h_i^2\Big)\Big/\Big(\sum_{i=1}^{K} m_i\Big) - \Big(\sum_{i=1}^{K} m_i h_i\Big)^2\Big/\Big(\sum_{i=1}^{K} m_i\Big)^2. \tag{4.20}$$

Finally, we observe that

$$p = \mathsf{E}_*[\mathsf{Var}\,(\overline{H}_{in}|\Theta_i)], \tag{4.21}$$

and the corresponding estimators can be given by

$$\widehat{p} = \Big(\sum_{i=1}^{K} m_i s_i^2\Big)\Big/\Big(\sum_{i=1}^{K} m_i\Big). \tag{4.22}$$

From the properties of bootstrap estimator, see, e.g., Hall (1992), the estimator $\widehat{u}$, $\widehat{q}$ and $\widehat{p}$ are consistent with respect to u, q and p when $B \to \infty$, respectively.

In real application, however, the estimator $\widehat{u}$ and $\widehat{q}$ may be negative value, we can take $\widehat{u}^* = \max(0, \widehat{u})$, $\widehat{q}^* = \max(0, \widehat{q})$.

§5. Simulation Study

In the section, we assume that X is a Bernoulli variable with $\mathsf{P}(X=1) = 1 - \mathsf{P}(X=0) = \Theta$ and $\Theta \backsim U(0,1)$, the uniform distribution on interval $(0,1)$. In this simulation we take $v(x) = e^{\lambda x}$ and $h(x) = x^2 + 1$ with $\lambda = 0.2$. Therefore, Then under the new probability P_*, the density functions of Θ and X_i, $i = 1, \ldots, n$ are

$$\pi_*(\theta) = \frac{\pi(\theta) m_h(\theta)}{m_h} = \frac{2}{3}(\theta + 1), \qquad \theta \in (0,1)$$

and

$$\mathsf{P}_*(X_i \in B|\Theta) = \mathsf{P}(X_i \in B|\Theta),$$

respectively. The risk premium is

$$R(\Theta_i) = \frac{\mathsf{E}[v(X_{ij})h(X_{ij})|\Theta]}{\mathsf{E}[h(X_{ij})|\Theta_i]} = \frac{2(e^{0.4} - 1)\Theta_i}{\Theta_i + 1} + 1.$$

However, it is quite difficult to work out a closed form of $h_i(\Theta_i) = \mathsf{E}[\overline{H}_{in}|\Theta_i]$, therefore, the structure parameter u, p and q can not be analytically calculated. Thus, instead, we use a Monte Carlo method to numerically compute them. We take sizes $n = 20$, $n = 100$, the risk premium and corresponding inhomogeneous credibility estimators and their MSE are given by the following tables:

Table 1 The simulation results

	$n=20$			$n=100$		
θ_i	0.2	0.4	0.6	0.2	0.4	0.6
$R(\theta_i)$	1.0738	1.1265	1.1661	1.0738	1.1265	1.1661
$\widehat{R(\theta_i)}^*$	1.0794	1.1267	1.1634	1.0741	1.1256	1.1650
Std.	0.1807	0.1615	0.1250	0.0384	0.0352	0.0267

The results in the table above shows that the inhomogeneous credibility estimator is very closed to risk premium. Especially in the large sample, for instance, $n = 100$, the mean square of the credibility estimator is small enough to use in practice.

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具有多合同的广义加权保费的信度估计

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本文讨论了广义加权保费原理下的信度估计, 并把结论推广到多合同模型. 通过概率分布的变换, 本文得到了多合同模型下广义加权保费的非齐次和齐次信度估计. 并且讨论了这些估计的统计性质. 最后, 运用重抽样方法讨论了信度因子中未知结构参数的估计. 数值模拟表明, 非齐次信度估计能运用于保险实际.

关键词: 损失函数, 广义加权保费, 信度估计, 齐次估计, 重抽样方法.

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