应用概率统计 第三十一卷
第二期 2015年4月

Chinese Journal of Applied Probability
and Statistics Vol.31 No.2 Apr. 2015

Precise Large Deviations for Compound Renewal Risk Model with Negative Dependence Claims *

Song Lixin Feng Jinghai Yuan Liangliang

(*School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024*)

Abstract

In this paper, we investigate the precise large deviations for a sum of claims in compound renewal risk model with negative dependence structure, in which we assume that $\{X_n, n \ge 1\}$ is a sequence of negative dependence rv's with distribution functions $\{F_n, n \ge 1\}$ and the average of right tails of distribution functions F_n is equivalent to some distribution function F with consistently varying tails. We try to build a platform for the classical large deviation theory and for the compound renewal risk model.

Keywords: Precise large deviations, negative dependence, random sums, consistently varying tails, compound renewal risk model.

AMS Subject Classification: 60F10, 60F05, 60G50.

§1. Introduction

Mainstream research on precise large deviation probabilities has been concentrated on the study of the asymptotics $\mathsf{P}(S(t) - \mathsf{E}S(t) > x) \sim \lambda(t)\overline{F}(x)$, which holds uniformly for all $x \ge \gamma\lambda(t)$ for every fixed $\gamma > 0$ as $t \to \infty$. Here $\{X_n, n \ge 1\}$ is a sequence of independent, identically distributed (i.i.d.) nonnegative heavy-tailed rv's with common distribution function F and finite expectation μ, independent of a process $\{N(t), t \ge 0\}$ driven by a sequence of nonnegative, integer-valued rv's. Assume that $\lambda(t) = \mathsf{E}N(t) < \infty$ for all $t \ge 0$ but $\lambda(t) \to \infty$, as $t \to \infty$. $S(t) = \sum\limits_{i=1}^{N(t)} X_i$, $t \ge 0$, denote randomly indexed sums (random sums). All limit relations, unless explicitly stated, are for $t \to \infty$ or consequently for $\lambda(t) \to \infty$.

We say X (or its df F) is heavy-tailed if it has no exponential moments. An important subclass of heavy-tailed distributions is $\mathcal{D}$, which consists of all distributions with dominated variation in the sense that the relation $\limsup\limits_{x\to\infty} \overline{F}(xy)/\overline{F}(x) < \infty$ holds for any $y \in (0,1)$

*The project was supported by the National Natural Sciences Foundation of China (11101061, 11371077 and 61175041).

Received May 6, 2014. Revised October 10, 2014.

doi: 10.3969/j.issn.1001-4268.2015.02.006

(or equivalently, for $y = 1/2$). Another slightly smaller subclass is $\mathcal{C}$, which consists of all distributions with consistent variation in the sense that $\lim\limits_{y\searrow 1}\liminf\limits_{x\to\infty}\overline{F}(xy)/\overline{F}(x) = 1$ or, equivalently, $\lim\limits_{y\nearrow 1}\limsup\limits_{x\to\infty}\overline{F}(xy)/\overline{F}(x) = 1$. For a distribution F, we know that if $F \in \mathcal{D}$, then for any $y > 0$, $\overline{F}(xy)$ and $\overline{F}(x)$ are of the same order as $x \to \infty$, that is

$$0 < \liminf_{x\to\infty}\overline{F}(xy)/\overline{F}(x) \le \limsup_{x\to\infty}\overline{F}(xy)/\overline{F}(x) < \infty.$$

See Cline and Samorodnitsky (1994) for more details. We denote this by $\overline{F}(xy) \asymp \overline{F}(x)$. Set

$$\gamma(y) := \liminf_{x\to\infty}\overline{F}(xy)/\overline{F}(x) \qquad \text{and} \qquad \gamma_{_F} := \inf\{-\log\gamma(y)/\log y : y > 1\};$$
$$\mu(y) := \limsup_{x\to\infty}\overline{F}(xy)/\overline{F}(x) \qquad \text{and} \qquad \mu_{_F} := \sup\{-\log\mu(y)/\log y : y > 1\}.$$

We call $\gamma_{_F}$ and $\mu_{_F}$ the upper and lower Matuszewska index of the distribution function F respectively. See Chapter 2.1 of Bingham et al. (1987) and Cline and Samorodnitsky (1994) for more details of the Matuszewska indices.

Strolling in past literature on precise large deviations, we find that most works were conducted only for independent rv's, though several dealing with non-identically distributed rv's. Here, we introduce some important dependence structures of the rv's. These dependence structures have been systematically investigated in the literature since they were introduced by Ghosh (1981) and Block et al. (1982).

Definition 1.1 We call rv's $\{X_k, k \ge 1\}$

(1) lower negatively dependent (LND) if for each n, $n \ge 1$ and all $x_1, \ldots, x_n$,

$$\mathsf{P}(X_1 \le x_1, \ldots, X_n \le x_n) \le \prod_{1\le k\le n}\mathsf{P}(X_k \le x_k); \tag{1.1}$$

(2) upper negatively dependent (UND) if for each n, $n \ge 1$ and all $x_1, \ldots, x_n$,

$$\mathsf{P}(X_1 > x_1, \ldots, X_n > x_n) \le \prod_{1\le k\le n}\mathsf{P}(X_k > x_k); \tag{1.2}$$

(3) negatively dependent (ND) if both (1.1) and (1.2) hold for each n, $n \ge 1$ and all $x_1, \ldots, x_n$.

Tang et al. (2001) studied the precise large deviations in the compound renewed model, and the model is as follows:

Definition 1.2 The compound renewal risk model

(a) the individual claim sizes $\{X_n, n \ge 1\}$ are i.i.d. nonnegative rv's with a common distribution function F and a finite mean $\mu = \mathsf{E}X_1$;

(b) the accident inter-arrival times $\{Y_n, n \geq 1\}$ are i.i.d. non-negative rv's with a finite mean $\mathsf{E}Y_1 = 1/\lambda$, independent of $\{X_n, n \geq 1\}$;

(c) the number of accidents in the interval $[0, t]$ is denoted by $\tau(t) = \sup\{n \geq 1 : T_n \leq t\}$, $t \geq 0$, where $T_n = \sum\limits_{i=1}^{n} Y_i$, $n \geq 1$, denote the arrival time of the nth accident;

(d) the number of individual claims caused by the nth accident is a non-negative, integer-valued rv Z_n, and $\{Z_n, n \geq 1\}$ constitutes a process of i.i.d. rv's with a common df G, independent of $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$;

(e) the total number of claims up to time t is given by $N'(t) = \sum\limits_{i=1}^{\tau(t)} Z_i$, $t \geq 0$;

(f) the total claim amount process $\{S'(t), t \geq 0\}$ is defined by $S'(t) = \sum\limits_{i=1}^{N'(t)} X_i$.

For more details in compound renewal risk model, Tang et al. (2001) proved the precise large deviations results, while Kaas and Tang (2005) proved again the precise large deviations results when the number of individual claims $\{Z_n, n \geq 1\}$ in Definition 1.2 are ND structure. Based on Definition 1.1 and Definition 1.2, we introduce the following more realistic model in the context of insurance.

Definition 1.3 The general compound renewal risk model is given by conditions (b)-(f) in Definition 1.2 and

(a$'$) the individual claim sizes $\{X_n, n \geq 1\}$ are ND non-negative rv's with distribution function $\{F_n, n \geq 1\}$ and a finite mean vector $\boldsymbol{\mu} = (\mathsf{E}X_1, \mathsf{E}X_2, \ldots, \mathsf{E}X_n, \ldots)$.

The goal of this paper is to study the precise large deviations in general compound renewal risk model. More precisely, we consider ND structures for the rv's $\{X_n, n \geq 1\}$ in the general compound renewal risk model, as the same time, we assume that the average of right tails of distribution functions F_n is equivalent to some distribution function F with consistently varying tails.

The rest of paper is organized as follows. In Section 2 we introduce some useful properties and lemmas in the paper. The main results are presented in Section 3. Finally the proofs of our results are given in Section 4.

§2. Preliminaries

The following properties of LND or UND rv's are direct consequences of Definition 1.1 and were mentioned by Block et al. (1982, p.769):

Property 2.1 For rv's $\{X_k, k \geq 1\}$ and real functions $\{f_k, k \geq 1\}$,

(1) if $\{X_k, k \geq 1\}$ are LND (UND) and $\{f_k, k \geq 1\}$ are all monotone increasing, then $\{f_k(X_k), k = 1, 2, \ldots\}$ are LND (UND);

(2) if $\{X_k, k \geq 1\}$ are LND (UND) and $\{f_k, k \geq 1\}$ are all monotone decreasing, then $\{f_k(X_k), k = 1, 2, \ldots\}$ are UND (LND);

(3) if $\{X_k, k \geq 1\}$ are ND and $\{f_k, k \geq 1\}$ are either all monotone increasing or all monotone decreasing, then $\{f_k(X_k), k = 1, 2, \ldots\}$ are ND;

(4) if $\{X_k, k \geq 1\}$ are nonnegative and UND, then for each $n = 1, 2, \ldots$,

$$\mathsf{E}\Big(\prod_{1 \leq k \leq n} X_k\Big) \leq \prod_{1 \leq k \leq n} \mathsf{E} X_k.$$

We restate following results that were obtained in the literature of the precise large deviations. We need the following lemmas to prove the main results behind.

Lemma 2.1 Let $\{X_k, k \geq 1\}$ be upper negative dependence (UND) rv's with distribution functions $\{F_k, k \geq 1\}$ and mean vector be $\mathbf{0}$, satisfying $\sup_{k \geq 1} \mathsf{E}(X_k^+)^r < \infty$ for some $r > 1$. Then for each fixed $\gamma > 0$ and $p > 0$, there exist positive numbers v and $C = C(v, \gamma)$ irrespective to x and n such that for all $x \geq \gamma n$ and $n = 1, 2, \ldots$,

$$\mathsf{P}\Big(\sum_{1 \leq k \leq n} X_k \geq x\Big) \leq \sum_{1 \leq k \leq n} \overline{F}_k(vx) + Cx^{-p}.$$

Remark 1 In fact, Lemma 2.1 is a modification of Lemma 2.3 of Tang (2006). We just need give some modifications as following:

(1) $n\overline{F}(vx)$ in Tang (2006) is replaced by $\sum_{k=1}^{n} \overline{F}_k(vx)$;

(2) $h' = (vx)^{-1} \log\big((v^{q-1}x^q)/(n\mathsf{E}(X_1^+))^q + 1\big)$ in Tang (2006) is replaced by

$$h = (vx)^{-1} \log\Big((v^{q-1}x^q)/\Big(\sum_{k=1}^{n} \mathsf{E}(X_k^+)\Big)^q + 1\Big);$$

(3) $C' = \sup_{x \geq 0} \exp\{v^{-1} + (v^{q-1}x^q\overline{F}(vx))/\mathsf{E}(X_1^+)^q\}\big((v^{q-1}\gamma)/\mathsf{E}(X_1^+)^q\big)^{-1/(2v)} < \infty$ in Tang (2006) is replaced by

$$C = \sup_{x \geq 0} \exp\Big\{v^{-1} + \Big(v^{q-1}x^q \sum_{1 \leq k \leq n} \overline{F}_k(vx)\Big)\Big/ \sum_{1 \leq k \leq n} \mathsf{E}(X_k^+)^q\Big\} \cdot \Big((v^{q-1}\gamma)\Big/ \max_{1 \leq k \leq n} \mathsf{E}(X_k^+)^q\Big)^{-1/(2v)} < \infty.$$

This lemma will be used in deriving the lower bound of the large-deviation probabilities in the proof of Theorem 3.1.

Lemma 2.2 and Lemma 2.3 can be found in Ng et al. (2004). These inequalities will play a key role in the proof of Theorem 3.1.

Lemma 2.2 For a distribution function $F \in \mathcal{D}$ with a finite expectation, $1 \leq \gamma_F < \infty$ and as $x \to \infty$, $x^{-p} = o(\overline{F}(x))$ for any $p > \gamma_F$.

Lemma 2.3 For a distribution function $F \in \mathcal{D}$ and every $\rho > \gamma_F$, there exist positive x_0 and B such that, for all $\theta \in (0,1]$ and all $x \geq \theta^{-1}x_0$, $\overline{F}(\theta x)/\overline{F}(x) \leq B\theta^{-\rho}$.

Lemma 2.4 and 2.5 are reformulations of Lemma 3.3 and 3.5 of Tang et al. (2001). We will need these two lemmas in the later part of this paper.

Lemma 2.4 Let $\{\zeta(t),\, t \geq 0\}$ be a stochastic process with a common expectation $\mathsf{E}\zeta(t) = 1$. If for any fixed $\delta > 0$, $\mathsf{E}\zeta(t)I_{\{\zeta(t)>1+\delta\}} = o(1)$, then $\zeta(t) \xrightarrow{\mathsf{P}} 1$.

Lemma 2.5 Suppose $\{Y_n, n \geq 1\}$ is a sequence of i.i.d. non-negative rv's with a common mean $\mathsf{E}Y_1 = 1/\lambda$, constituting a renewal counting process $\{N(t), t \geq 0\}$. We have for any positive constants δ and m, $\sum\limits_{k>(1+\delta)\lambda(t)} k^m \mathsf{P}(N(t) = k) = o(1)$.

Next, we give two useful lemmas, which are the popularization and application of Theorem 1 in Fuk and Nagaev (1971) and Lemma 2.3 in Skučaitė (2004) respectively. These inequalities will play a key role in the proof of Theorem 3.1 and Theorem 3.2.

Lemma 2.6 Let $\{X_n, n \geq 1\}$ be UND rv's with distribution functions $\{F_n, n \geq 1\}$, $x > 0$ be any positive constant, and let $(y_1, \ldots, y_n)$ be any set of positive numbers. Then for $y > \max\limits_{1\leq k\leq n}\{y_k\}$ and $0 < t \leq 1$, we have $\mathsf{P}(S_n > x) \leq \sum\limits_{k=1}^{n} \mathsf{P}(X_k \geq y_k) + P_1$, where

$$P_1 = \exp\Big\{xy^{-1} - xy^{-1}\ln\Big((xy^{t-1})/\sum_{k=1}^{n}\int_0^{y_k} u^t \mathrm{d}F_k(u) + 1\Big)\Big\}.$$

Lemma 2.7 Let $\{X_n, n \geq 1\}$ be UND or LND rv's with distribution functions $\{F_n, n \geq 1\}$ and finite expectations $\{\mu_n, n \geq 1\}$, and let $\{N(t), t \geq 0\}$ be a stochastic process generated by non-negative integer-valued rv's independent of the sequence $\{X_n, n \geq 1\}$. Assume that

(1) the expectations $\{\mu_n, n \geq 1\}$ satisfy for some $\overline{\mu} < \infty$, $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \mu_k/n = \overline{\mu}$;

(2) the stochastic process $N(t)$ satisfies that $N(t)/\lambda(t) \xrightarrow{\mathsf{P}} 1$, as $t \to \infty$.

Then $\mathsf{E}S(t) \sim \overline{\mu}\lambda(t)$, i.e. $\mathsf{E}S(t) = \overline{\mu}\lambda(t)(1 + o(1))$.

Remark 2 In fact, all the work that we need to do is just changing the independence property among the variables $\{X_n, n \geq 1\}$ which appears in Theorem 1 of Fuk and Nagaev (1971) and Lemma 2.3 of Skučaitė (2004) into negative dependence structure.

However, this kind of change of relationships will not effect the final results and proofs at all.

We also need the following result in the proof of Theorem 3.3.

Lemma 2.8 For a distribution function $F \in \mathcal{D}$ with a finite expectation and every $\mu < \mu_F$, there exist a positive number M such that,

$$\overline{F}(x) \le M x^{-\mu} \qquad \text{as } x \to \infty. \tag{2.1}$$

Proof Since $F \in \mathcal{D}$, we know that $\gamma_F \ge \mu_F \ge 0$. From Proposition 2.2.1 of Bingham et al. (1987) we know that, for any $\epsilon > 0$, there exist positive constants B' and x_0 such that the inequality

$$x^{-(\gamma_F+\epsilon)}/B' \le \overline{F}(xy)/\overline{F}(y) \le B' x^{-(\mu_F-\epsilon)} \tag{2.2}$$

holds uniformly for $x \ge 1$ and $y \ge x_0$. Hence, fixing the variable y in (2.2) leads to the result (2.1). □

§3. Main Results

In this section, using Lemma 2.1−2.8, we give the main results, which indicate that the precise large deviations are insensitive to the assumed ND structure. The following theorem is a result about precise large deviations of nonrandom sums:

Theorem 3.1 Let $\{X_n, n \ge 1\}$ be a sequence of nonnegative ND rv's with distribution functions $\{F_n, n \ge 1\}$ and finite expectations $\{\mu_n, n \ge 1\}$; X be a nonnegative random variable with a distribution function $F \in \mathcal{C}$ and a finite expectation $\overline{\mu}$. Assume that

(1) the distribution functions $\{F_n, n \ge 1\}$ and F satisfy Assumption (A):

$$\lim_{n\to\infty} \sum_{1\le k\le n} \overline{F}_k(x)/(n\overline{F}(x)) = 1$$

holds uniformly for $x \ge X_0$, for some $X_0 > 0$;

(2) the expectations $\overline{\mu}$ and $\{\mu_n, n \ge 1\}$ satisfy Assumption (B):

$$\lim_{n\to\infty} \sum_{1\le k\le n} \mu_k/n = \overline{\mu} < \infty, \qquad \text{and} \qquad \sup_{n\ge 1} \mu_n < \infty.$$

Then, for any fixed $\gamma > 0$,

$$\mathsf{P}(S_n - \mathsf{E}(S_n) > x) \sim n\overline{F}(x) \tag{3.1}$$

holds uniformly for $x \ge \gamma n$, where $S_n = \sum_{1\le k\le n} X_k$.

Based on Theorem 3.1, we have the asymptotic results for random sums as follows:

Theorem 3.2 Let $\{X_n, n \geq 1\}$ be a sequence of nonnegative ND rv's with distribution functions $\{F_n, n \geq 1\}$ and finite expectations $\{\mu_n, n \geq 1\}$; X be a nonnegative random variable with a distribution function $F \in \mathcal{C}$ and a finite expectation $\overline{\mu}$. Assume that

(1) the distribution functions $\{F_n, n \geq 1\}$ and F satisfy Assumption (A) in Theorem 3.1;

(2) the expectations $\overline{\mu}$ and $\{\mu_n, n \geq 1\}$ satisfy Assumption (B) in Theorem 3.1;

(3) $\{N(t), t \geq 0\}$ is a non-negative and integer-valued process independent of $\{X_n, n \geq 1\}$, and $\mathsf{E}N(t) = \lambda(t) \to \infty$ as $t \to \infty$. For any fixed $\delta > 0$, and some $p > \gamma_F$, $\{N(t), t \geq 0\}$ satisfies Assumption I: $\mathsf{E}N^p(t) I_{\{N(t)>(1+\delta)\lambda(t)\}} = O(\lambda(t))$.

Then, for any fixed $\gamma > 0$,

$$\mathsf{P}(S(t) - \mathsf{E}(S(t)) > x) \sim \lambda(t)\overline{F}(x) \tag{3.2}$$

holds uniformly for $x \geq \gamma\lambda(t)$, as $t \to \infty$.

Using Theorem 3.2, we obtain the main result in this paper.

Theorem 3.3 In the general compound renewal risk model in Definition 1.3, let $G \in \mathcal{C}$ and $\gamma_F < \mu_G$, Assumption (A) and (B) be satisfied. Then the compound renewal process $\{N'(t), t \geq 0\}$ satisfies Assumption I in Theorem 3.2 and for any fixed $\gamma > \mathsf{E}Z$, $\mathsf{P}(S'(t) - \mathsf{E}(S'(t)) > x) \sim \lambda'(t)\overline{F}(x)$ holds uniformly for $x \geq \gamma\lambda'(t)$, as $t \to \infty$, where $\mathsf{E}N'(t) = \lambda'(t)$.

§4. Proofs

4.1 The Proof of Theorem 3.1

Proof We modify the proof of Theorem 3.1 in Ng et al. (2004). At first, we estimate the lower bound. For any $\lambda > 1$,

$$\begin{aligned}
&\mathsf{P}\Big(S_n - \sum_{1\leq k\leq n} \mu_k > x\Big) \geq \mathsf{P}\Big(S_n - \sum_{1\leq k\leq n} \mu_k > x, \max_{1\leq j\leq n} X_j > \lambda x\Big) \\
\geq& \sum_{1\leq j\leq n} \mathsf{P}\Big(S_n - \sum_{1\leq k\leq n} \mu_k > x, X_j > \lambda x\Big) \\
&- \sum_{1\leq j<l\leq n} \mathsf{P}\Big(S_n - \sum_{1\leq k\leq n} \mu_k > x, X_j > \lambda x, X_l > \lambda x\Big)
\end{aligned}$$

$$\geq \sum_{1\leq j\leq n} \mathsf{P}\Big(S_n - X_j - \sum_{1\leq k\leq n} \mu_k > (1-\lambda)x, X_j > \lambda x\Big) - \Big(\sum_{1\leq j\leq n} \overline{F}_j(\lambda x)\Big)^2$$

$$\geq \sum_{1\leq j\leq n} \overline{F}_j(\lambda x)\Big(1 - \sum_{k=j}^{n} \overline{F}_j(\lambda x)\Big) - \sum_{1\leq j\leq n} \mathsf{P}\Big(S_n^{(j)} - \sum_{1\leq k\leq n} \mu_k \leq (1-\lambda)x\Big), \tag{4.1}$$

where $S_n^{(j)} = \sum_{1\leq k\neq j\leq n} X_k$. Here we use the definition of ND rv's (see (2) and (3) in Definition 1.1) in the third inequality, and use an elementary inequality $\mathsf{P}(AB) \geq \mathsf{P}(B) - \mathsf{P}(A^c)$ for all events A and B in the last inequality. For any $\delta_1 > 0$, using Assumption (A), for all large n, $x \geq X_0$, we have

$$(1-\delta_1)n\overline{F}(x) \leq \sum_{1\leq j\leq n} \overline{F}_j(x) \leq (1+\delta_1)n\overline{F}(x). \tag{4.2}$$

We estimate the second term in (4.1), for all large x, $x \geq X_0$, we have

$$\mathsf{P}\Big(S_n^{(j)} - \sum_{1\leq k\leq n} \mu_k \leq (1-\lambda)x\Big) \leq \mathsf{P}\Big(\sum_{1\leq k\neq j\leq n} (\mu_k - X_k) \geq (\lambda-1)x/2\Big).$$

By (2) and (3) in Property 2.1, the rv's $\{\mu_k - X_k, k \geq 1\}$ are UND. Then for arbitarily fixed $\gamma > 0$ and $p > \gamma_F$, by Lemma 2.1 there exist positive constants υ_0 and C irrespective to x and n such that

$$\begin{aligned} &\mathsf{P}\Big(\sum_{1\leq k\neq j\leq n} (\mu_k - X_k) \geq (\lambda-1)x/2\Big) \\ \leq& \sum_{1\leq k\neq j\leq n} \mathsf{P}(\mu_k - X_k \geq (\lambda-1)x/(2\upsilon_0)) + Cx^{-p} \\ \leq& \sum_{1\leq k\leq n} F_k(-(\lambda-1)x/(4\upsilon_0)) + Cx^{-p} \end{aligned}$$

holds for all $x \geq \gamma n$ and for all large n. Using the fact that $\{X_n, n \geq 1\}$ be non-negative rv's and Lemma 2.2, we know that

$$\mathsf{P}\Big(S_n^{(j)} - \sum_{1\leq k\leq n} \mu_k \leq (1-\lambda)x\Big) = o(\overline{F}(\lambda x)). \tag{4.3}$$

Plugging (4.2) and (4.3) into (4.1) yields that

$$\mathsf{P}\Big(S_n - \sum_{1\leq k\leq n} \mu_k > x\Big) \geq (1-\delta_1)n\overline{F}(\lambda x)(1 - (1+\delta_1)n\overline{F}(\lambda x)) - \delta_1(n\overline{F}(\lambda x)).$$

Let $\delta_1 \downarrow 0$, we have

$$\liminf_{n\to\infty} \inf_{x\geq\gamma n} \mathsf{P}\Big(S_n - \sum_{1\leq k\leq n} \mu_k > x\Big)/(n\overline{F}(\lambda x)) \geq \liminf_{n\to\infty} \inf_{x\geq\gamma n} (1 - n\overline{F}(\lambda x)) = 1. \tag{4.4}$$

Here, we use $n\overline{F}(\lambda x)\to 0$, as $n\to\infty$, uniformly for $x\geq\gamma n$ in the last step. Hence, using (4.4) we have

$$\begin{aligned}&\liminf_{n\to\infty}\inf_{x\geq\gamma n}\mathsf{P}\Big(S_n-\sum_{1\leq k\leq n}\mu_k>x\Big)/(n\overline{F}(x))\\ &\geq\Big(\liminf_{n\to\infty}\inf_{x\geq\gamma n}(1-n\overline{F}(\lambda x))\Big)\liminf_{x\to\infty}\overline{F}(\lambda x)/\overline{F}(x).\end{aligned}$$

Since $F\in\mathcal{C}$ and $\lambda>1$ is arbitrary, we can conclude that

$$\liminf_{n\to\infty}\inf_{x\geq\gamma n}\mathsf{P}\Big(S_n-\sum_{1\leq k\leq n}\mu_k>x\Big)/(n\overline{F}(x))\geq\lim_{\lambda\searrow 1}\liminf_{x\to\infty}\overline{F}(\lambda x)/\overline{F}(x)=1. \tag{4.5}$$

Now we start to estimate the upper bound. For any $\theta\in(0,1)$, we define

$$\widetilde{X}_k:=X_kI_{(X_k\leq\theta x)}\quad\text{for}\quad k\geq 1,\qquad \widetilde{S}_n:=\sum_{1\leq k\leq n}\widetilde{X}_k\quad\text{and}\quad \widetilde{x}:=x+\sum_{1\leq k\leq n}\mu_k.$$

By a standard truncation argument, we can show that

$$\begin{aligned}&\mathsf{P}\Big(S_n-\sum_{1\leq k\leq n}\mu_k>x\Big)\\ &\leq\mathsf{P}\Big(\max_{1\leq k\leq n}X_k>\theta x\Big)+\mathsf{P}\Big(\max_{1\leq k\leq n}X_k\leq\theta x,S_n-\sum_{1\leq k\leq n}\mu_k>x\Big)\\ &\leq\sum_{1\leq k\leq n}\mathsf{P}(X_k>\theta x)+\mathsf{P}(\widetilde{S}_n>\widetilde{x}).\end{aligned} \tag{4.6}$$

Applying (4.2) to the first term in (4.6), we can conclude that, for any $\delta_2>0$,

$$\begin{aligned}\mathsf{P}\Big(S_n-\sum_{1\leq k\leq n}\mu_k>x\Big)&\leq\sum_{1\leq k\leq n}\overline{F}_k(\theta x)+\mathsf{P}(\widetilde{S}_n>\widetilde{x})\\ &\leq(1+\delta_2)n\overline{F}(\theta x)+\mathsf{P}(\widetilde{S}_n>\widetilde{x}).\end{aligned} \tag{4.7}$$

We estimate the second term in (4.7). Let $a=\{-\log(n\overline{F}(\theta x)),1\}$, which tends to ∞ uniformly for $x\geq\gamma n$. For arbitrarily fixed $h=h(x,n)>0$, we have

$$\begin{aligned}\mathsf{P}(\widetilde{S}_n>\widetilde{x})/(n\overline{F}(\theta x))&\leq\mathsf{E}\mathrm{e}^{h(\widetilde{S}_n-\widetilde{x})+a}\\ &\leq\exp\Big\{\sum_{1\leq k\leq n}\int_0^{\theta x}(\mathrm{e}^{ht}-1)\mathrm{d}F_k(t)-h\widetilde{x}+a\Big\}.\end{aligned} \tag{4.8}$$

Here we use the property of ND rv's (see (4) in Property 2.1) in the second inequality. The value of h above will be specified later. Splitting the integral on the right-hand side of (4.8) into two terms, and applying an inequality $\mathrm{e}^x-1\leq x\mathrm{e}^x$ for all x, we obtain that for every $k\geq 1$,

$$\begin{aligned}\int_0^{\theta x}(\mathrm{e}^{ht}-1)\mathrm{d}F_k(t)&\leq\mathrm{e}^{h\theta x/a^2}\int_0^{\theta x/a^2}ht\mathrm{d}F_k(t)+\mathrm{e}^{h\theta x}\overline{F}_k(\theta x/a^2)\\ &\leq h\mu_k\mathrm{e}^{h\theta x/a^2}+\mathrm{e}^{h\theta x}\overline{F}_k(\theta x/a^2).\end{aligned} \tag{4.9}$$

Plugging (4.9) into (4.8) yields that, for all large n, for any $\delta_3, \delta_4 > 0$, we have

$$\begin{aligned}
&\mathsf{P}(\widetilde{S}_n > \widetilde{x})/(n\overline{F}(\theta x)) \\
\le\ &\exp\Big\{h \sum_{1\le k\le n} \mu_k \mathrm{e}^{h\theta x/a^2} + \mathrm{e}^{h\theta x} \sum_{1\le k\le n} \overline{F}_k(\theta x/a^2) - h\widetilde{x} + a\Big\} \\
\le\ &\exp\Big\{h \sum_{1\le k\le n} \mu_k (\mathrm{e}^{h\theta x/a^2} - 1) + (1+\delta_3)\mathrm{e}^{h\theta x} n\overline{F}(\theta x/a^2) - hx + a\Big\} \\
\le\ &\exp\big\{(1+\delta_4)hn\overline{\mu}(\mathrm{e}^{h\theta x/a^2} - 1) + (1+\delta_3)Ba^{2\rho}\mathrm{e}^{h\theta x} n\overline{F}(\theta x) - hx + a\big\}. \qquad (4.10)
\end{aligned}$$

Here we use Assumption (A) in the second inequality, and use Assumption (B) and Lemma 2.3 in the third inequality. Let $h = (a - 2\rho\log a)/(\theta x)$ in (4.10), we obtain that, for all large n, for any $\delta_5 > 0$,

$$\begin{aligned}
&\mathsf{P}(\widetilde{S}_n > \widetilde{x})/(n\overline{F}(\theta x)) \\
\le\ &\exp\big\{nh\overline{\mu}(1+\delta_4)(\mathrm{e}^{a^{-1}} - 1) + (1+\delta_3)B - (a - 2\rho\log a)\theta^{-1} + a\big\} \\
\le\ &\mathrm{e}^{(1+\delta_3)B}\exp\{(1-\theta^{-1} + (1+\delta_4)\delta_5)a\}. \qquad (4.11)
\end{aligned}$$

Let $\delta_2 \downarrow 0$, $\delta_3 \downarrow 0$, $\delta_4 \downarrow 0$, $\delta_5 \downarrow 0$, combining (4.11) with (4.7) we have

$$\limsup_{n\to\infty} \sup_{x\ge\gamma n} \mathsf{P}\Big(S_n - \sum_{1\le k\le n}\mu_k > x\Big)/(n\overline{F}(\theta x)) \le 1 + \limsup_{n\to\infty}\sup_{x\ge\gamma n} \mathsf{P}(\widetilde{S}_n > \widetilde{x})/(n\overline{F}(\theta x)) = 1.$$

Since $F \in \mathcal{C}$ and the arbitrariness of $\theta \in (0,1)$ we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\sup_{x\ge\gamma n} \mathsf{P}\Big(S_n - \sum_{1\le k\le n}\mu_k > x\Big)/(n\overline{F}(x)) \\
=\ &\lim_{\theta\nearrow 1}\limsup_{n\to\infty}\sup_{x\ge\gamma n} \mathsf{P}\Big(S_n - \sum_{1\le k\le n}\mu_k > x\Big)/(n\overline{F}(\theta x))(\overline{F}(\theta x)/\overline{F}(x)) \le 1. \qquad (4.12)
\end{aligned}$$

The result (3.1) follows from (4.5) and (4.12). □

4.2 The Proof of Theorem 3.2

Proof By Lemma 2.4 with $\zeta(t) = N(t)/\lambda(t)$, we can easily see that Assumption I implies

$$N(t)/\lambda(t) \xrightarrow{\mathsf{P}} 1. \qquad (4.13)$$

By the same approach as used in the proof of Lemma 4.2 of Klüppelberg and Mikosch (1997) and Theorem 4.1 of Ng et al. (2004), we know that, for any $\delta > 0$, we have

$$\begin{aligned}
&\mathsf{P}(S(t) - \mathsf{E}(S(t)) > x) \\
=\ &\Big(\sum_{n\le(1+\delta)\lambda(t)} + \sum_{n>(1+\delta)\lambda(t)}\Big)\mathsf{P}(N(t)=n)\mathsf{P}(S_n - \mathsf{E}(S(t)) > x). \qquad (4.14)
\end{aligned}$$

First, we estimate the first term in (4.14), clearly,

$$\begin{aligned}&\sum_{n\le(1+\delta)\lambda(t)}\mathsf{P}(N(t)=n)\mathsf{P}(S_n-\mathsf{E}(S(t))>x)\\&=\sum_{|n-\lambda(t)|<\epsilon(t)\lambda(t)}+\sum_{n-\lambda(t)<-\epsilon(t)\lambda(t)}+\sum_{\epsilon(t)\lambda(t)<n-\lambda(t)<\delta\lambda(t)}\\&=:K_1+K_2+K_3.\end{aligned}\tag{4.15}$$

Here $\epsilon(t)$ is a positive function, such that $\epsilon(t)\to 0$ as $t\to\infty$. By Lemma 2.7 and Assumption (B), we know that, for any $\delta_6>0$, $t\to\infty$,

$$(1-\delta_6)\lambda(t)\overline{\mu}\le \mathsf{E}S(t)\le(1+\delta_6)\lambda(t)\overline{\mu}.$$

Start with the estimation of K_1, for any $\delta_7>0$,

$$\begin{aligned}K_1&\le\sum_{|n-\lambda(t)|<\epsilon(t)\lambda(t)}\mathsf{P}(N(t)=n)\mathsf{P}(S_n-\mathsf{E}(S_n)>x-(1+\delta_6)n\overline{\mu}+(1-\delta_6)\overline{\mu}\lambda(t))\\&\le\sum_{|n-\lambda(t)|<\epsilon(t)\lambda(t)}\mathsf{P}(N(t)=n)\mathsf{P}(S_n-\mathsf{E}(S_n)>x-\epsilon(t)\lambda(t)\overline{\mu}-\delta_6(n+\lambda(t))\overline{\mu})\\&\le(1+\delta_7)(1+\epsilon(t))\lambda(t)\overline{F}(x)\sum_{|n-\lambda(t)|<\epsilon(t)\lambda(t)}\mathsf{P}(N(t)=n).\end{aligned}$$

Here, in the last inequality, we use Theorem 3.1 and the fact

$$\overline{F}(x-\epsilon(t)\lambda(t)\overline{\mu}-\delta_6(n+\lambda(t))\overline{\mu})\le(1+\delta_7)\overline{F}(x)$$

for any fixed $\gamma>0$, uniformly for $x>\gamma\lambda(t)$, as $t\to\infty$, since $F\in\mathcal{C}$. For any $\delta_8>0$, by the same treatment we obtain the corresponding asymptotic lower bound as

$$K_1\ge(1-\delta_8)(1-\epsilon(t))\lambda(t)\overline{F}(x)\sum_{|n-\lambda(t)|<\epsilon(t)\lambda(t)}\mathsf{P}(N(t)=n)$$

for any fixed $\gamma>0$, $x>\gamma\lambda(t)$, as $t\to\infty$. Furthermore, according to (4.13),

$$\sum_{|n-\lambda(t)|<\epsilon(t)\lambda(t)}\mathsf{P}(N(t)=n)=\mathsf{P}(|N(t)-\lambda(t)|<\epsilon(t)\lambda(t))\to 1,\qquad\text{as }\ t\to\infty.$$

Thus, we can obtain

$$\limsup_{t\to\infty}\sup_{x\ge\gamma\lambda(t)}|K_1/(\lambda(t)\overline{F}(x))-1|=0.\tag{4.16}$$

Next, we estimate K_2, for any $\delta_9,\delta_{10}>0$,

$$\begin{aligned}K_2&\le\mathsf{P}\big(S_{[(1-\epsilon(t))\lambda(t)]}-\mathsf{E}(S(t))>x\big)\sum_{n-\lambda(t)<-\epsilon(t)\lambda(t)}\mathsf{P}(N(t)=n)\\&\le\mathsf{P}\big(S_{[(1-\epsilon(t))\lambda(t)]}-\mathsf{E}S_{[(1-\epsilon(t))\lambda(t)]}>x+(1-\delta_9)\lambda(t)\overline{\mu}-(1+\delta_9)[(1-\epsilon(t))\lambda(t)]\overline{\mu}\big)\\&\quad\times\mathsf{P}(N(t)-\lambda(t)<-\epsilon(t)\lambda(t))\\&\le\delta_{10}(1+\delta_{10})^2[(1-\epsilon(t))\lambda(t)]\overline{F}(x)=o(\lambda(t)\overline{F}(x)),\end{aligned}\tag{4.17}$$

where δ_9, and δ_{10} are small enough. By the same approach as used in the proof of K_2, we know that

$$K_3 = o(\lambda(t)\overline{F}(x)). \tag{4.18}$$

Plugging (4.16), (4.17) and (4.18) into (4.15) we can obtain

$$\sum_{n\le(1+\delta)\lambda(t)} \mathsf{P}(N(t)=n)\mathsf{P}(S_n - \mathsf{E}(S(t)) > x) \sim \lambda(t)\overline{F}(x), \tag{4.19}$$

for any fixed $\gamma > 0$, uniformly for $x > \gamma\lambda(t)$, as $t \to \infty$.

To complete the proof, it remains to estimate the second term in (4.14). We use Lemma 2.6 and set $t = 1$, $y_k = x/(2v)$, $v > 1$, $y = x/v$, $v > 1$, ($y > \max\limits_{1\le k\le n}\{y_k\}$ for large x), for any $\delta_{11}, \delta_{12} > 0$, we obtain

$$\begin{aligned}
\mathsf{P}(S_n \ge x) &\le \sum_{1\le k\le n} \mathsf{P}(X_k \ge y_k) + \exp\left\{xy^{-1} - xy^{-1}\ln\left(x/\sum_{1\le k\le n}\int_0^{y_k} u\mathrm{d}F_k(u) + 1\right)\right\} \\
&\le \sum_{1\le k\le n} \mathsf{P}(X_k \ge x/(2v)) + \exp\{v - v\ln(x/(n\overline{\mu}(1+\delta_{11})))\} \\
&\le (1+\delta_{12})n\overline{F}(x/(2v)) + \mathrm{e}^v(n\overline{\mu}(1+\delta_{11}))^v x^{-v}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\sum_{n>(1+\delta)\lambda(t)} \mathsf{P}(N(t)=n)\mathsf{P}(S_n - \mathsf{E}(S(t)) > x) \\
&\le (1+\delta_{12})\overline{F}(x/(2v)) \sum_{n>(1+\delta)\lambda(t)} n\mathsf{P}(N(t)=n) \\
&\quad + (\mathrm{e}\overline{\mu}(1+\delta_{11}))^v x^{-v} \sum_{n>(1+\delta)\lambda(t)} n^v\mathsf{P}(N(t)=n) =: J_1 + J_2.
\end{aligned} \tag{4.20}$$

Firstly, we estimate J_1. From Assumption I, we know that

$$\sum_{n>(1+\delta)\lambda(t)} n\mathsf{P}(N(t)=n) = o(\lambda(t)).$$

So, we have

$$\begin{aligned}
J_1 &= (1+\delta_{12})\overline{F}(x/(2v)) \sum_{n>(1+\delta)\lambda(t)} n\mathsf{P}(N(t)=n) \\
&\le (1+\delta_{12})B(2v)^\rho\overline{F}(x)o(\lambda(t)) = o(\lambda(t)\overline{F}(x)).
\end{aligned} \tag{4.21}$$

Here we use Lemma 2.3 in the second step. Where B is a positive number and $\rho > \gamma_{_F}$. Next, we estimate J_2. Setting v in J_2 equals to p, where $p > \gamma_{_F} \ge 1$,

$$\begin{aligned}
J_2 &= (\mathrm{e}\overline{\mu}(1+\delta_{11}))^p x^{-p} \sum_{n>(1+\delta)\lambda(t)} n^p\mathsf{P}(N(t)=n) \\
&= O(\lambda(t))(\mathrm{e}\overline{\mu}(1+\delta_{11}))^p x^{-p} = o(\lambda(t)\overline{F}(x)),
\end{aligned} \tag{4.22}$$

where we use Assumption I in the second equality, and use Lemma 2.2 in the last equality. Plugging (4.21), (4.22) into (4.20), we know that

$$\sum_{n>(1+\delta)\lambda(t)} \mathsf{P}(N(t)=n)\mathsf{P}(S_n-\mathsf{E}(S(t))>x)=o(\lambda(t)\overline{F}(x)). \tag{4.23}$$

Plugging (4.19) and (4.23) into (4.14), we know that (3.2) holds. □

4.3 The Proof of Theorem 3.3

Proof Note $N'(t)=\sum\limits_{i=1}^{\tau(t)} Z_i$, using Lemma 2.5, we know that the renewal counting process $\tau(t)$ satisfies Assumption I: $\mathsf{E}\tau^p(t)I_{\{\tau(t)>(1+\delta)\mathsf{E}\tau(t)\}}=O(\tau(t))$. Since rv's sequence $\{Z_n, n\ge 1\}$ are i.i.d. rv's with a common df $G\in\mathcal{C}$, it is easy to see that Assumption (A) and Assumption (B) in Theorem 3.2 are satisfied for $\{Z_n, n\ge 1\}$. Then, by Theorem 3.2, we have

$$\mathsf{P}(N'(t)\ge x)=\mathsf{P}(N'(t)-\lambda'(t)\ge x-\lambda'(t))\sim \mathsf{E}\tau(t)\overline{G}(x-\lambda'(t)) \tag{4.24}$$

uniformly for $x\ge\gamma\mathsf{E}\tau(t)$ for any fixed $\gamma>\mathsf{E}Z$. Since $\gamma_F<\mu_G$, we can choose a suitable p such that $\gamma_F<p<\mu_G$. Using Abel's idea, for any $\delta>0$, we have

$$\begin{aligned} &\sum_{k>(1+\delta)\lambda'(t)} k^p\mathsf{P}(N'(t)=k)\\ =&\sum_{k>(1+\delta)\lambda'(t)+1}(k^p-(k-1)^p)\mathsf{P}(N'(t)\ge k)\\ &+([(1+\delta)\lambda'(t)]+1)^p\mathsf{P}(N'(t)\ge[(1+\delta)\lambda'(t)]+1)=:A_1+A_2. \end{aligned} \tag{4.25}$$

Choosing some $\epsilon>0$, such that $\gamma_F<p+\epsilon<\mu_G$. For any $\delta_{13}>0$,

$$\begin{aligned} A_1&\le p\sum_{k>(1+\delta)\lambda'(t)+1}k^{p-1}\mathsf{P}(N'(t)\ge k)\\ &\le(1+\delta_{13})p\mathsf{E}\tau(t)\sum_{k>(1+\delta)\lambda'(t)+1}k^{p-1}\overline{G}(k-\lambda'(t))\\ &\le(1+\delta_{13})p\mathsf{E}\tau(t)\sum_{k>(1+\delta)\lambda'(t)+1}k^{p-1}\overline{G}((\delta k)/(1+\delta))\\ &=o(\mathsf{E}\tau(t))=o(\lambda'(t)). \end{aligned} \tag{4.26}$$

Here, we use (4.24) in the second inequality and use Lemma 2.8 in the last inequality. Similarly, for any $\delta_{14}>0$,

$$\begin{aligned} A_2&\le(1+\delta_{14})([(1+\delta)\lambda'(t)]+1)^p\mathsf{E}\tau(t)\overline{G}([(1+\delta)\lambda'(t)]+1-\lambda'(t))\\ &\le(1+\delta_{14})([(1+\delta)\lambda'(t)]+1)^p\mathsf{E}\tau(t)M_2(\delta\lambda'(t))^{-(p+\epsilon)}\\ &=o(\lambda'(t)), \end{aligned} \tag{4.27}$$

where M_1 and M_2 are nonnegative constants irrespective to k. Plugging (4.26) and (4.27) into (4.25) we obtain that $N'(t)$ satisfies Assumption I in Theorem 3.2. Hence by Theorem 3.2 the proof of Theorem 3.3 is completed. □

Acknowledgements We thank the referees for their valuable comments which help the authors improve the exposition of this paper significantly.

References

[1] Tang, Q., Insensitivity to negative dependence of the asymptotic behavior of precise large deviations, *Electronic Journal of Probability*, **11(4)**(2006), 107–120.

[2] Ng, K.W., Tang, Q., Yan, J.A. and Yang, H., Precise large deviations for sums of random variables with consistently varying tails, *Journal of Applied Probability*, **41(1)**(2004), 93–107.

[3] Skučaitė, A., Large deviations for sums of independent heavy-tailed random variables, *Lithuanian Mathematical Journal*, **44(2)**(2004), 198–208.

[4] Tang, Q., Su, C., Jiang, T. and Zhang, J.S., Large deviations for heavy-tailed random sums in compound renewal model, *Statistics & Probability Letters*, **52(1)**(2001), 91–100.

[5] Fuk, D.K. and Nagaev, S.V., Probability inequalities for sums of independent random variables, *Theory of Probability and Its Applications*, **16(4)**(1971), 643–660.

[6] Tang, Q., Insensitivity to negative dependence of asymptotic tail probabilities of sums and maxima of sums, *Stochastic Analysis and Applications*, **26(3)**(2008), 435–450.

[7] Bingham, N.H., Goldie, C.M. and Teugels, J.L., Regular Variation, Cambridge University Press, 1987.

[8] Cline, D.B.H. and Samorodnitsky, G., Subexponentiality of the product of independent random variables, *Stochastic Processes and Their Applications*, **49(1)**(1994), 75–98.

[9] Klüppelberg, C. and Mikosch, T., Large deviations of heavy-tailed random sums with applications in insurance and finance, *Journal of Applied Probability*, **34(2)**(1997), 293–308.

[10] Joag-Dev, K. and Proschan, F., Negative association of random variables with applications, *The Annals of Statistics*, **11(1)**(1983), 286–295.

[11] Cline, D.B.H. and Hsing, T., *Large Deviation Probabilities for Sums and Maxima of Random Variables with Heavy or Subexponential Tails*, Preprint, Texaxs A&M University, 1991.

[12] Kaas, R. and Tang, Q., A large deviation result for aggregate claims with dependent claim occurrences, *Insurance: Mathematics and Economics*, **36(3)**(2005), 251–259.

[13] Wang, S.J. and Wang, W.S., Precise large deviations for sums of random variables with consistently varying tails in multi-risk models, *Journal of Applied Probability*, **44(4)**(2007), 889–900.

[14] Block, H.W., Savits, T.H. and Shaked, M., Some concepts of negative dependence, *The Annals of Probability*, **10(3)**(1982), 765–772.

[15] Ghosh, M., Multivariate negative dependence, *Communications in Statistics - Theory and Methods*, **10(4)**(1981), 307–337.

[16] Wang, D. and Tang, Q., Maxima of sums and random sums for negatively associated random variables with heavy tails, *Statistics & Probability Letters*, **68(3)**(2004), 287–295.

[17] He, W., Cheng, D. and Wang, Y., Asymptotic lower bounds of precise large deviations with nonnegative and dependent random variables, *Statistics & Probability Letters*, **83(1)**(2013), 331–338.

[18] Lu, D., Song, L. and Xu, Y., Precise large deviations for sums of independent random variables with consistently varying tails, *Communications in Statistics - Theory and Methods*, **43(1)**(2014), 28–43.

复合更新风险模型中负相依索赔额下的精细大偏差

宋立新 冯敬海 袁亮亮

(大连理工大学数学科学学院, 大连, 116024)

本文考虑了在复合更新风险模型当中, 负相依索赔额情形下与之相关的精细大偏差的若干问题. 文中假设$\{X_n, n \geq 1\}$是一列负相依的随机变量, 其对应分布列为$\{F_n, n \geq 1\}$, 并假定F_n的右尾分布等同于某个具有一致变化尾的分布. 根据所得的结果试图建立与经典大偏差相似的结论, 并将其应用到改进后的复合更新风险模型当中.

关键词: 精细大偏差, 负相依, 随机和, 一致变化的尾部, 复合更新风险模型.

学科分类号: O211.4.