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# The Best Linear Unbiased Estimation of Regression Coefficient under Weighted Balanced Loss Function *

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**Abstract**

In this paper, the best linear unbiased estimator of regression coefficients in the linear model is studied. Under weighted balanced loss function the minimum risk properties of linear estimator of regression coefficients in the class of linear unbiased estimator is discussed. Furthermore, some kinds of relative efficiencies of the best linear unbiased estimator and ordinary least squares estimator are given, and the lower bound or upper bound of these relative efficiencies are also given.

**Keywords:** Weighted balanced loss function, linear estimators, best liner unbiased estimator, relative efficiency.

**AMS Subject Classification:** 62J05.

## §1. Introduction

Consider the following linear model

$$\begin{cases} y = X\beta + \epsilon, \\ \mathsf{E}(\epsilon) = 0, \\ \mathsf{Cov}\,(\epsilon) = \sigma^2 I_n, \end{cases} \tag{1.1}$$

where $y$ is an $n \times 1$ vector of observation, $X$ is an $n \times p$ known matrix of rank $p$, $\beta$ is a $p \times 1$ vector of unknown parameters, $\epsilon$ is an $n \times 1$ vector of disturbances with expectation $\mathsf{E}(\epsilon) = 0$ and variance-covariance matrix $\mathsf{Cov}\,(\epsilon) = \sigma^2 I_n$.

Let $\widetilde{\beta}$ stands for any estimator of $\beta$, then the quadratic loss function which reflects the goodness of the fitted model is

$$(y - X\widetilde{\beta})'(y - X\widetilde{\beta}), \tag{1.2}$$

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while the commonly employed loss function for the precision of estimation is squared error loss function

$$(\widetilde{\beta}-\beta)'(\widetilde{\beta}-\beta) \tag{1.3}$$

or weighted squared error loss function

$$(\widetilde{\beta}-\beta)'X'X(\widetilde{\beta}-\beta). \tag{1.4}$$

Both the criteria are important and it may be desirable to employ both the criteria simultaneously in practice. Accordingly; considering both the criteria of goodness of fit and precision of estimation together, Zellner (1994) has introduced the following balanced loss function:

$$w(y-X\widetilde{\beta})'(y-X\widetilde{\beta})+(1-w)(\widetilde{\beta}-\beta)'X'X(\widetilde{\beta}-\beta), \tag{1.5}$$

where $w$ is a scalar between 0 and 1. When $w=0$, the loss function of (1.5) reflects the precision of estimation and when $w=1$, the loss function of (1.5) reflects the goodness of fitted model.

Furthermore, using the idea of simultaneous prediction of actual and average values of study variable, Shalabh (1995) has presented the following loss function as

$$w^2(y-X\widetilde{\beta})'(y-X\widetilde{\beta})+(1-w)^2(\widetilde{\beta}-\beta)'X'X(\widetilde{\beta}-\beta)+2w(1-w)(X\widetilde{\beta}-y)'X(\widetilde{\beta}-\beta), \tag{1.6}$$

where $w$ is a scalar between 0 and 1. Such loss function is an extension of the balanced loss function of (1.5) and also take care of the covariability between the goodness of fitted model and precision of estimation.

The balanced loss function has been received considerable attention in the literature. Rodrigues and Zellner (1994) have discussed the balanced loss function in the estimation of mean time to failure. Gruber (2004) has studied the empirical Bayes and approximate minimum mean square error estimator under a general balanced loss function. Zhu et al. (2010) derived the best linear unbiased estimator under the balanced loss function. Hu et al. (2010) obtained the optimal estimator under the balanced loss function and discussed the relative efficiency of the optimal estimator with the ordinary least squares estimator. Jozani et al. (2006) presented the weighted balance-type loss function and considered the issues of the admissibility, dominance, Bayesianity and minimality.

Since the weighted balanced loss function is very popular, we consider it in this paper. We will obtain the optimal estimator under the weighted balanced loss function. Some relative efficiencies are also given in this paper.

The rest of the paper is organized as follows: In Section 2, we present the model and the weighted balanced loss function. In Section 3, we obtain the best linear unbiased estimator and consider some relative efficiencies. Some conclusion remarks are discussed in Section 4.

## §2. Linear Model and Loss Function

Consider the following general linear model

$$\begin{cases} y = X\beta + \epsilon, \\ \mathsf{E}(\epsilon) = 0, \\ \mathsf{Cov}\,(\epsilon) = \sigma^2 V, \end{cases} \tag{2.1}$$

where $y$ is an $n\times 1$ observable random vector, $\epsilon$ denotes an $n\times 1$ random error vector, $X$ represents an $n\times p$ known matrix with $\mathrm{rank}(X)=p$, $V$ shows an $n\times n$ known positive definite matrix, $\beta$ shows a $p\times 1$ unknown parameters. A lot of statisticians have proposed many methods to obtain the best linear estimator of regression coefficient of this model.

According to Shalabh (1995)'s thought of weighted balanced loss, we propose the following weighted balanced loss function:

$$\begin{aligned} W(\widetilde{\beta},\beta,\sigma^2) &= w^2(y-X\widetilde{\beta})'V^{-1}(y-X\widetilde{\beta}) + (1-w)^2(\widetilde{\beta}-\beta)'S(\widetilde{\beta}-\beta) \\ &\quad + 2w(1-w)(X\widetilde{\beta}-y)'V^{-1}X(\widetilde{\beta}-\beta), \end{aligned} \tag{2.2}$$

where $w$ is a scalar between 0 and 1, $S$ is a positive definite matrix and $\widetilde{\beta}$ is an any estimator of $\beta$. The corresponding risk function is defined by

$$R(\widetilde{\beta},\beta,\sigma^2) = \mathsf{E}\{W(\widetilde{\beta},\beta,\sigma^2)\}. \tag{2.3}$$

## §3. The Best Linear Unbiased Estimator

Write $\Im_1 = \{Ly; L \text{ is a } p\times n \text{ constant matrix and } LX = I_p\}$, the optimal estimator in $\Im_1$ is obtained when $V$ is a positive definite matrix.

**Definition 3.1** If $Ly$ in $\Im_1$, let

$$\begin{aligned} R(Ly,\beta,\sigma^2) &= \mathsf{E}\{W(Ly,\beta,\sigma^2)\} \\ &= \mathsf{E}\{w^2(y-XLy)'V^{-1}(y-XLy) + (1-w)^2(Ly-\beta)'S(Ly-\beta) \\ &\quad + 2w(1-w)(XLy-y)'V^{-1}X(Ly-\beta)\} \end{aligned} \tag{3.1}$$

reach the minimum, then $Ly$ is the best linear unbiased estimator.

**Lemma 3.1** (Wang, 1987) Let $X$ be an $n\times p$ matrix, $L$ is a $p\times n$ matrix, then $\partial \mathrm{tr} L'X'XL/\partial L = 2X'XL$.

**Lemma 3.2** (Yang, 1988) Let $A$ be an $n\times n$ positive definite matrix, and $\lambda_1 \geq \cdots \geq \lambda_n > 0$ be the ordered eigenvalues of $A$, $P$ is an $n\times k$ matrix with $P'P = I_k$, $n > k$. Denote $l = \min(k, n-k)$, then

$$1 \leq \frac{\mathrm{tr}(P'AP)}{\mathrm{tr}(P'A^{-1}P)^{-1}} \leq \Big[\sum_{i=1}^{l}(\lambda_i + \lambda_{n-i+1}) \Big/ \Big(2\sum_{i=1}^{l}(\sqrt{\lambda_i\lambda_{n-i+1}})\Big)\Big]^2.$$

**Lemma 3.3** (Wang et al., 2006) Let $A$ be an $n\times n$ positive definite matrix, and $\lambda_1 \geq \cdots \geq \lambda_n > 0$ be the ordered eigenvalues of $A$, $P$ is an $n\times k$ matrix with $P'P = I_k$, $n > k$. Denote $l = \min(k, n-k)$, then

$$\mathrm{tr}(P'AP - (P'A^{-1}P)^{-1}) \leq \sum_{i=1}^{l}(\sqrt{\lambda_i} - \sqrt{\lambda_{n-i+1}})^2.$$

### 3.1 The Optimal Estimator

In this subsection, we will present the linear unbiased estimator of $\beta$ in $\Im_1$. Now, we begin with the following theorem.

**Theorem 3.1** For the model (2.1), when $V$ is a positive definite matrix, if $L = (X'V^{-1}X)^{-1}X'V^{-1}$, then $Ly$ is the best linear unbiased estimator of $\beta$ under the weighted balanced loss function.

**Proof** Consider the weighted loss function

$$\begin{aligned} W(\widetilde{\beta}, \beta, \sigma^2) &= w^2(y - X\widetilde{\beta})'V^{-1}(y - X\widetilde{\beta}) + (1-w)^2(\widetilde{\beta} - \beta)'S(\widetilde{\beta} - \beta) \\ &\quad + 2w(1-w)(X\widetilde{\beta} - y)'V^{-1}X(\widetilde{\beta} - \beta). \end{aligned} \tag{3.2}$$

Assume that $Ly \in \Im_1$ and according to the weighted balanced loss function of (3.2), then its risk function is given by

$$\begin{aligned} & R(Ly, \beta, \sigma^2) = \mathsf{E}\{W(Ly, \beta, \sigma^2)\} \\ = \ & \mathsf{E}\{w^2(y - XLy)'V^{-1}(y - XLy) + (1-w)^2(Ly - \beta)'S(Ly - \beta) \\ & + 2w(1-w)(XLy - y)'V^{-1}X(Ly - \beta)\} \\ = \ & \sigma^2[w^2\mathrm{tr}(I_n - XL)'V^{-1}(I_n - XL)V + (1-w)^2\mathrm{tr}L'SLV \\ & + 2w(1-w)\mathrm{tr}(XL - I_n)'V^{-1}XLV] \\ & + w^2\beta'X'(I_n - XL)'V^{-1}(I_n - XL)X\beta + (1-w)^2\beta'(LX - I_p)'S(LX - I_p)\beta \\ & + 2w(1-w)\beta'X'(XL - I_n)'V^{-1}X(LX - I_p)\beta, \end{aligned} \tag{3.3}$$

where $S$ is a positive definite matrix. Since $LX = I_p$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, then we obtain

$$\begin{aligned} & R(Ly,\beta,\sigma^2) = \mathsf{E}\{W(Ly,\beta,\sigma^2)\} \\ = \;& \sigma^2[w^2\mathrm{tr}(I_n - XL)'V^{-1}(I_n - XL)V + (1-w)^2\mathrm{tr}L'SLV \\ & + 2w(1-w)\mathrm{tr}(XL - I_n)'V^{-1}XLV] \\ = \;& \sigma^2[w^2\mathrm{tr}(V^{-1} - 2L'X'V^{-1} + L'X'V^{-1}XL)V + (1-w)^2\mathrm{tr}L'SLV \\ & + 2w(1-w)\mathrm{tr}(L'X'V^{-1}XLV - V^{-1}XLV)] \\ = \;& w\sigma^2(nw - 2p) + \sigma^2\mathrm{tr}[L'(w(2-w)X'V^{-1}X + (1-w)^2S)LV]. \end{aligned} \tag{3.4}$$

Now, we define $M = w(2-w)X'V^{-1}X + (1-w)^2S$. If we want $Ly$ to be the best linear unbiased estimator of $\beta$ that is equal to

$$\begin{cases} \min\ \mathrm{tr}(L'MLV) \\ \text{s.t. } LX = I_p \end{cases}. \tag{3.5}$$

Using the Lagrange method, put

$$F(L,\lambda) = \mathrm{tr}(L'MLV) - 2\mathrm{tr}[\lambda'(LX - I_p)], \tag{3.6}$$

where $\lambda$ is a $p \times p$ matrix of Lagrangian multipliers. By Lemma 3.1, we get

$$MLV - \lambda X' = 0, \tag{3.7}$$

$$LX - I_p = 0. \tag{3.8}$$

From Equation (3.7) and $M = w(2-w)X'V^{-1}X + (1-w)^2S > 0$, we have $L = M^{-1}\lambda X'V^{-1}$, then substitute it into (3.8), we obtain $\lambda = M(X'V^{-1}X)^{-1}$. Then substitute it into (3.7), we get

$$L = (X'V^{-1}X)^{-1}X'V^{-1}. \tag{3.9}$$

Next, we prove that $(X'V^{-1}X)^{-1}X'V^{-1}y$ obtain the minimum risk in $\Im_1$.

Let $\widetilde{L}y$ be an any estimator of $\beta$ in $\Im_1$. By $LX = I_p$, we get $\widetilde{L} = (X'V^{-1}X)^{-1}X'V^{-1} + \mu N$, where $\mu$ is an any $p \times n$ matrix and $N = (I - X(X'V^{-1}X)^{-1}X'V^{-1})$, thus the risk function of $Ly$ is

$$\begin{aligned} & R(\widetilde{L}y,\beta,\sigma^2) \\ = \;& w\sigma^2(nw-2p) + \sigma^2\mathrm{tr}\{((X'V^{-1}X)^{-1}X'V^{-1} + \mu N)'M((X'V^{-1}X)^{-1}X'V^{-1} + \mu N)V\} \\ = \;& w\sigma^2(nw-2p) + \sigma^2\mathrm{tr}\{V^{-1}X(X'V^{-1}X)^{-1}M(X'V^{-1}X)^{-1}X\} \\ & + 2\sigma^2\mathrm{tr}\{X(X'V^{-1}X)^{-1}M(\mu N)V\} + \sigma^2\mathrm{tr}(\mu N)'M(\mu N)V \\ = \;& R((X'V^{-1}X)^{-1}X'V^{-1}y,\beta,\sigma^2) + \sigma^2\mathrm{tr}(\mu NV^{1/2})'M(\mu NV^{1/2}). \end{aligned} \tag{3.10}$$

Since $M>0$, then we have $R(\widetilde{L}y,\beta,\sigma^2)\geq R((X'V^{-1}X)^{-1}X'V^{-1}y,\beta,\sigma^2)$ and the equality holds if and only if $L=(X'V^{-1}X)^{-1}X'V^{-1}$. □

**Remark 1** Under the weighted balanced loss function, the best linear unbiased estimator of $\beta$ in $\Im_1$ is $\widehat{\beta}=(X'V^{-1}X)^{-1}X'V^{-1}y$, which is same as the estimator got by Zhu et al. (2010) under the balanced loss function.

**Remark 2** Under the weighted balanced loss function, the best linear unbiased estimator of $\beta$ in $\Im_1$ is $\widehat{\beta}=(X'V^{-1}X)^{-1}X'V^{-1}y$. By Wang and Yang (1995), the best linear unbiased estimator is superior over the ordinary least squares estimator $\widehat{\beta}_{\text{OLS}}=(X'X)^{-1}X'y$ under the Pitman's closeness criterion.

### 3.2 Relative Efficiencies of the Best Linear Unbiased Estimator

In Subsection 3.1, we got the best linear unbiased estimator $\widehat{\beta}=(X'V^{-1}X)^{-1}X'V^{-1}y$ under the weighted balanced loss function in the linear regression model. However, as we all know, the covariance matrix is usually unknown, at this time, we use the ordinary least squares estimator $\widehat{\beta}_{\text{OLS}}=(X'X)^{-1}X'y$ to replace the best linear unbiased estimator. We know this will lead to some loss. There are many papers have discussed this loss, such as: Rao (1985), Yang and Wang (2009), Yang and Wu (2011), Liu (2000), Liu et al. (2009), Wang and Yang (2012). In this subsection, we define two relative efficiencies to measure the loss under the weighted balanced loss risk function.

Now we define two relative efficiencies:

$$e_1(\widehat{\beta}|\widehat{\beta}_{\text{OLS}})=R(\widehat{\beta}_{\text{OLS}},\beta,\sigma^2)-R(\widehat{\beta},\beta,\sigma^2) \tag{3.11}$$

and

$$e_2(\widehat{\beta}|\widehat{\beta}_{\text{OLS}})=\frac{R(\widehat{\beta},\beta,\sigma^2)}{R(\widehat{\beta}_{\text{OLS}},\beta,\sigma^2)}, \tag{3.12}$$

where $R(\widetilde{\beta},\beta,\sigma^2)$ is defined in (2.3), $\widehat{\beta}=(X'V^{-1}X)^{-1}X'V^{-1}y$ and $\widehat{\beta}_{\text{OLS}}=(X'X)^{-1}X'y$.

Based on the definition of weighted balanced loss risk function, we obtain

$$\begin{aligned}
&R(\widehat{\beta}_{\text{OLS}},\beta,\sigma^2)=\mathsf{E}\{W((X'X)^{-1}X'y,\beta,\sigma^2)\}\\
=\;&\mathsf{E}\{w^2(y-X(X'X)^{-1}X'y)'V^{-1}(y-X(X'X)^{-1}X'y)\\
&+(1-w)^2((X'X)^{-1}X'y-\beta)'S((X'X)^{-1}X'y-\beta)\\
&+2w(1-w)(X(X'X)^{-1}X'y-y)'V^{-1}X((X'X)^{-1}X'y-\beta)\}
\end{aligned}$$

$$\begin{aligned}
&= \sigma^2[w^2\mathrm{tr}(V^{-1}-2X(X'X)^{-1}X'V^{-1}+X(X'X)^{-1}X'V^{-1}X(X'X)^{-1}X')V \\
&\quad +(1-w)^2\mathrm{tr}X(X'X)^{-1}X'V^{-1}X(X'X)^{-1}X'V \\
&\quad +2w(1-w)\mathrm{tr}(X(X'X)^{-1}X'V^{-1}X(X'X)^{-1}X'V-V^{-1}X(X'X)^{-1}X'V)] \\
&= w\sigma^2(nw-2p)+\sigma^2\mathrm{tr}M(X'X)^{-1}X'VX(X'X)^{-1} \qquad (3.13)
\end{aligned}$$

and

$$\begin{aligned}
&R(\widehat{\beta},\beta,\sigma^2)=\mathsf{E}\{W((X'V^{-1}X)^{-1}X'V^{-1}y,\beta,\sigma^2)\} \\
&= \mathsf{E}\{w^2(y-X(X'V^{-1}X)^{-1}X'V^{-1}y)'V^{-1}(y-X(X'V^{-1}X)^{-1}X'V^{-1}y) \\
&\quad +(1-w)^2((X'V^{-1}X)^{-1}X'V^{-1}y-\beta)'S((X'V^{-1}X)^{-1}X'V^{-1}y-\beta) \\
&\quad +2w(1-w)(X(X'V^{-1}X)^{-1}X'V^{-1}y-y)'V^{-1}X((X'V^{-1}X)^{-1}X'V^{-1}y-\beta)\} \\
&= w\sigma^2(nw-2p)+\sigma^2\mathrm{tr}M(X'V^{-1}X)^{-1}, \qquad (3.14)
\end{aligned}$$

where $M=w(2-w)X'V^{-1}X+(1-w)^2S$.

Now, we give the bounds of the two relative efficiencies.

**Theorem 3.2** For the model (2.1), when $V$ is a positive definite matrix, let $V=Q'\Lambda Q$, $Q$ is an $n\times n$ orthogonal matrix, $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$, and $\lambda_1\ge\cdots\ge\lambda_n>0$. $c_1\ge\cdots\ge c_p>0=c_{p+1}=\cdots=c_n$ is the eigenvalues of $QXM^{-1}X'Q'$, then we have

$$e_1(\widehat{\beta}|\widehat{\beta}_{\mathrm{OLS}})\le\frac{\sigma^2\sum\limits_{i=1}^{m}(\sqrt{\lambda_i}-\sqrt{\lambda_{n-i+1}})^2}{c_p},$$

where $\mathrm{rank}(X)=p$ and $n>p$, $m=\min(p,n-p)$.

**Proof** Denote $QX=\gamma$, then we have $\mathrm{rank}(\gamma)=p$ and

$$\begin{aligned}
e_1(\widehat{\beta}|\widehat{\beta}_{\mathrm{OLS}}) &= R(\widehat{\beta}_{\mathrm{OLS}},\beta,\sigma^2)-R(\widehat{\beta},\beta,\sigma^2) \\
&= \{w\sigma^2(nw-2p)+\sigma^2\mathrm{tr}M(X'X)^{-1}X'VX(X'X)^{-1}\} \\
&\quad -\{w\sigma^2(nw-2p)+\sigma^2\mathrm{tr}M(X'V^{-1}X)^{-1}\} \\
&= \sigma^2\mathrm{tr}M(X'X)^{-1}X'VX(X'X)^{-1}-\sigma^2\mathrm{tr}M(X'V^{-1}X)^{-1} \\
&= \sigma^2\mathrm{tr}M(\gamma'\gamma)^{-1}\gamma'\Lambda\gamma(\gamma'\gamma)^{-1}-\sigma^2\mathrm{tr}M(\gamma'\Lambda^{-1}\gamma)^{-1} \\
&= \sigma^2\mathrm{tr}[(\rho'\rho)^{-1}\rho'\Lambda\rho(\rho'\rho)^{-1}-\mathrm{tr}(\rho'\Lambda^{-1}\rho)^{-1}], \qquad (3.15)
\end{aligned}$$

where $\rho=\gamma M^{-1/2}$, $\mathrm{rank}(\rho)=p$. Then do singular value decomposition for $\rho$, we obtain $\rho=Q_1\Gamma^{1/2}Q_2'$, where $Q_1$ is an $n\times p$ orthogonal matrix and $Q_2$ is a $p\times p$ orthogonal matrix. $\Gamma=\mathrm{diag}(c_1,\ldots,c_p)$, then by Lemma 3.3 and Equation (3.15), we obtain

$$\begin{aligned} e_1(\widehat{\beta}|\widehat{\beta}_{\mathrm{OLS}}) &= \sigma^2\mathrm{tr}\Gamma^{-1}[Q_1'\Lambda Q_1-(Q_1'\Lambda^{-1}Q_1)^{-1}] \\ &\le \frac{\sigma^2}{c_p}\mathrm{tr}[Q_1'\Lambda Q_1-(Q_1'\Lambda^{-1}Q_1)^{-1}] \\ &\le \frac{\sigma^2\sum\limits_{i=1}^{m}(\sqrt{\lambda_i}-\sqrt{\lambda_{n-i+1}})^2}{c_p}. \end{aligned} \tag{3.16}$$

The proof of Theorem 3.2 is completed. □

Now, we present another theorem of this paper.

**Theorem 3.3** For the model (2.1), when $V$ is a positive definite matrix, let $V=Q'\Lambda Q$, $Q$ is an $n\times n$ orthogonal matrix, $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$, where $\lambda_1\ge\cdots\ge\lambda_n>0$ is the ordered eigenvalues of $V$. $c_1\ge\cdots\ge c_p>0=c_{p+1}=\cdots=c_n$ is the eigenvalues of $QXM^{-1}X'Q'$. If $wn-2p=0$, then we have

$$e_2(\widehat{\beta}|\widehat{\beta}_{\mathrm{OLS}})\ge\frac{c_p}{c_1}\Big[2\sum_{i=1}^{m}(\sqrt{\lambda_i\lambda_{n-i+1}})\Big/\sum_{i=1}^{m}(\lambda_i+\lambda_{n-i+1})\Big]^2,$$

where $\mathrm{rank}(X)=p$ and $n>p$, $m=\min(p,n-p)$.

**Proof** By the proof of Theorem 3.2 and Lemma 3.2, we have

$$\begin{aligned} e_2(\widehat{\beta}|\widehat{\beta}_{\mathrm{OLS}}) &= \frac{R(\widehat{\beta},\beta,\sigma^2)}{R(\widehat{\beta}_{\mathrm{OLS}},\beta,\sigma^2)} \\ &= \frac{\mathrm{tr}M(X'V^{-1}X)^{-1}}{\mathrm{tr}M(X'X)^{-1}X'VX(X'X)^{-1}} \\ &= \frac{\mathrm{tr}(\rho'\Lambda^{-1}\rho)^{-1}}{\mathrm{tr}(\rho'\rho)^{-1}\rho'\Lambda\rho(\rho'\rho)^{-1}} \\ &= \frac{\mathrm{tr}\Gamma^{-1}(Q_1'\Lambda^{-1}Q_1)^{-1}}{\mathrm{tr}\Gamma^{-1}Q_1'\Lambda Q_1} \\ &\ge \frac{c_p}{c_1}\Big[2\sum_{i=1}^{m}(\sqrt{\lambda_i\lambda_{n-i+1}})\Big/\sum_{i=1}^{m}(\lambda_i+\lambda_{n-i+1})\Big]^2. \end{aligned} \tag{3.17}$$

The proof of Theorem 3.3 is completed. □

**Theorem 3.4** For the model (2.1), when $V$ is a positive definite matrix, let $V=Q'\Lambda Q$, $Q$ is an $n\times n$ orthogonal matrix, $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$, where $\lambda_1\ge\cdots\ge\lambda_n>0$ is the eigenvalues of $V$, $c_1\ge\cdots\ge c_p>0=c_{p+1}=\cdots=c_n$ is the ordered eigenvalues of $QXM^{-1}X'Q'$. If $wn-2p\ne0$, then

$$e_2(\widehat{\beta}|\widehat{\beta}_{\mathrm{OLS}})\ge 1-\frac{\sigma^2\sum\limits_{i=1}^{m}(\sqrt{\lambda_i}-\sqrt{\lambda_{n-i+1}})^2}{w^2(n-p)c_p},$$

where $\mathrm{rank}(X)=p$ and $n>p$, $m=\min(p,n-p)$.

**Proof** By the proof of Theorem 3.2 we have

$$
\begin{aligned}
e_2(\widehat{\beta}|\widehat{\beta}_{\rm OLS}) &= \frac{R(\widehat{\beta},\beta,\sigma^2)}{R(\widehat{\beta}_{\rm OLS},\beta,\sigma^2)} \\
&= \frac{w(nw-2p)+{\rm tr}M(X'V^{-1}X)^{-1}}{w(nw-2p)+{\rm tr}M(X'X)^{-1}X'VX(X'X)^{-1}} \\
&= 1-\frac{{\rm tr}MX(X'X)^{-1}X'VX(X'X)^{-1}-{\rm tr}M(X'V^{-1}X)^{-1}}{w(nw-2p)+{\rm tr}M(X'X)^{-1}X'VX(X'X)^{-1}} \\
&\ge 1-\frac{\Big[\sigma^2\sum\limits_{i=1}^{m}(\sqrt{\lambda_i}-\sqrt{\lambda_{n-i+1}})^2\Big]/c_p}{w(nw-2p)+{\rm tr}M(X'V^{-1}X)^{-1}} \\
&= 1-\frac{\sigma^2\sum\limits_{i=1}^{m}(\sqrt{\lambda_i}-\sqrt{\lambda_{n-i+1}})^2}{(w(nw-2p)+w(2-w)p+(1-w)^2{\rm tr}(X'V^{-1}X)^{-1}S)c_p} \\
&\ge 1-\frac{\sigma^2\sum\limits_{i=1}^{m}(\sqrt{\lambda_i}-\sqrt{\lambda_{n-i+1}})^2}{w^2(n-p)c_p}. \qquad (3.18)
\end{aligned}
$$

The proof of Theorem 3.4 is completed. □

In this section, we obtain the best linear unbiased estimator when $V$ is a positive definite matrix, and we also discuss the relative efficiencies for the best linear unbiased estimator with the ordinary least squares estimator.

## §4. Conclusions

In this paper, we discussed the best linear unbiased estimator in linear model under the weighted balanced loss function. The relative efficiency of the best linear unbiased estimator relative to the ordinary least squares estimator is also discussed.

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# 加权平衡损失函数下回归系数的最佳线性无偏估计

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这篇文章我们研究了回归系数的最佳线性无偏估计. 在加权平衡损失函数下, 我们得到了回归系数的最佳线性无偏估计. 同时提出了度量最佳线性无偏估计和最小二乘估计的相对效率. 并且我们给出了它们的上下界.

**关键词**: 加权平衡损失, 线性估计, 最佳线性无偏估计, 相对效率.

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