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A Note on the Borel-Cantelli Lemma for Capacity *

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Abstract

In this note, we prove the Borel-Cantelli lemma for capacity without pairwise independent assumption. The best lower bound about union for capacity is obtained. Classical Borel-Cantelli lemma is extended to the case of capacity.

Keywords: Capacity, Borel-Cantelli lemma, sublinear expectation, pairwise independence.

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§1. Introduction

The celebrated Borel-Cantelli lemma states that if $A_1, A_2, \ldots$ is a sequence of events on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and if $\sum\limits_{i=1}^{\infty} \mathsf{P}(A_i) < \infty$, then $\mathsf{P}\Big(\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{i=n}^{\infty} A_i\Big) = 0$; if $A_1, A_2, \ldots$ is a sequence of independent events and if $\sum\limits_{i=1}^{\infty} \mathsf{P}(A_i) = \infty$, then $\mathsf{P}\Big(\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{i=n}^{\infty} A_i\Big) = 1$. This lemma plays an important role in proving all theorems of the strong type in probability theory. But since the assumption of independence in the second part, its application is limited. Many investigations tried to replace the independence condition, for example, Chung and Erdös (1952), Erdös and Rényi (1959), Kochen and Stone (1964), Bruss (1980), Petrov (2002), etc. Chen et al. (2013) obtained Borel-Cantelli lemma for capacity to prove strong laws of large numbers for capacities. Specifically, let $\{A_n, n \geq 1\}$ be a sequence of events in $\mathcal{F}$ and $(\overline{C}, \underline{C})$ be a pair of capacities generated by sublinear expectation $\mathsf{E}_{\rm sl}$.

(A) If $\sum\limits_{i=1}^{\infty} \overline{C}(A_i) < \infty$, then $\overline{C}\Big(\limsup\limits_{i\to\infty} A_i\Big) = 0$;

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(B) If $\{A_n, n \geq 1\}$ are pairwise independent with respect to $\underline{C}$, namely $\underline{C}\Big(\bigcap_{i=1}^{\infty} A_i^c\Big) = \prod_{i=1}^{\infty} \underline{C}(A_i^c)$ and if $\sum_{i=1}^{\infty} \overline{C}(A_i) = \infty$, then $\overline{C}\Big(\limsup_{i\to\infty} A_i\Big) = 1$.

Motivated by the work of Chung and Erdös (1952), we investigate the Borel-Cantelli lemma for capacity without the pairwise independent condition with respect to $\underline{C}$. In this note, we also give the best lower bound about union for capacity to prove our result.

§2. Preliminaries

In this section, we present some preliminaries in the theory of sublinear expectation and capacity. More details of this section can be found in Peng (2006, 2010) and Denis et al. (2011).

Definition 2.1 (see Peng, 2010) Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω. We assume that all constants are in $\mathcal{H}$ and that $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of our "random variables". A nonlinear expectation $\mathsf{E}_{\rm sl}$ on $\mathcal{H}$ is a functional $\mathsf{E}_{\rm sl}$: $\mathcal{H} \mapsto \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geq Y$ then $\mathsf{E}_{\rm sl}[X] \geq \mathsf{E}_{\rm sl}[Y]$.

(b) Constant preserving: $\mathsf{E}_{\rm sl}[c] = c$.

The triple $(\Omega, \mathcal{H}, \mathsf{E}_{\rm sl})$ is called a nonlinear expectation space (compare with a probability space $(\Omega, \mathcal{F}, \mathsf{P})$). We are mainly concerned with sublinear expectation where the expectation $\mathsf{E}_{\rm sl}$ satisfies also

(c) Sub-additivity: $\mathsf{E}_{\rm sl}[X + Y] \leq \mathsf{E}_{\rm sl}[X] + \mathsf{E}_{\rm sl}[Y]$.

(d) Positive homogeneity: $\mathsf{E}_{\rm sl}[\lambda X] = \lambda \mathsf{E}_{\rm sl}[X]$, $\forall\, \lambda \geq 0$.

If only (c) and (d) are satisfied, $\mathsf{E}_{\rm sl}$ is called a sublinear functional.

Hu and Peng (2009) showed that a sublinear expectation can be expressed as a supremum of linear expectation, namely there exists a family of probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that

$$\mathsf{E}_{\rm sl}[X] = \sup_{\mathsf{P}\in\mathbb{P}} \mathsf{E}_{\mathsf{P}}[X], \qquad X \in \mathcal{H}.$$

For this $\mathbb{P}$, we define

$$\overline{C}(A) := \sup_{\mathsf{P}\in\mathbb{P}} \mathsf{E}_{\mathsf{P}}[1_A]; \qquad \underline{C}(A) := \inf_{\mathsf{P}\in\mathbb{P}} \mathsf{E}_{\mathsf{P}}[1_A], \qquad \forall\, A \in \mathcal{F}.$$

Obviously,

$$\overline{C}(A) + \underline{C}(A^c) = 1,$$

where A^c is complement set of A.

Similarly, we define the minimum expectation as follows:

$$\mathsf{E}_{\min}[X] = \inf_{\mathsf{P}\in\mathbb{P}} \mathsf{E}_{\mathsf{P}}[X], \qquad X \in \mathcal{H}.$$

It is easy to check that $\overline{C}$ and $\underline{C}$ are two continuous capacities (see Choquet, 1954) in the sense that

Definition 2.2 A set function V: $\mathcal{F} \to [0,1]$ is called a continuous capacity if it satisfies

(1) $V(\emptyset) = 0$, $V(\Omega) = 1$.

(2) $V(A) \le V(B)$, whenever $A \subset B$ and $A, B \in \mathcal{F}$.

(3) $V(A_n) \uparrow (\downarrow) V(A)$, if $A_n \uparrow (\downarrow) A$, where $A_n, A \in \mathcal{F}$.

Borel-Cantelli lemma is still true for capacities $\overline{C}$ and $\underline{C}$ under some assumptions.

Lemma 2.1 Let $\{A_n, n \ge 1\}$ be a sequence of events in $\mathcal{F}$.

(1) If $\sum\limits_{n=1}^{\infty} \overline{C}(A_n) < \infty$, then $\overline{C}\Big(\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{i=n}^{\infty} A_i\Big) = 0$.

(2) If $\sum\limits_{n=1}^{\infty} \mathsf{P}(A_n) < \infty$ for some $\mathsf{P} \in \mathbb{P}$, then $\underline{C}\Big(\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{i=n}^{\infty} A_i\Big) = 0$.

(3) Suppose that $\{A_n, n \ge 1\}$ are pairwise independent with respect to $\overline{C}$, i.e.,

$$\overline{C}\Big(\bigcap_{i=1}^{\infty} A_i^c\Big) = \prod_{i=1}^{\infty} \overline{C}(A_i^c).$$

If $\sum\limits_{n=1}^{\infty} \underline{C}(A_n) = \infty$, then $\underline{C}\Big(\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{i=n}^{\infty} A_i\Big) = 1$.

(4) Suppose that $\{A_n, n \ge 1\}$ are pairwise independent with respect to $\underline{C}$, i.e.,

$$\underline{C}\Big(\bigcap_{i=1}^{\infty} A_i^c\Big) = \prod_{i=1}^{\infty} \underline{C}(A_i^c).$$

If $\sum\limits_{n=1}^{\infty} \overline{C}(A_n) = \infty$, then $\overline{C}\Big(\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{i=n}^{\infty} A_i\Big) = 1$.

The proof of Lemma 2.1 is similar to that of Chen et al. (2013), we omit it.

§3. Main Results

In this section, we shall present the Borel-Cantelli lemma for capacity without the pairwise independent assumption.

Theorem 3.1 Let $\{A_k\}$ be a sequence of events satisfying:

(I) $\sum\limits_{k=1}^{\infty} \overline{C}(A_k) = \infty$.

(II) For every pair of positive integers m, n with $n \geq m$ there exist $\lambda(m)$ and $M(n, m) > m$ such that for every $k \geq M$ we have

$$\overline{C}(A_k A_m^c \cdots A_n^c) > \lambda \overline{C}(A_k)\overline{C}(A_m^c \cdots A_n^c). \tag{3.1}$$

(III) There exist two absolute constants λ_1 and λ_2 with the following property: to each A_j there corresponds a set of events $A_{j_1}, \ldots, A_{j_s}$ belonging to $\{A_k\}$ such that

$$\sum_{i=1}^{s} \overline{C}(A_j A_{j_i}) < \lambda_1 \overline{C}(A_j) \tag{3.2}$$

and if $k > j$ but A_k is not among the A_{j_i} $(1 \leq i \leq s)$ then

$$\overline{C}(A_j A_k) < \lambda_2 \overline{C}(A_i)\overline{C}(A_k). \tag{3.3}$$

Then

$$\underline{C}\Big(\limsup_{k\to\infty} A_k\Big) = 1. \tag{3.4}$$

Before proceeding to the proof we first prove an important lemma, which is the best lower bound for union with respect to $\overline{C}$.

Lemma 3.1 (Lower bound for union) Let $\{B_k\}$, $k = 1, \ldots, n$, be an arbitrary sequence of events in $(\Omega, \mathcal{F}, \overline{C})$. If $\overline{C}\Big(\bigcup\limits_{k=1}^{n} B_k\Big) > 0$, then

$$\overline{C}\Big(\bigcup_{k=1}^{n} B_k\Big) \geq \frac{\Big(\sum\limits_{k=1}^{n} \underline{C}(B_k)\Big)^2}{\sum\limits_{k=1}^{n} \overline{C}(B_k) + 2\sum\limits_{1\leq j<k\leq n} \overline{C}(B_j B_k)}. \tag{3.5}$$

Proof Define random variables $X_k(\omega)$, $\omega \in \Omega$, as follows:

$$X_k(\omega) = \begin{cases} 1, & \text{if } \omega \in B_k, \\ 0, & \text{if } \omega \notin B_k. \end{cases} \tag{3.6}$$

We have the following inequality:

$$\begin{aligned} \mathsf{E}_{\rm sl}\Big[\Big(\sum_{k=1}^{n} X_k\Big)^2\Big] &= \mathsf{E}_{\rm sl}\Big[\sum_{k=1}^{n} X_k^2 + 2\sum_{1\leq j<k\leq n} X_j X_k\Big] \\ &\leq \sum_{k=1}^{n} \mathsf{E}_{\rm sl}[X_k^2] + 2\sum_{1\leq j<k\leq n} \mathsf{E}_{\rm sl}[X_j X_k]. \end{aligned}$$

Note that the definition of X_k, we have

$$2\sum_{1\leq j<k\leq n} \overline{C}(B_j B_k) \geq \mathsf{E}_{\rm sl}\Big[\Big(\sum_{k=1}^{n} X_k\Big)^2\Big] - \sum_{k=1}^{n} \mathsf{E}_{\rm sl}[X_k^2]. \tag{3.7}$$

Now by the Schwarz inequality we have

$$\begin{aligned}\Big[\mathsf{E}_{\rm sl}\Big[\sum_{k=1}^n X_k\Big]\Big]^2 &= \Big[\mathsf{E}_{\rm sl}\Big[\sum_{k=1}^n X_k I\Big(\sum_{k=1}^n X_k>0\Big)\Big]\Big]^2\\ &\le \overline{C}\Big(\sum_{k=1}^n X_k>0\Big)\mathsf{E}_{\rm sl}\Big[\Big(\sum_{k=1}^n X_k\Big)^2\Big].\end{aligned}$$

On the other hand, we have

$$\Big[\mathsf{E}_{\rm sl}\Big[\sum_{k=1}^n X_k\Big]\Big]^2 \ge \Big[\mathsf{E}_{\min}\Big[\sum_{k=1}^n X_k\Big]\Big]^2 \ge \Big[\sum_{k=1}^n \mathsf{E}_{\min}[X_k]\Big]^2,$$

thus

$$\Big[\sum_{k=1}^n \mathsf{E}_{\min}[X_k]\Big]^2 \le \overline{C}\Big(\sum_{k=1}^n X_k>0\Big)\mathsf{E}_{\rm sl}\Big[\Big(\sum_{k=1}^n X_k\Big)^2\Big]. \tag{3.8}$$

Since $\mathsf{E}_{\rm sl}[X_k]=\mathsf{E}_{\rm sl}[X_k^2]=\overline{C}(B_k)$, $\mathsf{E}_{\min}[X_k]=\underline{C}(B_k)$, $\overline{C}\Big(\sum_{k=1}^n X_k>0\Big)=\overline{C}\Big(\bigcup_{k=1}^n B_k\Big)$ by definition, (3.5) follows from (3.7) and (3.8). □

Proof of Theorem 3.1 Let

$$D_m=\bigcup_{k=m}^\infty A_k.$$

Since $\underline{C}\Big(\limsup_{k\to\infty} A_k\Big)=\lim_{m\to\infty}\underline{C}(D_m)$, it is sufficient to prove that $\underline{C}(D_m)=1$ for every m. Suppose that this is not true for a certain m; let $\underline{C}(D_m)=1-\alpha$, $\alpha>0$. Thus

$$\overline{C}\Big(\bigcap_{k=m}^\infty A_k^c\Big)=\alpha>0. \tag{3.9}$$

From (3.1) and (3.9), if $k>M(n)$, we have

$$\overline{C}(A_kA_m^c\cdots A_n^c)>\lambda\alpha\overline{C}(A_k). \tag{3.10}$$

Hence by (I) of Theorem 3.1, $\sum_{k=M(n)}^\infty \overline{C}(A_kA_m^c\cdots A_n^c)=\infty$. Therefore there exists an integer $M'(n)>M(n)$ such that

$$0<\epsilon_0<\sum_{k=M}^{M'}\underline{C}(A_kA_m^c\cdots A_n^c)<\sum_{k=M}^{M'}\overline{C}(A_kA_m^c\cdots A_n^c)\le\beta<\infty. \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\sum_{k=M}^{M'}\overline{C}(A_k)<\frac{\beta}{\lambda\alpha} \quad\text{and}\quad \sum_{M\le j<k\le M'}\overline{C}(A_j)\overline{C}(A_k)<\frac12\Big(\frac{\beta}{\lambda\alpha}\Big)^2. \tag{3.12}$$

From (3.2) and (3.3) of the Theorem 3.1 as well as (3.12), we have

$$\begin{aligned}\sum_{M\le j<k\le M'}\overline{C}(A_jA_k) &\le \lambda_1\sum_{k=M}^{M'}\overline{C}(A_k)+\lambda_2\sum_{M\le j<k\le M'}\overline{C}(A_j)\overline{C}(A_k)\\ &< \frac{\lambda_1\beta}{\lambda\alpha}+\frac{\lambda_2}{2}\Big(\frac{\beta}{\lambda\alpha}\Big)^2. \end{aligned}\tag{3.13}$$

Let $F_k = A_k A_m^c \cdots A_n^c$, $M \le k \le M'$. It is obvious that $F_k \subseteq A_k$, hence by (3.12), we obtain

$$\sum_{k=M}^{M'} \overline{C}(F_k) \le \sum_{k=M}^{M'} \overline{C}(A_k) < \frac{\beta}{\lambda\alpha}. \tag{3.14}$$

Applying Lemma 3.1 to $\{F_k\}$, $M \le k \le M'$, and using (3.11) and (3.14), we obtain

$$\begin{aligned} 2\sum_{M\le j<k\le M'} \overline{C}(A_jA_k) &\ge 2\sum_{M\le j<k\le M'} \overline{C}(F_jF_k) \\ &\ge \Big[\overline{C}\Big(\bigcup_{k=M}^{M'} F_k\Big)\Big]^{-1}\Big(\sum_{k=M}^{M'} \underline{C}(F_k)\Big)^2 - \sum_{k=M}^{M'} \overline{C}(F_k) \\ &> 1\cdot \epsilon_0^2 - \frac{\beta}{\lambda\alpha}, \end{aligned}$$

namely

$$\sum_{M\le j<k\le M'} \overline{C}(A_jA_k) > \frac{\epsilon_0^2}{2} - \frac{\beta}{2\lambda\alpha}. \tag{3.15}$$

But if we set $\epsilon_0 = 1$, $\beta/(\lambda\alpha) = 1/2$, $\lambda_1 = 1/8$, $\lambda_2 = 1/2$, combining (3.13) and (3.15), then

$$\frac{1}{4} = \frac{\epsilon_0^2}{2} - \frac{\beta}{2\lambda\alpha} < \sum_{M\le j<k\le M'} \overline{C}(A_jA_k) < \frac{\lambda_1\beta}{\lambda\alpha} + \frac{\lambda_2}{2}\Big(\frac{\beta}{\lambda\alpha}\Big)^2 = \frac{1}{8}. \tag{3.16}$$

We note (3.16) is contradictory. This contradiction proves that $\alpha = 0$. Hence $\underline{C}(D_m) = 1$, that is $\underline{C}\Big(\limsup_{k\to\infty} A_k\Big) = 1$. The proof is complete. □

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关于容度下Borel-Cantelli引理的一点注记

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这个注记中我们证明了在没有两两独立假设条件下对容度的Borel-Cantelli引理, 获得了容度对并事件的最优下界. 这些结果推广了经典的Borel-Cantelli引理.

关键词: 容度, Borel-Cantelli引理, 次线性期望, 两两独立.

学科分类号: O211.1, O211.9.