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A Local Limit Theorem for the Minimum of a Random Walk with Markovian Increments *

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Abstract: The local limit theorems for the minimum of a random walk with Markovian increments is given, with using Presman's factorization theory. This result implies the asymptotic behaviour of the survival probability for a critical branching process in Markovian depended random environment.

Keywords: local limit theorem; Markov chain; semi-Markov chain; random walk

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§1. Introduction and Main Results

Let $(\Omega, \mathscr{F}, \mathsf{P})$ be a probability space and E be a finite set with N elements. Assume that $X = (X_n)_{n\geqslant 0}$ is an irreducible and aperiodic Markov chain, defined on $(\Omega, \mathscr{F}, \mathsf{P})$, with values in E and with transition probability $P = (p_{i,j})_{i,j\in E}$. The chain X admits a unique invariant probability denoted by ν. Let $(F(i,j,\mathrm{d}t))_{i,j\in E}$ be a family of probability measures on $\mathbb{R}$. Consider a sequence of $\mathbb{R}$-valued random variables $(Y_n)_{n\geqslant 0}$ defined on $(\Omega, \mathscr{F}, \mathsf{P})$, such that $(Y_n, X_n)_{n\geqslant 0}$ is a Markov chain on $\mathbb{R}\times E$ with transition probability $\widetilde{P}$, defined by: for any $(u,i)\in\mathbb{R}\times E$, any Borel set $A\subset\mathbb{R}$ and $j\in E$,

$$\widetilde{P}((u,i), A\times\{j\}) = \mathsf{P}(Y_{n+1}\in A, X_{n+1} = j \,|\, Y_n = u, X_n = i) = p_{i,j}F(i,j,A).$$

Such a chain $(Y_n, X_n)_{n\geqslant 0}$ is called a semi-Markovian chain: once the family $(F(i,j,\cdot))_{i,j\in E}$ is fixed, the transitions of this chain is controlled by $(X_n)_{n\geqslant 0}$. We thus consider the canonical probability space $((\mathbb{R}\times E)^{\mathbb{N}}, (\mathscr{B}(\mathbb{R})\otimes\mathscr{P}(E))^{\otimes\mathbb{N}}, (\mathsf{P}_{(u,i)})_{(u,i)\in\mathbb{R}\times E})$ associated with $(Y_n, X_n)_{n\geqslant 0}$ and, for any $(u,i)\in\mathbb{R}\times E$, we denoted by $\mathsf{E}_{(u,i)}$ the expectation with respect to $\mathsf{P}_{(u,i)}$. To simplify our notations, we will denote $\mathsf{P}_{(0,i)}$ by P_i and $\mathsf{E}_{(0,i)}$ by E_i.

Set $S_0 = 0$, $S_n = S_0 + Y_1 + \cdots + Y_n$ and $m_n = \min(S_0, S_1, \ldots, S_n)$. In the case when E reduces to one point, the random variable S_n is the sum of n i.i.d. random variables

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on $\mathbb{R}$. In this case, if $(S_n)_{n\geqslant 0}$ is supposed to be centered, aperiodic with a finite variance, then for all continuous functions with compact support on $\mathbb{R}^-$, one gets

$$\lim_{n\to+\infty}\sqrt{n}\,\mathsf{E}(\varphi(m_n))=C>0,$$

where C is a constant depending on φ (see [13] for instance).

The first goal of this paper is to extend the so-called local limit theorem for the process $(m_n,X_n)_{n\geqslant 0}$ associated with the semi-Markovian chain $(S_n,X_n)_{n\geqslant 0}$ defined above. We assume once and for all the following hypotheses H:

H1 there exists $\alpha>0$, such that for all $\lambda\in\mathbb{C}$ with $|\mathrm{Re}\,\lambda|\leqslant\alpha$, we have

$$\sup_{(i,j)\in E\times E}|\widehat{F}(i,j,\lambda)|<+\infty,\qquad \text{where } \widehat{F}(i,j,\lambda)=\int_{\mathbb{R}}\mathrm{e}^{\lambda t}F(i,j,\mathrm{d}t);$$

H2 there exist $n_0\geqslant 1$ and $(i_0,j_0)\in E\times E$, such that the measure $\mathsf{P}_{i_0}(X_{n_0}=j_0,S_{n_0}\in \mathrm{d}x)$ has an absolutely continuous component with respect to the Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}$;

H3 $\mathsf{E}_\nu(S_n)=\sum_{(i,j)\in E\times E}\nu_i p_{i,j}\int_{\mathbb{R}}tF(i,j,\mathrm{d}t)=0$;

H4 there exist $\varepsilon_0,\delta_0>0$ such that

$$\inf_{i\in E}\sum_{j\in E}p_{i,j}F_{i,j}(]\varepsilon_0,+\infty[)=\delta_0>0.$$

The hypothesis H1 guarantees the existence of Laplace transform of F. H2 is related to 'Cramer's condition', in the case when $(S_n)_{n\geqslant 0}$ is a random walk on $\mathbb{R}$ with i.i.d. increments $(Y_i)_{i\geqslant 1}$, i.e. $\limsup_{t\to+\infty}|\widehat{\mu}(t)|<1$, where $\widehat{\mu}$ is the characteristic function of the common probability law μ of $(Y_i)_{i\geqslant 1}$. H3 assumes that S_n is a central random walk. And H4 will be used to prove positivity of the limit in (3).

Theorem 1 Under the hypotheses H, there exists a constant $\sigma^2>0$ such that, for all $(i,j)\in E\times E$,

$$\sqrt{n}\,\mathsf{E}_i(\mathrm{e}^{\lambda m_n},X_n=j)\overset{n\to+\infty}{\longrightarrow}\frac{H_{i,j}(\lambda)}{\sqrt{\pi}},\tag{1}$$

where $H_{i,j}(\lambda)>0$ for all $\lambda>0$ and

$$\lim_{\lambda\to 0^+}\lambda H_{i,j}(\lambda)=\sqrt{\frac{2}{\pi\sigma^2}}\,\nu_j.\tag{2}$$

It will be also convenient to state this result under the following form:

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Theorem 2 Under the hypotheses H, for all $(i,j) \in E \times E$, one gets

$$\lim_{n\to+\infty} \sqrt{n}\,\mathsf{P}_i(m_n \geqslant -x, X_n = j) = h_{i,j}(x), \tag{3}$$

where the function $h_{i,j} : x \mapsto h_{i,j}(x)$ is increasing and positive on $\mathbb{R}^+$. For any $i,j \in E$ and $x \geqslant 0$, $h_{i,j}(x)$ satisfies the following harmonic relation:

$$h_{i,j}(x) = \mathsf{E}_i[1_{[x+S_1\geqslant 0]} h_{X_1,j}(x+S_1)].$$

Furthermore,

$$h_{i,j}(x) \sim x\sqrt{\frac{2}{\sigma^2}}\,\nu_j, \qquad \text{as } x \to +\infty.$$

The question of local limit theorem for the minimum (or maximum) of random walk on $\mathbb{R}$ consists of determining an equivalence of the probability that the minimum (or maximum) of this random walk locates in a bounded interval in $\mathbb{R}$ by time n, as n tending to infinity. Theorems 1 and 2 have several simple consequences which are of interest. A typical such application is to study the asymptotic behaviour of the survival probability of a critical branching process in an i.i.d. random environment ([11], [12]). However, there are few results where the increments of random walk are Markovian depended. The aim of this article is therefore to extend the local limit theorems for random walks with i.i.d. increments to Markovian depended ones. The applications and analogous results, under appropriate conditions, hold therefore for a branching process in a Markovian environment ([16]).

The local central limit theorem, shorten by local limit theorem, is a kind of results as stated in [1], [2] and [7]. In particular (see also [3]), consider a probability measure μ on $\mathbb{R}$, such that $\int x\mathrm{d}\mu(x) = 0$ and $\int x^2\mathrm{d}\mu(x) = \sigma^2 > 0$. The local limit theorem asserts that the sequence $(\sqrt{n}\mu^{*n})_{n\geqslant 1}$ converges weakly to the measure $\mathrm{d}x/\sigma\sqrt{2\pi}$, that is, for any continuous function f with compact support on $\mathbb{R}$,

$$\sqrt{n}\int_{\mathbb{R}} f\mathrm{d}\mu^n \overset{n\to+\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi\sigma^2}}\int_{\mathbb{R}} f(x)\mathrm{d}x. \tag{4}$$

Such kind of theorems on some specific spaces are studied since the late 1970s. P. Bougerol developed this result in 1981 ([3]) on certain groups of Lie, more precisely, he extended (4) on certain locally compact groups G which processes a compact sub-group K, such that (G,K) is a Guelfand couple. As a consequence, he proved in 1983 ([4]) that when μ belongs to a class R of centered probability measures satisfying some invariance properties, the sequence $(n^{3/2}\mu^{*n})_{n\geqslant 0}$ converges weakly to a non-degenerate measure on a non unimodular solvable group. Based on the heat kernel on a Lie group which is not necessary unimodular

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([18]) and the development on asymptotic behaviour of the convolution powers μ^{*n} of a large class of probability measures ([19]), E. Le Page and M. Peigné obtained in [14] and [15] such a local limit theorem on the semi-direct product $G_d = \mathbb{R}^{*+} \times \mathbb{R}^d$ and $G_{d,N} = G_d \times \mathbb{R}^N$ respectively, where the measure μ does not belong to the class R.

§2. Proofs of the Main Results

This section is devoted to the proofs of two main theorems 1 and 2, the sketch proofs of these theorems will be shown at the end of this section. In particular, it is shown in Lemma 3 that the Laplace transform of m_n can be factorized and its proof is given in Section 4.1. The two fundamental preparation theorems 4 and 5 are proven respectively in Section 5.2 and 5.3. And some preliminary lemmas such as Lemma 7 and Lemma 8 will be proven in Section 6.

Suppose that $V_N[-\alpha_0, \alpha_0]$ is the algebra of $N \times N$ matrices whose terms are Laplace transforms of Radon measures σ on $\mathbb{R}$, satisfying $\int_{\mathbb{R}} \mathrm{e}^{\lambda x} \mathrm{d}|\sigma|(x) < +\infty$, for $|\mathrm{Re}\,\lambda| \leqslant \alpha_0$. Let $G \in V_N[-\alpha_0, \alpha_0]$, defined by

$$G(\lambda) = \Big(\int_{\mathbb{R}} \mathrm{e}^{\lambda x} \mathrm{d}\sigma_{i,j}(x) \Big)_{1 \leqslant i,j \leqslant N}.$$

For $|\mathrm{Re}\,\lambda| \leqslant \alpha_0$, set[1]

$$\mathscr{N} G(\lambda) = \Big(\int_{]-\infty,0]} \mathrm{e}^{\lambda x} \mathrm{d}\sigma_{i,j}(x) \Big)_{i,j}, \qquad \mathscr{N}^* G(\lambda) = \Big(\int_{]-\infty,0[} \mathrm{e}^{\lambda x} \mathrm{d}\sigma_{i,j}(x) \Big)_{i,j};$$
$$\mathscr{P} G(\lambda) = \Big(\int_{[0,+\infty[} \mathrm{e}^{\lambda x} \mathrm{d}\sigma_{i,j}(x) \Big)_{i,j}, \qquad \mathscr{P}^* G(\lambda) = \Big(\int_{]0,+\infty[} \mathrm{e}^{\lambda x} \mathrm{d}\sigma_{i,j}(x) \Big)_{i,j}.$$

For $|z| < 1$, consider the following matrices of measures on $\mathbb{R}$:

$$B_z(\mathrm{d}y) = \Big(\sum_{n=1}^{+\infty} z^n \mathsf{P}_i\{S_1 \geqslant S_n, S_2 \geqslant S_n, \ldots, S_{n-1} \geqslant S_n, S_n \in \mathrm{d}y, X_n = j\} \Big)_{i,j},$$
$$B_z^*(\mathrm{d}y) = \Big(\sum_{n=1}^{+\infty} z^n \mathsf{P}_i\{S_1 > S_n, S_2 > S_n, \ldots, S_{n-1} > S_n, S_n \in \mathrm{d}y, X_n = j\} \Big)_{i,j},$$
$$C_z(\mathrm{d}y) = \Big(\sum_{n=1}^{+\infty} z^n \mathsf{P}_i\{S_1 \geqslant 0, S_2 \geqslant 0, \ldots, S_{n-1} \geqslant 0, S_n \in \mathrm{d}y, X_n = j\} \Big)_{i,j},$$
$$C_z^*(\mathrm{d}y) = \Big(\sum_{n=1}^{+\infty} z^n \mathsf{P}_i\{S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n \in \mathrm{d}y, X_n = j\} \Big)_{i,j}.$$

[1]The sign $\mathscr{N}$ (resp. $\mathscr{N}^*$) corresponds to the restriction of the Radon measure to the negative half-line $\mathbb{R}^-$ (resp. $\mathbb{R}^{*-}$) and the sign $\mathscr{P}$ (resp. $\mathscr{P}^*$) corresponds to the positive half-line $\mathbb{R}^+$ (resp. $\mathbb{R}^{*+}$).

For $\operatorname{Re}\lambda = 0$, the related Laplace transforms of the above measures, denoted respectively by $B_z(\lambda)$, $B_z^*(\lambda)$, $C_z(\lambda)$ and $C_z^*(\lambda)$, are defined as follows:

$$B_z(\lambda) = \int_{-\infty}^{+\infty} \mathrm{e}^{\lambda y} B_z(\mathrm{d}y), \qquad B_z^*(\lambda) = \int_{-\infty}^{+\infty} \mathrm{e}^{\lambda y} B_z^*(\mathrm{d}y);$$
$$C_z(\lambda) = \int_{-\infty}^{+\infty} \mathrm{e}^{\lambda y} C_z(\mathrm{d}y), \qquad C_z^*(\lambda) = \int_{-\infty}^{+\infty} \mathrm{e}^{\lambda y} C_z^*(\mathrm{d}y).$$

Notice that the series which appear in these formulas do converge for $|z| < 1$ and that the matrices $B_z(\lambda)$, $B_z^*(\lambda)$, $C_z(\lambda)$ and $C_z^*(\lambda)$ belong to $V_N[0,0]$.

Denote the open unit disk in complex plane by $\mathbb{D}$, i.e. $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Set $D_{\rho,\theta} := \{z; z \neq 1, |\arg(z-1)| > \theta > 0, |z| < \rho\}$. For $\lambda > 0$, any $i, j \in E$ and $z \in \mathbb{D}$, let

$$G_{i,j}(z,\lambda) := \sum_{n=0}^{+\infty} z^n \mathsf{E}_i(\mathrm{e}^{\lambda m_n}; X_n = j),$$
$$H_{i,j}(z,\lambda) := \sqrt{1-z}\, G_{i,j}(z,\lambda),$$

and

$$H_{i,j}(\lambda) := \lim_{z\to 1} H_{i,j}(z,\lambda),$$

if the limits above exist.

The Laplace transform $G_{i,j}(z,\lambda)$ of m_n may be expressed in terms of the operators $\mathscr{N}^*$ and $\mathscr{P}$ and the matrices B_z^* and C_z; we have the following lemma, which is obtained by the Markov property of the process (S_n, X_n).

Lemma 3 For $\lambda > 0$ and $|z| < 1$,

$$G_{i,j}(z,\lambda) = \{[I + \mathscr{N}^* B_z^*(\lambda)][I + \mathscr{P} C_z(0)]\}_{i,j}. \tag{5}$$

From now on we fix certain $\rho > 1$ and $\theta \in]0, \pi/2[$, such that the following two theorems are true (the existence of such ρ and θ are assumed and are proven later in Section 4.3, Corollary 24).

Theorem 4 1. For $\lambda > 0$ (resp. $\lambda < 0$) closed to 0, the function $I + \mathscr{N}^* B_z^*(\lambda)$ (resp. $I + \mathscr{P} C_z(\lambda)$) admits an analytic expansion on $D_{\rho,\theta}$.

2. We have

$$\lim_{\lambda\to 0^+} \lambda(I + \mathscr{N}^* B_1^*(\lambda)) = A_-, \tag{6}$$
$$\lim_{\lambda\to 0^-} \lambda(I + \mathscr{P} C_1(\lambda)) = A_+, \tag{7}$$

with

$$-\frac{k''(0)}{2} A_- A_+ = \Pi(0). \tag{8}$$

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Theorem 5 The functions $\sqrt{1-z}\,(I+\mathscr{N}^*B_z(0))$ and $\sqrt{1-z}\,(I+\mathscr{P}C_z(0))$ admit an analytic expansion on $D_{\rho,\theta}$ and may be continuously extended on $\overline{D_{\rho,\theta}}$. Furthermore, one gets

$$\lim_{z\to 1}\sqrt{1-z}\,(I+\mathscr{N}^*B_z^*(0))=\sqrt{\frac{k''(0)}{2}}\,A_-, \tag{9}$$

$$\lim_{z\to 1}\sqrt{1-z}\,(I+\mathscr{P}C_z(0))=-\sqrt{\frac{k''(0)}{2}}\,A_+. \tag{10}$$

By Lemma 3, it is obvious that $H_{i,j}(z,\lambda)$ can be factorized as follows:

$$H_{i,j}(z,\lambda)=\{[I+\mathscr{N}^*B_z^*(\lambda)]\sqrt{1-z}\,[I+\mathscr{P}C_z(0)]\}_{i,j}.$$

And by Theorem 5, one gets the limit of $H_{i,j}(z,\lambda)$ exists, as $z\to 1$, its limit is explicit and can be written as

$$H_{i,j}(\lambda)=-\sqrt{\frac{k''(0)}{2}}\,\{(I+\mathscr{N}^*B_1^*(\lambda))A_+\}_{i,j}. \tag{11}$$

The following fundamental lemma, which is a classical result in the theory of complex variables, will be used to prove both Theorem 1 and Theorem 2.

Lemma 6 ([10]) If a function $z\mapsto G(z)$ satisfies the following conditions:

- G is analytic on $D_{\rho,\theta}$ and can be written as $G(z)=\sum\limits_{n=0}^{+\infty}g_nz^n$;
- $\sqrt{1-z}\,G(z)$ is bounded in $D_{\rho,\theta}$;
- $\lim\limits_{\substack{z\to 1\\ z\in D_{\rho,\theta}}}\sqrt{1-z}\,G(z)=C\neq 0$,

then

$$g_n\sim\frac{C}{\sqrt{\pi n}},\qquad \text{as } n\to+\infty.$$

Proof of Theorem 1 By (11) and (8), one gets

$$\lim_{\lambda\to 0^+}\lambda H_{i,j}(\lambda)=-\sqrt{\frac{k''(0)}{2}}\,(A_-A_+)_{i,j}=\sqrt{\frac{2}{k''(0)}}\,\nu_j>0. \tag{12}$$

It's now sufficient to prove that the limit of $\sqrt{n}\,\mathsf{E}_i(\mathrm{e}^{\lambda m_n};X_n=j)$ exists and is equal to $H_{i,j}(\lambda)/\sqrt{\pi}$, as $n\to+\infty$. Indeed, the coefficients of A_+ in (11) are $\leqslant 0$; the function $H_{i,j}$ is in fact the Laplace transform of a positive measure $\mu_{i,j}$ on $\mathbb{R}^-$ and this measure is $\neq 0$ by (12). And therefore for all $\lambda>0$, there exists an interval $[a,b]\subset\mathbb{R}^-$ such that $\mu_{i,j}([a,b])>0$, so that the following inequality holds:

$$H_{i,j}(\lambda)=\int_{-\infty}^{0}\mathrm{e}^{\lambda x}\mathrm{d}\mu_{i,j}(x)\geqslant\int_a^b\mathrm{e}^{\lambda x}\mathrm{d}\mu_{i,j}(x)\geqslant\mathrm{e}^{\lambda a}\mu_{i,j}([a,b])>0. \tag{13}$$

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Consequently, for any $\lambda > 0$, the function $z \mapsto G_{i,j}(z,\lambda)$ is analytic on $D_{\rho,\theta}$, the function $z \mapsto \sqrt{1-z}\, G_{i,j}(z,\lambda)$ is bounded on $D_{\rho,\theta}$ and $\lim\limits_{z\to 1} H_{i,j}(z,\lambda) = H_{i,j}(\lambda) > 0$, which yield the result by applying Lemma 6 to the function $z \to G_{i,j}(z,\lambda)$, for a fixed $\lambda > 0$. □

Before showing Theorem 2, we first need some notations. For any $(i,j) \in E \times E$, the distribution function $h_{i,j} : \mathbb{R}^+ \to \mathbb{R}$ of the measure $\mu_{i,j}$, is defined by

$$h_{i,j}(x) = \begin{cases} -\sqrt{\dfrac{k''(0)}{2\pi}}\, \{[I + \mathscr{N}^* B_1^*(1_{[-x,0]})]A_+\}_{i,j}, & x > 0; \\ -\sqrt{\dfrac{k''(0)}{2\pi}}\, (A_+)_{i,j}, & x = 0, \end{cases}$$

where for $x > 0$,

$$\mathscr{N}^* B_1^*(1_{[-x,0]}) = \sum_{n=1}^{+\infty} \mathsf{P}_i(S_1 > S_n, S_2 > S_n, \ldots, S_{n-1} > S_n, -x \leqslant S_n \leqslant 0, X_n = j).$$

Decompose now the 'potential' $\mathscr{N}^* B_1^*(1_{[-x,0]})$ in terms of the ladder epochs $\{\tau_j\}_{j\geqslant 0}$ of the random walk $(S_n)_{n\geqslant 0}$, defined recursively by

$$\tau_0 = 0 \qquad \text{and} \qquad \tau_j = \inf\{n;\ \text{for all } n \geqslant \tau_{j-1}, S_n < S_{\tau_{j-1}}\}, \quad \text{for } j \geqslant 1.$$

Notice that $\{\tau_1 > n\} = \{m_n = 0\}$, for any $n \geqslant 1$. For any $x \in \mathbb{R}^{*+}$ and $l \geqslant 0$, consider the matrix $B_l(x)$ defined by

$$B_l(x) = (B_l(x)_{i,j})_{i,j},$$

with $B_l(x)_{i,j} = \sum\limits_{k\in E} \mathsf{P}_i(S_{\tau_l} \geqslant -x, X_{\tau_l} = k)(A_+)_{k,j}$. One gets

$$h_{i,j}(x) = -\sqrt{\frac{k''(0)}{2\pi}} \sum_{l\geqslant 0} B_l(x)_{i,j} = -\sqrt{\frac{k''(0)}{2\pi}} \sum_{k\in E} \mathsf{E}_i\Big[\sum_{l\geqslant 0} 1_{[-x,0]}(S_{\tau_l}), X_{\tau_l} = k\Big](A_+)_{k,j}.$$

Notice that, for x large enough, $\mathsf{E}_i\big[\sum\limits_{l\geqslant 0} 1_{[-x,0]}(S_{\tau_l}), X_{\tau_l} = k\big] > 0$ for any $i,k \in E$ since S_{τ_1} is finite P_i-a.s.; so is $h_{i,j}(x)$, since by (8) at least one of the terms $(A_+)_{k,j}$ is non-null. We will see that this property holds in fact for any $x \geqslant 0$. In particular, we have

Lemma 7 For any $l \geqslant 0$, any $i,j \in E$ and $x > 0$ such that $h_{i,j}$ is discontinuous at x, we have

$$\limsup_{n\to+\infty} \sqrt{n}\, \mathsf{P}_i(m_n \geqslant -x, X_n = j) \leqslant h_{i,j}(x) \tag{14}$$

and

$$\liminf_{n\to+\infty} \sqrt{n}\, \mathsf{P}_i(S_{\tau_l} \geqslant -x, \tau_l \leqslant n, \tau_{l+1} > n, X_n = j) \geqslant -\sqrt{\frac{k''(0)}{2\pi}}\, B_l(x)_{i,j}. \tag{15}$$

Lemma 8 There exists a constant $C > 0$, such that $\forall\, i, j \in E$, $\forall\, x \geqslant 0$,

$$\sqrt{n}\,\mathsf{P}_i(m_n \geqslant -x, X_n = j) \leqslant C(x+1). \tag{16}$$

The arguments that we use here to prove (3) is more or less similar to the ones used in the proof of Theorem A in [13], where the so-called 'extended continuity theorem' (Theorem 2a of XIII.1 in [9]) is applied to the points x at which the map $h_{i,j}$ is continuous. While, for $x = 0$ and x such that $h_{i,j}$ is discontinuous (useful results are detailed in Lemma 7), other new arguments related to factorization method are established in Section 4, as well as the regularity of the factors on $D_{\rho,\theta}$ is introduced in Theorem 5.

Lemma 8 is a foundational statement that we need to prove the harmonic of $h_{i,j}$ for (S_n, X_n). And the behaviour of $h_{i,j}(x)$, as $x \to +\infty$, is obtained by using (2) and Tauberian Theorem (see for instance Theorem 1 of XIII.5 in [9]).

Proof of Theorem 2

(a) Proof of the equality (3): For $x > 0$, $h_{i,j}$ is continuous, (3) is an immediate consequence of the extended continuity theorem ([9]).

For $x = 0$, by (10) in Theorem 5 and Lemma 6 (when $-(A_+)_{i,j} > 0$),

$$\sqrt{n}\,\mathsf{P}_i(m_n = 0, X_n = j) \overset{n\to+\infty}{\longrightarrow} -\sqrt{\frac{k''(0)}{2\pi}}\,(A_+)_{i,j}. \tag{17}$$

The same result holds when $-(A_+)_{i,j} = 0$, by Corollary 1 in [10].

For $x > 0$ at which $h_{i,j}$ is discontinuous, one gets

$$\begin{aligned}
&\liminf_{n\to+\infty} \sqrt{n}\,\mathsf{P}_i(m_n \geqslant -x, X_n = j)\\
&\geqslant \sum_{l=0}^{N} \liminf_{n\to+\infty} \sqrt{n}\,\mathsf{P}_i(S_{\tau_l} \geqslant -x, \tau_l \leqslant n, \tau_{l+1} > n, X_n = j)\\
&\geqslant \Big(-\sqrt{\frac{k''(0)}{2\pi}}\Big) \sum_{l=0}^{N} B_l(x)_{i,j}\\
&\overset{N\to+\infty}{\longrightarrow} \Big(-\sqrt{\frac{k''(0)}{2\pi}}\Big)[(I + \mathscr{N}^* B_1^*(1_{[-x,0]}))A_+]_{i,j} = h_{i,j}(x).
\end{aligned} \tag{18}$$

Combining the above inequality and (14) in Lemma 7, one gets the expected result.

(b) The harmonic of $h_{i,j}$: One gets

$$\begin{aligned}
&\sqrt{n+1}\,\mathsf{P}_i(m_{n+1} \geqslant -x, X_{n+1} = j)\\
&= \sqrt{\frac{n+1}{n}} \sum_{i_1\in E} \int p_{i,i_1} 1_{[x+y_1\geqslant 0]} \sqrt{n}\,\mathsf{P}_{i_1}(m_n \geqslant -x - y_1, X_n = j) F(i, i_1, \mathrm{d}y_1),
\end{aligned}$$

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with

$$\mathsf{E}_i(x+|Y_1|) = x + \sum_{j\in E} p_{i,j}\int_{\mathbb{R}} |u|F(i,j,\mathrm{d}u) < +\infty.$$

By Lemma 8 and dominated convergence theorem, tending $n \to +\infty$ in the equality above, one thus gets $\forall\, x \geqslant 0$,

$$\begin{aligned} h_{i,j}(x) &= \sum_{i_1\in E}\int p_{i,i_1}1_{[x+y_1\geqslant 0]}h_{i_1,j}(y_1+x)F(i,i_1,\mathrm{d}y_1) \\ &= \mathsf{E}_i[1_{[x+Y_1\geqslant 0]}h_{X_1,j}(x+Y_1)]. \end{aligned} \tag{19}$$

(c) The asymptotic behaviour of $h_{i,j}(x)$, as $x \to +\infty$: By (2) in Theorem 1, for $\lambda > 0$, $\lim\limits_{\tau\to 0^+}[H_{i,j}(\tau\lambda)/H_{i,j}(\tau)] = 1/\lambda$. According to Tauberian theorem (see for instance Theorem 1 of XIII.5 in [9]), we get

$$h_{i,j}(x) = \mu_{i,j}([-x,0]) \sim \frac{H_{i,j}(1/x)}{\Gamma(2)} \sim \sqrt{\frac{2}{k''(0)}}\,\nu_j x, \qquad \text{as } x\to+\infty. \tag{20}$$

(d) Monotonicity and positivity of $h_{i,j}$ on $\mathbb{R}^+$: for any $i,j\in E$, $h_{i,j}$ is increasing because it is the limit of a sequence of increasing functions. By (20), there exists $M_j \geqslant 1$ such that for $x \geqslant M_j$, $h_{i,j}(x) \geqslant \sqrt{2/k''(0)}\,\nu_j/2$. Choose now n_0 such that $n_0\varepsilon_0 \geqslant M_j$, where ε_0 is defined in hypothesis H4. Iterating Formula (19), we have

$$\begin{aligned} h_{i,j}(0) &= \mathsf{E}_i\Big[1_{\left[\bigcap\limits_{n=1}^{n_0}\{S_n\geqslant 0\}\right]}h_{X_{n_0},j}(S_{n_0})\Big] \\ &\geqslant \mathsf{E}_i\Big[1_{\left[\bigcap\limits_{n=1}^{n_0}\{S_n\geqslant n\varepsilon_0\}\right]}h_{X_{n_0},j}(S_{n_0})\Big] \geqslant \frac{1}{2}\sqrt{\frac{2}{k''(0)}}\,\nu_j\mathsf{P}_i\Big(\bigcap_{n=1}^{n_0}\{S_n\geqslant n\varepsilon_0\}\Big). \end{aligned}$$

By hypothesis H4, there exists $\delta_0 > 0$ such that $\mathsf{P}_i\big(\bigcap\limits_{n=1}^{n_0}\{S_n \geqslant n\varepsilon_0\}\big) \geqslant \delta_0^{n_0}$, which implies $h_{i,j}(0) > 0$. Then for any $i,j\in E$ and $x\geqslant 0$, one gets $h_{i,j}(x) \geqslant h_{i,j}(0) > 0$. □

§3. On the Spectrum of the Semi-Markovian Chain

For any $\lambda\in\mathbb{C}$, consider an complex-valued $N\times N$ matrix $P(\lambda)$ defined by

$$P(\lambda) = (P(\lambda)_{i,j})_{i,j\in E}, \qquad \text{with } P(\lambda)_{i,j} = p_{i,j}\widehat{F}(i,j,\lambda) = p_{i,j}\int_{\mathbb{R}} \mathrm{e}^{\lambda t}F(i,j,\mathrm{d}t).$$

It is easy to verify that for any $n\geqslant 1$, $|\mathrm{Re}\,\lambda| < \alpha$,

$$P^n(\lambda) = (P^n(\lambda)_{i,j})_{i,j} = (\mathsf{E}_i[\mathrm{e}^{\lambda S_n}, X_n = j])_{i,j}.$$

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In particular, $P(0)$ is equal to the transition matrix P of the Markov chain $(X_n)_{n\geqslant 0}$ (and $P^n(0) = P^n = \left(p_{i,j}^{(n)}\right)_{i,j\in E}$). Its spectral radius is equal to 1 since P is stochastic; moreover, since $P(0)$ is aperiodic, the eigenvalue 1 is the unique simple eigenvalue with modulus 1 and its associated eigenvector is $e = (1,\ldots,1)^{\mathsf{T}}$. According to Perron-Frobenius theorem, there thus exists a unique vector $\nu = (\nu_1,\ldots,\nu_N)^{\mathsf{T}}$ with positive coefficients such that $\sum_{i=1}^{N}\nu_i = 1$ and ${}^t\nu P(0) = {}^t\nu$ (the vector ${}^t\nu$ may be identified as a probability measure on E). So we have

$$P = \Pi + R,$$

where

- Π is a matrix of rank 1 given by $\Pi = (\Pi_{i,j})_{i,j\in E} = \begin{pmatrix} \nu_1 & \nu_2 & \cdots & \nu_N \\ \vdots & \vdots & \vdots & \vdots \\ \nu_1 & \nu_2 & \cdots & \nu_N \end{pmatrix}$,
- R is a matrix with spectral radius < 1,
- Π and R satisfy the relation $\Pi R = R\Pi = 0$.

In order to study the spectrum of the bounded linear operator $P(\lambda)$, one may expect an analogue decomposition as for P above. Indeed, the following theorem (see Theorem 9 of Chapter 7 in [8] for the complete version) indicates such an analytical decomposition with respect to λ is possible:

Theorem 9 (Perturbation theorem) Assume that $\mathfrak{X}$ is an finite dimensional complex Banach space and for any complex value λ, $P(\lambda) : \mathfrak{X} \to \mathfrak{X}$ is a bounded linear map ($P(\lambda)$ is also called an operator in $\mathfrak{X}$). Let $\sigma(P(\lambda))$ be the spectrum of the operator $P(\lambda)$ and k_0 be an isolated point of $\sigma(P(0))$. Suppose that for $\gamma > 0$, $P(\lambda)$ is analytic for $|\lambda| < \gamma$ and the eigenspace $E(k_0; P(0))\mathfrak{X}$ has finite dimension m. If there exists an open set U with $\overline{U} \cap \sigma(P(0)) = \{k_0\}$. Then there exists a positive $\delta > \gamma$, an integer $l \leqslant m$ and an integer n, such that for $|\lambda| < \delta$, $U \cap \sigma(P(\lambda))$ is a finite set $\{k_1(\lambda),\ldots,k_l(\lambda)\}$. Each function $k_i(\lambda)$ depends analytically on the principal value of the fractional power $\lambda^{1/n}$ of λ and satisfies $k_i(0) = k_0$.

According to this theorem, for $|\lambda|$ small enough, $P(\lambda)$ has a unique eigenvalue $k(\lambda)$ of modulus equal to the spectral radius of $P(\lambda)$ and this eigenvalue is simple. Therefore, there exists a unique vector $\nu(\lambda) = (\nu_1(\lambda),\ldots,\nu_N(\lambda))^{\mathsf{T}}$ such that $\sum_{i=1}^{N}\nu_i(\lambda) = 1$ and ${}^t\nu(\lambda)P(\lambda) = k(\lambda){}^t\nu(\lambda)$; we can thus also define a unique vector $e(\lambda) = (e_1(\lambda),\ldots,e_N(\lambda))^{\mathsf{T}}$ such that $P(\lambda)e(\lambda) = k(\lambda)e(\lambda)$ and ${}^t\nu(\lambda)e(\lambda) = 1$. More precisely, we have the following theorem, which permits us to develop the resolvent of $P(\lambda)$ as in (25).

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Theorem 10 Under hypotheses H1 and H2, there exist $0 < \gamma_0 < 1/3$ and $0 < \alpha_0 \leqslant \alpha$ such that

1. If $\lambda \in \Delta_{\alpha_0} := \{\lambda \in \mathbb{C}; |\mathrm{Re}\,\lambda|, |\mathrm{Im}\,\lambda| \leqslant \alpha_0\}$, then

$$P(\lambda) = k(\lambda)\Pi(\lambda) + R(\lambda), \tag{21}$$

where

- $k(\lambda) \in \mathbb{C}$ is the dominant eigenvalue of $P(\lambda)$, and satisfies $|1 - k(\lambda)| \leqslant \gamma_0$;
- $\Pi(\lambda)$ is a rank 1 matrix, which corresponds to the projector on the 1-dimensional eigenspace associated with $k(\lambda)$ and is given by $\Pi(\lambda) = (e_i(\lambda)\nu_j(\lambda))_{i,j\in E}$;
- $R(\lambda)$ is a matrix with spectral radius $r(R(\lambda)) < 1 - 2\gamma_0$;
- The matrices $\Pi(\lambda)$and $R(\lambda)$ satisfy the following relation:

$$\Pi(\lambda)R(\lambda) = R(\lambda)\Pi(\lambda) = 0. \tag{22}$$

Furthermore, the maps $\lambda \mapsto k(\lambda)$, $\lambda \mapsto \Pi(\lambda)$ and $\lambda \mapsto R(\lambda)$ are analytic on the set Δ_{α_0}.

2. There exist $\alpha_0' \leqslant \alpha_0$ and $\chi \in]0,1[$ such that if $|\mathrm{Re}\,\lambda| \leqslant \alpha_0'$ and $|\mathrm{Im}\,\lambda| \geqslant \alpha_0$, the spectral radius of $P(\lambda)$ satisfies the inequality

$$r(P(\lambda)) \leqslant \chi < 1. \tag{23}$$

Proof The first assertion is a direct consequence of the perturbation theorem with $k_0 = 1$, $U = \mathbb{D}$ and $m = 1$. To prove the second assertion, we will use the following lemmas:

Lemma 11 There exist $\alpha_1 > 0$, $\beta_1 > 0$ and $\chi_1 \in]0,1[$ such that $r(P(\lambda)) \leqslant \chi_1$ for any $\lambda \in \mathbb{C}$ satisfying $|\mathrm{Re}\,\lambda| \leqslant \alpha_1$ and $|\mathrm{Im}\,\lambda| \geqslant \beta_1$.

Lemma 12 For any $0 < a < b$, there exist $\alpha_{a,b} > 0$ and $\chi_{a,b} \in]0,1[$ such that $r(P(\lambda)) \leqslant \chi_{a,b}$ for any $\lambda \in \mathbb{C}$ satisfying $|\mathrm{Re}\,\lambda| \leqslant \alpha_{a,b}$ and $a \leqslant |\mathrm{Im}\,\lambda| \leqslant b$.

Theorem 10 can thus be proved easily. Indeed, it is sufficient to fix a, b in Lemma 12 in the following way: $a = \alpha_0$, $b = \beta_1$ given by Lemma 11 and $\alpha_0' = \inf(\alpha_0, \alpha_1, \alpha_{a,b})$, $\chi = \inf(\chi_1, \chi_{a,b})$. □

It remains to prove Lemma 11 and Lemma 12. We first need the following simple fact:

Fact 13 Fix $\gamma > 0$ and let $f : \mathbb{R} \mapsto \mathbb{R}$ be such that the function $y \mapsto \mathrm{e}^{\gamma|y|} f(y)$ belongs to $\mathbb{L}^1(\mathbb{R}, \mathrm{d}x)$. Then

$$\lim_{\substack{|t|\to+\infty \\ t\in\mathbb{R}}} \sup_{|a|\leqslant\gamma} \left| \int_{\mathbb{R}} \mathrm{e}^{(a+it)y} f(y)\mathrm{d}y \right| = 0.$$

Proof of Lemma 11 Set $M = (p_{i,j}F(i,j,\mathrm{d}x))_{i,j}$. By Hypotheses H, there exists $k_1 \geqslant 0$ such that all the terms of the matrix $M^{\bullet k_1}$ have absolutely continuous components. Using the fact that

$$M_{i,j}^{\bullet k_1}(\mathrm{d}x) = \varphi_{k_1,i,j}(x)\mathrm{d}x + \theta_{k_1,i,j}(\mathrm{d}x),$$

where for any $(i,j) \in E \times E$,

- the function $\varphi_{k_1,i,j}$ is strictly positive, belongs to $\mathbb{L}^1(\mathbb{R}, \mathrm{d}x)$ and satisfies

$$0 < \int \varphi_{k_1,i,j}(x)\mathrm{d}x \leqslant 1;$$

- $\theta_{k_1,i,j}(\mathrm{d}x)$ is a singulary measure with respect to the Lebesgue measure such that

$$0 \leqslant \int \theta_{k_1,i,j}(\mathrm{d}x) < 1.$$

The matrix containing the absolutely continuous components $\varphi_{k_1,i,j}(x)\mathrm{d}x$ (resp. the singulary measures $\theta_{k_1,i,j}$) is denoted by $H_{k_1}(\mathrm{d}x)$ (resp. $\Theta_{k_1}(\mathrm{d}x)$) and its relative Laplace transform term by term is denoted by $\mathfrak{L}(H_{k_1})(\lambda)$ (resp. $\mathfrak{L}(\Theta_{k_1})(\lambda)$), for $|\mathrm{Re}\,\lambda| \leqslant \alpha_0$.

By Fact 13, we have

$$\limsup_{\substack{|t|\to+\infty \\ t\in\mathbb{R}}} \sup_{|a|\leqslant\alpha_0} \|P^{k_1}(a+it)\| \leqslant \lim_{\substack{|t|\to\infty \\ t\in\mathbb{R}}} \sup_{|a|\leqslant\alpha_0} \|\mathfrak{L}(H_{k_1})(a+it)\| + \limsup_{\substack{|t|\to+\infty \\ t\in\mathbb{R}}} \sup_{|a|\leqslant\alpha_0} \|\mathfrak{L}(\Theta_{k_1})(a+it)\|$$

$$\leqslant \sup_{|a|\leqslant\alpha_0} \|\mathfrak{L}(\Theta_{k_1})(a)\|.$$

Moreover, $\|\mathfrak{L}(\Theta_{k_1})(0)\| = 1-\delta$ with $\delta \in]0,1[$; by continuity of the map $x \in \mathbb{R} \mapsto \mathfrak{L}(\Theta_{k_1})(x)$, there thus exists a real number α_1 such that

$$\sup_{|a|\leqslant\alpha_1} \|\mathfrak{L}(\Theta_{k_1})(a)\| \leqslant 1 - \delta/2 < 1.$$

Set $\chi_1 = 1 - \delta/4$ and choose $\beta_1 > 0$ such that for any $\lambda \in \mathbb{C}$ satisfying $|\mathrm{Re}\,\lambda| \leqslant \alpha_1$ and $|\mathrm{Im}\,\lambda| \geqslant \beta_1$ one gets $\|P^{k_1}(\lambda)\| \leqslant \chi_1$, which implies $r(P(\lambda)) \leqslant \chi_1^{1/k_1} < 1$. □

Proof of Lemma 12 Fix $\lambda \in \mathbb{C}$ such that $|\mathrm{Re}\,\lambda| \leqslant \alpha_1$ and $|\mathrm{Im}\,\lambda| \in [a,b]$. Since for any $i,j \in E$ the measure $P_{i,j}^{\bullet k_1}$ has an absolutely continuous component, one gets

$$|P_{i,j}^{k_1}(\lambda)| < P_{i,j}^{k_1}(\mathrm{Re}\,\lambda),$$

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i.e. $|P_{i,j}^{k_1}(\lambda)| \leqslant \rho_\lambda P_{i,j}^{k_1}(\operatorname{Re}\lambda)$ with $0 < \rho_\lambda < 1$; by continuity of the map $\lambda \mapsto |P_{i,j}^{k_1}(\lambda)|$, one gets $\rho_{a,b} := \sup_{\substack{|\operatorname{Re}\lambda|\leqslant\alpha_1 \\ |\operatorname{Im}\lambda|\in[a,b]}} \rho_\lambda \in]0,1[$. There thus exists $0 < \rho < 1$ such that

$$|P_{i,j}^{k_1}(\lambda)| \leqslant \rho P_{i,j}^{k_1}(\operatorname{Re}\lambda).$$

Therefore, for any λ such that $|\operatorname{Re}\lambda| \leqslant \alpha_1$ and $|\operatorname{Im}\lambda| \in [a,b]$, one gets

$$r(P(\lambda)) \leqslant \rho_{a,b}^{1/k_1} k(\operatorname{Re}\lambda). \tag{24}$$

But, one gets

$$|k(\operatorname{Re}\lambda) - 1| \leqslant |\operatorname{Re}\lambda| \sup_{-\alpha_1\leqslant u\leqslant\alpha_1} |k'(u)| \leqslant \alpha_1 M_{\alpha_1},$$

whith $M_{\alpha_1} = \sup_{-\alpha_1\leqslant u\leqslant\alpha_1} |k'(u)| < +\infty$. Finally, for α_1 small enough, one gets

$$\chi_{a,b} := \sup_{\substack{|\operatorname{Re}\lambda|\leqslant\alpha_1 \\ |\operatorname{Im}\lambda|\in[a,b]}} r(P(\lambda)) \in]0,1[. \qquad \square$$

Proposition 14 Under the hypotheses H1–H3, one gets

$$k'(0) = 0 \qquad \text{and} \qquad \sigma^2 := k''(0) > 0.$$

Remark 15 From now on, we will assume $\alpha_0 = \alpha_0'$; by (21), for $\lambda \in \mathbb{C}$ such that $|\operatorname{Re}\lambda| \leqslant \alpha_0$, one gets

- if $|\operatorname{Im}\lambda| \leqslant \alpha_0$ (i.e. $\lambda \in \Delta_{\alpha_0}$), then

$$(I - zP(\lambda))^{-1} = \frac{zk(\lambda)}{1 - zk(\lambda)}\Pi(\lambda) + \sum_{n=0}^{+\infty} z^n R^n(\lambda). \tag{25}$$

- if $|\operatorname{Im}\lambda| \geqslant \alpha_0$, then for some $\chi \in]0,1[$,

$$r(P(\lambda)) \leqslant \chi. \tag{26}$$

In this expression, one can see that, for any fixed $\lambda \in \Delta_{\alpha_0}$, the function $z \mapsto (I - zP(\lambda))^{-1}$ is analytic on the set of all complex numbers $\mathbb{C}$, excepted the points z satisfying the equation $zk(\lambda) = 1$.

In the following subsections, the singularity of $z \mapsto (I - zP(\lambda))^{-1}$ will be studied specifically and it will be seen that the solutions $\lambda_-(z)$ and $\lambda_+(z)$ of the equation $zk(\lambda) = 1$ are indeed its removable singularities.

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3.1 The Equation $zk(\lambda) = 1$ for $z \in \mathbb{C}$ and $|\mathrm{Re}\,\lambda| \leqslant \alpha_0$

We follow [5] and consider the equation

$$zk(\lambda) = 1, \qquad \text{for } z \in \mathbb{C} \text{ and } |\mathrm{Re}\,\lambda| \leqslant \alpha_0. \tag{27}$$

In this section, we will study the local expansion of the roots $\lambda_-(z)$ and $\lambda_+(z)$ for the equation above when z is near 1.

Since $\lambda \mapsto k(\lambda)$ is analytic on the open set Δ_{α_0}, one may assume that $k''(\lambda) > 0$ for any $\lambda \in]-\alpha_0, \alpha_0[$. By the implicit function theorem, for $z \in \mathbb{R}$, the equation (27) has at most two roots in a sub interval of $[-\alpha_0, \alpha_0]$ (still denoted by $[-\alpha_0, \alpha_0]$ in order to simplify the notation). Since $k'(0) = 0$, one gets $\min\limits_{-\alpha_0 \leqslant \lambda \leqslant \alpha_0} k(\lambda) = k(0) = 1$. Set $q = 1/\inf(k(-\alpha_0), k(\alpha_0))$, then when $z \in [q, 1]$, the equation (27) has exactly two solutions: one is $\lambda_-(z) \in [-\alpha_0, 0]$ and another is $\lambda_+(z) \in [0, \alpha_0]$; furthermore, these two solutions coincide if and only if $z = 1$, and $\lambda_-(1) = \lambda_+(1) = 0$.

For any $\delta_1, \delta_2 > 0$ such that $q + \delta_1 < 1$, set

$$K(\delta_1, \delta_2) := \{z : q + \delta_1 < |z| < 1 + \delta_2, \mathrm{Re}\, z > 0, |\mathrm{Im}\, z| < \delta_1\}.$$

We will describe in the following sections the local behavior of some functions of the complex variable $z \in K(\delta_1, \delta_2)$ but with respect to the variable $t := \sqrt{1 - z}$. In order to fix a principal determination of the function $\sqrt{\ }$, we introduce the subset $K^*(\delta_1, \delta_2) \subset K(\delta_1, \delta_2)$ defined by

$$K^*(\delta_1, \delta_2) := \{z, q + \delta_1 < |z| < 1 + \delta_2, \mathrm{Re}\, z > 0, |\mathrm{Im}\, z| < \delta_1, z \notin [1, 1 + \delta_2[\,\}.$$

Note that the map $z \mapsto \sqrt{1 - z}$ is well defined on $K^*(\delta_1, \delta_2)$.

By the local inversion theorem, since $k'(0) = 0$ and $k''(0) > 0$, one may choose $\delta_1 \in]0, 1-q[$ and $\delta_2 > 0$ in such a way that the two functions $z \mapsto \lambda_+(z)$ and $z \mapsto \lambda_-(z)$, defined a priori on $]q + \delta_1, 1 + \delta_2[$, admit an analytic expansion to the region $K(\delta_1, \delta_2) \setminus \{1\}$ and these functions remain to be the solutions of (27) for $z \in K(\delta_1, \delta_2) \setminus \{1\}$ and $|\,\mathrm{Re}\,\lambda\,| \leqslant \alpha_0$.

We may thus compute the local expansion of these two functions $\lambda_\pm$; one gets the identity

$$\lambda_\pm(z) = \pm\sqrt{\frac{2}{k''(0)}}(1 - z)^{1/2} - \frac{k^{(3)}(0)}{3(k''(0))^2}(1 - z) + O((1 - z)^{3/2}). \tag{28}$$

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3.2 The Resolvent of $P(\lambda)$

We denote by $V_N[-\alpha_0,\alpha_0]$ the algebra of $N\times N$ matrices whose terms are Laplace transforms of Radon measures σ on $\mathbb{R}$, satisfying

$$\int_{\mathbb{R}} \mathrm{e}^{\lambda x}\mathrm{d}|\sigma|(x)<+\infty, \qquad \text{for } |\mathrm{Re}\,\lambda|\leqslant\alpha_0.$$

The following result is stated in [5]; we refer the reader to [20] for a detailed proof.

Theorem 16 There exist δ_1,δ_2 and $\varepsilon>0$ such that

1) The function $A(z,\lambda)$ defined by

$$A(z,\lambda):=(I-zP(\lambda))^{-1}+\frac{\Pi_+(z)}{(\lambda-\lambda_+(z))\beta_+(z)}+\frac{\Pi_-(z)}{(\lambda-\lambda_-(z))\beta_-(z)} \tag{29}$$

is analytic for (z,λ) in the open set

$$E(\delta_1,\delta_2,\varepsilon):=\{(z,\lambda);z\in K(\delta_1,\delta_2),\lambda\in S_z(\varepsilon)\},$$

with $S_z(\varepsilon):=\{\lambda:\lambda_-(z,\varepsilon)<\mathrm{Re}\,\lambda<\lambda_+(z,\varepsilon)\}$, where $\beta_\pm(z):=zk'(\lambda_\pm(z))$, $\Pi_\pm(z):=\Pi(\lambda_\pm(z))$, $\lambda_-(z,\varepsilon):=\mathrm{Re}\,\lambda_-(z)-\varepsilon$ and $\lambda_+(z,\varepsilon):=\mathrm{Re}\,\lambda_+(z)+\varepsilon$.

2) For $(z,\lambda)\in E(\delta_1,\delta_2,\varepsilon)$, one gets

$$\begin{aligned}(I-zP(\lambda))^{-1}=I&-\frac{\Pi_+(z)}{(\lambda-\lambda_+(z))\beta_+(z)}-\frac{\Pi_-(z)}{(\lambda-\lambda_-(z))\beta_-(z)}\\&+\int 1_{]-\infty,0[}\,\mathrm{e}^{\lambda x}\mathrm{d}a_-(z,x)+\int 1_{[0,+\infty[}\,\mathrm{e}^{\lambda x}\mathrm{d}a_+(z,x),\end{aligned} \tag{30}$$

where $a_+(z,\cdot)$ (resp. $a_-(z,\cdot)$) is a Radon measure on $\mathbb{R}_+$ (resp. $\mathbb{R}^-$), with values in $V_{N\times N}[-\alpha_0,\alpha_0]$.

Furthermore, for $x\geqslant 0$ (resp. $x<0$), the function $z\mapsto a_+(z,x)$ (resp. $z\mapsto a_-(z,x)$) is analytic on $K(\delta_1,\delta_2)$, and satisfy: for any $z\in\overline{K}(\delta_1,\delta_2)$:

$$\|a_+(z,+\infty)-a_+(z,x)\|\leqslant C\mathrm{e}^{-(\mathrm{Re}\,\lambda_+(z)+\varepsilon)x}, \qquad x\geqslant 0, \tag{31}$$

$$\|a_-(z,-\infty)-a_-(z,x)\|\leqslant C\mathrm{e}^{-(\mathrm{Re}\,\lambda_-(z)-\varepsilon)x}, \qquad x<0. \tag{32}$$

§4. On the Factorization of $I-zP(\lambda)$

4.1 Preliminaries

Let's first explain briefly how we factorize $G_{i,j}(z,\lambda)$:

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Proof of Lemma 3 Applying Markov property to the process (X_n, S_n), one gets

$$\begin{aligned}
G_{i,j}(z,\lambda) &= \sum_{n=0}^{+\infty} z^n \sum_{k=0}^{n} \mathsf{E}_i(\mathrm{e}^{\lambda S_k}; S_0 > S_k, \ldots, S_{k-1} > S_k, S_{k+1} \geqslant S_k, \ldots, S_n \geqslant S_k, X_n = j) \\
&= \sum_{n=0}^{+\infty} z^n \sum_{k=0}^{n} \sum_{l\in E} \mathsf{E}_i(\mathrm{e}^{\lambda S_k}; S_1 > S_k, \ldots, S_{k-1} > S_k, S_k < 0, X_k = l) \\
&\quad \times \mathsf{E}_l(S_1 \geqslant 0, \ldots, S_{n-k} \geqslant 0, X_{n-k} = j) \\
&= \sum_{l\in E} \Big[\sum_{k=0}^{+\infty} z^k \mathsf{E}_i(\mathrm{e}^{\lambda S_k}; S_1 > S_k, \ldots, S_{k-1} > S_k, S_k < 0, X_k = l) \Big] \\
&\quad \times \Big[\sum_{p=0}^{+\infty} z^p \mathsf{E}_l(S_1 \geqslant 0, \ldots, S_p \geqslant 0, X_p = j) \Big] \\
&= \{[I + \mathscr{N}^* B_z^*(\lambda)][I + \mathscr{P} C_z(0)]\}_{i,j}. \qquad \square
\end{aligned}$$

We will therefore study the regularity with respect to z and λ of each factor $I + \mathscr{N}^* B_z^*(\lambda)$ and $I + \mathscr{P} C_z(0)$. To this end, we use a classical approach based on the so-called Wiener-Hopf factorization; one thus state the following proposition whose proof is based on similar arguments as the ones developed above (see for instance [20]):

Proposition 17 For any $z \in \mathbb{D}$, one gets

1. For $\mathrm{Re}\,\lambda = 0$,
$$I - zP(\lambda) = (I - \mathscr{P} B_z^*(\lambda))(I - \mathscr{N}^* C_z(\lambda)), \tag{33}$$
2. For $\mathrm{Re}\,\lambda \leqslant 0$,
$$(I - \mathscr{P} B_z^*(\lambda))^{-1} = I + \mathscr{P} C_z(\lambda), \tag{34}$$
3. For $\mathrm{Re}\,\lambda \geqslant 0$,
$$(I - \mathscr{N}^* C_z(\lambda))^{-1} = I + \mathscr{N}^* B_z^*(\lambda). \tag{35}$$

Remark 18 1) When E reduces to one point, the sequence $(S_n)_{n\geqslant 0}$ is a random walk on $\mathbb{R}$ and Proposition 17 corresponds to the classical Wiener-Hopf factorization ([9]).

2) Consider the following stopping times, which correspond to the first entrance time of the random walk $(S_n)_{n\geqslant 1}$ inside one of the semi-groups $\mathbb{R}^+$, $\mathbb{R}^{*+}$, $\mathbb{R}^-$ and $\mathbb{R}^{*-}$:

$$\begin{aligned}
T_+ &= \inf\{n \geqslant 1, S_n \geqslant 0\}; \qquad & T_+^* &= \inf\{n \geqslant 1, S_n > 0\}; \\
T_- &= \inf\{n \geqslant 1, S_n \leqslant 0\}; \qquad & T_-^* &= \inf\{n \geqslant 1, S_n < 0\}.
\end{aligned}$$

There is another way to express the matrices $\mathscr{N}^* C_z(\lambda)$ and $\mathscr{P} B_z^*(\lambda)$: for $|z| < 1$,

$$\mathscr{N}^* C_z(\lambda) = \big\{\mathsf{E}_i\big(z^{T_-^*} \mathrm{e}^{\lambda S_{T_-^*}}; X_{T_-^*} = j\big)\big\}_{i,j} \quad \text{when} \quad \mathrm{Re}\,\lambda \geqslant 0,$$

$$\mathscr{P}B_z^*(\lambda) = X^{-1}\big\{\mathsf{E}_i\big(z^{\widetilde{T}_+^*}\mathrm{e}^{\lambda\widetilde{S}_{\widetilde{T}_+^*}};X_{\widetilde{T}_+^*}=j\big)\big\}_{i,j}^{\mathsf{T}}X \quad \text{when} \quad \operatorname{Re}\lambda\leqslant 0;\ ^{2}$$

where X is the diagonal matrix $X := \begin{pmatrix} \nu_1 & & (0) \\ & \ddots & \\ (0) & & \nu_N \end{pmatrix}$ and $(\widetilde{S}_n, \widetilde{X}_n)$ is the dual chain of (S_n, X_n) with transition probability given by

$$\widetilde{P}_{(i,x)}(\{j\}\times A) = \frac{\nu_j}{\nu_i}p_{j,i}F(A-x,j,i).$$

In the sequel, we will extend this factorization to a larger set of parameters. We will first prove, by arguments of elementary type, that this identity is valid for $|z|\leqslant 1$ and $\operatorname{Re}\lambda\in[-\alpha_0,\alpha_0]$. In a second step, we will extend this identity for $\operatorname{Re}\lambda=0$ and z in a neighbourhood of the unit disc, excepted the point 1; this is much more delicate and it relies on a general argument of algebraic type, due to Presman ([17]).

4.2 General Factorization Theory of Presman

Let $\mathfrak{R}$ be an arbitrary algebraic ring with unit element e and $\mathscr{I}$ be the identity operator in $\mathfrak{R}$. Let the additive operator $\mathfrak{N}$ be defined on a two-side ideal $\mathfrak{R}'$ of the ring $\mathfrak{R}$, with

$$(\mathfrak{N}f)(\mathfrak{N}g) = \mathfrak{N}[(\mathfrak{N}f)g + f(\mathfrak{N}g) - fg] \tag{36}$$

holding for any $f,g\in\mathfrak{R}'$. It is easy to check that the operator $\mathfrak{P}=\mathscr{I}-\mathfrak{N}$ also satisfies the relation (36).

Definition 19 We say that the element $e-a$ of a ring $\mathfrak{R}$ admits a left canonical factorization with respect to the operator $\mathfrak{N}$ (l.c.f. $\mathfrak{N}$) if $a\in\mathfrak{R}'$ and if there exist $b,c\in\mathfrak{R}'$ such that

$$e-a = (e-\mathfrak{P}b)(e-\mathfrak{N}c), \tag{37}$$

$$(e-\mathfrak{P}b)^{-1} = e+\mathfrak{P}c, \tag{38}$$

$$(e-\mathfrak{N}c)^{-1} = e+\mathfrak{N}b. \tag{39}$$

In this case, we say that b and c provide a l.c.f. $\mathfrak{N}$. We call $e-\mathfrak{P}b$ and $e-\mathfrak{N}c$ respectively, the positive and negative components of the l.c.f. $\mathfrak{N}$.

Now, we assume that a depends analytically on the complex variable z in a neighbourhood of some z_0 and describe the regularity of the two components of the l.c.f. $\mathfrak{N}$; namely, we get the following

[2] For any $N\times N$ complex matrix A, we denote by A^{T} the transposed matrix of A.

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Lemma 20 ([17; Lemma 1.2]) Let $a(z)$ be an analytic function in a neighborhood of the point z_0, taking values in an ideal $\mathfrak{R}'$ of the Banach algebra $\mathfrak{R}$ and suppose that b_0 and c_0 provide a l.c.f. $\mathfrak{N}$ of the element $e-a(z_0)$. Then $e-a(z)$ admits l.c.f. $\mathfrak{N}$ in a neighborhood of the point z_0, where the elements $b(z)$ and $c(z)$ which provide the l.c.f. $\mathfrak{N}$ of the element $e-a(z)$ are analytic functions of z taking values in $\mathfrak{R}'$.

We achieve this paragraph explaining how one will use this general result in our context. Consider the algebraic ring $V_N[-\alpha_0,\alpha_0]$ of $N\times N$ matrices whose terms are Laplace transforms of Radon measures σ on $\mathbb{R}$, with exponential moment of order α_0. The operator $\mathfrak{N}$ will be here the operator $\mathscr{N}^*$ defined above and acting on $V_N[-\alpha_0,\alpha_0]$ and $\mathfrak{P}$ will be equal to $\mathscr{P}$.

If ν, μ are two Radon measures on $\mathbb{R}$, we have the following identity:

$$\nu^{*-}*\mu^{*-}=(\nu^{*-}*\mu+\nu*\mu^{*-}-\nu*\mu)^{*-}.\ ^{3}$$

Taking into account this equality, we obtain that $\mathscr{N}^*$ and $\mathscr{P}$ both satisfy the identity (36) for any f, $g\in V_N[a,b]$.

For $|z|<1$ and $|\mathrm{Re}\,\lambda|\leqslant\alpha_0$, we will consider the following $\mathbb{C}$-valued $N\times N$ matrices:

$$B_z^*(\lambda):=\Big(\sum_{n=1}^{+\infty}z^n\int_{-\infty}^{+\infty}\mathrm{e}^{\lambda y}\mathrm{dP}_i\{S_1>S_n,S_2>S_n,\ldots,S_{n-1}>S_n,S_n\leqslant y,X_n=j\}\Big)_{i,j},$$
$$C_z(\lambda):=\Big(\sum_{n=1}^{+\infty}z^n\int_{-\infty}^{+\infty}\mathrm{e}^{\lambda y}\mathrm{dP}_i\{S_1\geqslant 0,S_2\geqslant 0,\ldots,S_{n-1}\geqslant 0,S_n\leqslant y,X_n=j\}\Big)_{i,j}.$$

Recall now that $P(\lambda)$ belongs to $V_N[-\alpha_0,\alpha_0]$; furthermore, by Proposition 17, for any complex number z with modulus <1 and any $\lambda\in\mathbb{C}$ such that $\mathrm{Re}\,\lambda=0$, the operator $I-zP(\lambda)$ admits a l.c.f. $\mathscr{N}^*$ on $V_N[0,0]$ provided with B_z^* and C_z.

The above general Presman's result are therefore applicable to $z\mapsto A_z:=zP(\lambda)$ with values in $V_N[-\alpha_0,\alpha_0]$ for $|z|<1$ and analytic on the unit open disc in the complex plane.

In particular, the elements B_z^* and C_z belong to $V_N[0,0]$. In fact, one may precise this last statement, with the following lemma due to Presman (Lemma 1.3 in [17]):

Lemma 21 If $I-A_z$ is an analytic function of z in a neighbourhood of the point z_0, taking values in the ring $V_N[-\alpha_0,\alpha_0]$ and if in this neighbourhood $I-A_z$, as an element of $V_N[0,0]$, admits a l.c.f. with respect to $\mathscr{N}^*$ with corresponding elements B_z^* and C_z, then

[3] For any Radon measure γ on $\mathbb{R}$, we denote by γ^{*-} its restriction to $\mathbb{R}^{*-}$ given by

$$\gamma^{*-}(\mathrm{d}x)=1_{]-\infty,0[}(x)\gamma(\mathrm{d}x).$$

$\mathscr{P}B_z^*$ (resp. $\mathscr{N}^*C_z$) is analytic in z in this neighbourhood, with values in $\mathscr{P}V_N\,]-\infty,\alpha_0]$ (resp. $\mathscr{N}^*V_N[-\alpha_0,+\infty[$).

As a direct consequence of Proposition 17 and above Presman's theory, one gets

Corollary 22 The maps $z\mapsto \mathscr{P}B_z^*(\lambda)$ and $z\mapsto \mathscr{N}^*C_z(\lambda)$ are analytic on $\mathbb{D}$ with values $\mathscr{P}V_N\,]-\infty,\alpha_0]$ (resp. $\mathscr{N}^*V_N[-\alpha_0,+\infty[$); furthermore, the equality (33) remains valid for $-\alpha_0\leqslant \operatorname{Re}\lambda\leqslant\alpha_0$.

4.3 Expansion of the Factorization Outside the Unit Disc and Far from $z=1$

We study here the extension of the preceding factorization when $\operatorname{Re}\lambda=0$ and z lives in a neighbourhood of $\mathbb{D}\setminus\{1\}$. We have the

Theorem 23 There exists a neighbourhood U of $\mathbb{D}\setminus\{1\}$ such that, for $\operatorname{Re}\lambda=0$, the two maps $z\mapsto B_z^*(\lambda)$ and $z\mapsto C_z(\lambda)$ may be continuously expanded on U in such a way

1. for any $z\in U$, the formulas (33), (34) and (35) hold.
2. the maps $z\mapsto\mathscr{P}B_z^*$ and $z\mapsto\mathscr{N}^*C_z$ are analytic on U, with values in $V_N\,]-\infty,\alpha_0]$ and $V_N[-\alpha_0,+\infty[$ respectively.

Proof We fix λ such that $\operatorname{Re}\lambda=0$, $z_0\in\mathbb{C}\setminus\{1\}$ of modulus 1 and choose a sequence $(z_n)_{n\geqslant1}$ of complex numbers in $\mathbb{D}$ which converges to z_0.

By 2) of Remark 18, the two limits

$$B_{z_0}^+(\lambda):=\lim_{n\to+\infty}\mathscr{P}B_{z_n}^*(\lambda)\qquad\text{and}\qquad C_{z_0}^{*-}(\lambda):=\lim_{n\to+\infty}\mathscr{N}^*C_{z_n}(\lambda)$$

do exist; furthermore, (33) holds at any point z_n and letting $n\to+\infty$ one gets

$$I-z_0P(\lambda)=(I-B_{z_0}^+(\lambda))(I-C_{z_0}^{*-}(\lambda)).$$

Since $z_0\neq1$, the matrix $I-z_0P(\lambda)$ is invertible, so is $I-B_{z_0}^+(\lambda)$; by (34), the limit $\lim_{n\to+\infty}\mathscr{P}C_{z_n}(\lambda)$ does also exist (and is equal to $C_{z_0}^+(\lambda):=-I+(I-B_{z_0}^+(\lambda))^{-1}$).

Consequently,

$$C_{z_0}(\lambda):=\lim_{n\to+\infty}C_{z_n}(\lambda)=\lim_{n\to+\infty}\mathscr{N}^*C_{z_n}(\lambda)+\lim_{n\to+\infty}\mathscr{P}C_{z_n}(\lambda)=C_{z_0}^{*-}(\lambda)+C_{z_0}^+(\lambda)$$

does exist and one gets $C_{z_0}^{*-}=\mathscr{N}^*C_{z_0}(\lambda)$ and $C_{z_0}^+(\lambda)=\mathscr{P}C_{z_0}(\lambda)$.

By the same argument, one shows that $B_{z_0}^*(\lambda):=\lim_{n\to+\infty}B_{z_n}^*(\lambda)$ does exist and (35) holds at z_0.

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Finally $B^*_{z_0}(\lambda)$ and $C_{z_0}(\lambda)$ provide a l.c.f. $\mathscr{N}^*$ of $I - z_0P(\lambda)$; since $z \mapsto I - z_0P(\lambda)$ is analytic in a neighbourhood of z_0, so are the maps $z \mapsto \mathscr{P}B^*_z(\lambda)$ and $z \mapsto \mathscr{N}^*C_z(\lambda)$ by Lemma 20, with values in $\mathscr{P}V_N\,]-\infty, \alpha_0]$ and $\mathscr{N}^*V_N[-\alpha_0, +\infty[$ respectively, by Lemma 21. □

Corollary 24 There exist $\rho > 1$ and $\theta \in]0, \pi/2[$, such that

- the formulas (33), (34) and (35) hold for $\operatorname{Re}\lambda = 0$ and $z \in D_{\rho,\theta} \setminus K(\delta_1, \delta_2)$;
- for $|\operatorname{Re}\lambda| \leqslant \alpha_0$, the map $z \mapsto \mathscr{P}B^*_z$ (resp. $z \mapsto \mathscr{N}^*C_z$) is analytic on $D_{\rho,\theta} \setminus K(\delta_1, \delta_2)$; furthermore, $I - \mathscr{P}B^*_z$ (resp. $I - \mathscr{N}^*C_z$) is invertible (and their inverses are also analytic) on this domain.

Therefore, as mentioned before, the neighbourhood U is specified as $D_{\rho,\theta}$, defined in Section 2, throughout the proofs of the related results.

§5. On the Local Behavior near $z = 1$ of the Factors of the Laplace Transform of the Minimum

Recall that the Laplace transform of the minimum m_n may be decomposed as the product of two factors as written in (5). In this section, we will study each behavior of these two factors near $z = 1$. More precisely, we will first consider the case when $|z| \leqslant 1$ and then investigate the case when $|z| > 1$.

5.1 A Local Factorization near $z = 1$

As mentioned in the previous section, the matrices $I + \mathscr{N}^*B^*_z(\lambda)$ and $I + \mathscr{P}C_z(0)$ could be seen as the inverse of two factors for the matrix $I - zP(\lambda)$ and we will first study the regularities of these quantities for $z \in \overline{K}(\delta_1, \delta_2)$.

We now precise the type of regularity of $(I - \mathscr{P}B^*_z(\lambda))^{-1}$ and $(I - \mathscr{N}^*C_z(\lambda))^{-1}$ on the domain $K^*(\delta_1, \delta_2)$ for small enough $\delta_1, \delta_2 > 0$ (by Corollary 24, we already know that they are analytic on $D_{\rho,\theta} \cap (K(\delta_1, \delta_2))^c$ for some suitable $\rho > 1$ and $\theta > 0$). Following M.S. Bratiichuk ([6]), we set

$$F_\pm(z, \lambda) = I + \frac{\lambda_\pm(z) - a_\pm}{\lambda - \lambda_\pm(z)}\Pi_\pm(z),$$

where $a_+ = \alpha_0 + 1$ and $a_- = -\alpha_0 - 1$. Note that $F_\pm(z, \lambda)$ are analytic with respect to $z \in K^*(\delta_1, \delta_2)$, excepted at $1/k(\lambda)$ (so that $\lambda \neq \lambda_\pm(z)$). On the other hand, one gets

$$F^{-1}_+(z, \lambda) = I - \frac{\lambda_+(z) - a_+}{\lambda - a_+}\Pi_+(z)$$

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(and similarly $F_-^{-1}(z,\lambda) = I - \{[\lambda_-(z) - a_-]/[\lambda - a_-]\}\Pi_-(z)$)[4]. Let us emphasize that $F_\pm^{-1}(z,\lambda)$ are analytic on $K^*(\delta_1,\delta_2)$ (even at point $1/k(\lambda)$).

Set $B(z,\lambda) = F_+(z,\lambda)(I - zP(\lambda))F_-(z,\lambda)$; by the above, the matrix $B(z,\lambda)$ is invertible, we denote by $B^{-1}(z,\lambda)$ its inverse; we also set $B_+(z,\lambda) = F_+(z,\lambda)(I - \mathscr{P}B_z^*(\lambda))$ and $B_-(z,\lambda) = (I - \mathscr{N}^*C_z(\lambda))F_-(z,\lambda)$.

For $z \in \overline{K}(\delta_1, 0)$, according to the relation (33), we have

$$B(z,\lambda) = B_+(z,\lambda)B_-(z,\lambda), \qquad |\mathrm{Re}\,\lambda| \leqslant \alpha_0, \tag{40}$$

$$B^{-1}(z,\lambda) = B_-^{-1}(z,\lambda)B_+^{-1}(z,\lambda), \qquad \lambda \in S_z(\varepsilon). \tag{41}$$

Proposition 25 For $\delta_1, \delta_2, \varepsilon > 0$ small enough and $z \in K^*(\delta_1,\delta_2)$, one gets

$$B(z,\lambda) \in V_N[-\alpha_0, +\alpha_0] \qquad \text{and} \qquad B^{-1}(z,\lambda) = V_N[\lambda_-(z,\varepsilon), \lambda_+(z,\varepsilon)].$$

Furthermore, the maps

- $z \mapsto B(z,\lambda)$, $z \mapsto B_-(z,\lambda)$, $z \mapsto B_+(z,\lambda)$;
- $z \mapsto B^{-1}(z,\lambda)$, $z \mapsto B_-^{-1}(z,\lambda)$, $z \mapsto B_+^{-1}(z,\lambda)$;
- $z \mapsto \mathscr{P}B_z^*(\lambda)$, $z \mapsto \mathscr{N}^*C_z(\lambda)$,

admit an analytic expansion on $K^*(\delta_1,\delta_2)$ and with respect to the variable $t = \sqrt{1-z}$ for $z \in K^*(\delta_1,\delta_2)$.

In addition, the maps $z \mapsto (I - \mathscr{P}B_z^*(\lambda))^{-1}$ and $z \mapsto (I - \mathscr{N}^*C_z(\lambda))^{-1}$ are analytic on $K^*(\delta_1,\delta_2)$ excepted at point $1/k(\lambda)$; in particular, they are analytic on $D_{\rho,\theta}$.

Proof We follow here [17]. The relations (40) and (41) show that $B^{\pm 1}(z,\lambda)$ admit a canonical factorization for all z on the unit circle such that $|\mathrm{Im}\,z| < \delta_1$. Since these functions are regular with respect to the variable $t = \sqrt{1-z}$ for $z \in K^*(\delta_1,\delta_2)$, we may by Lemma 20 adapt the choice of δ_1 and δ_2 in such a way that the components of factorizations (40) and (41), regarded as functions of t, admit an analytic expansion with respect to the variable t. By the identity

$$\mathscr{P}B_z^*(\lambda) = I - F_+^{-1}(z,\lambda)B_+(z,\lambda), \tag{42}$$

one obtains the expected regularity of the functions $z \mapsto \mathscr{P}B_z^*(\lambda)$.

[4]Remark that for any column vector a and row vector b, setting $ba = \beta \in \mathbb{C}$, then $\det(I - ab) = 1 - \beta$ and $(I - ab)^{-1} = I + (1-\beta)^{-1}ab$. Applying these formulae to $a = -\{[\lambda_+(z) - a_+]/[\lambda - \lambda_+(z)]\}(e_1(\lambda_+(z)), \ldots, e_N(\lambda_+(z)))^{\mathsf{T}}$ and $b = (\nu_1(\lambda_+(z)), \ldots, \nu_N(\lambda_+(z)))$ to obtain the announced expression of $F_+^{-1}(z,\lambda)$; and the expression of $F_-^{-1}(z,\lambda)$ can be obtained analogously.

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At last, for $z \neq 1/k(\lambda)$, one gets by the previous equality

$$(I - \mathscr{P}B_z^*(\lambda))^{-1} = B_+^{-1}(z,\lambda)F_+(z,\lambda), \tag{43}$$

with $F_+(z,\lambda)$ well defined and analytic in z since $\lambda \neq \lambda_\pm(z)$ and one concludes.

The same holds similarly for $\lambda \mapsto \mathscr{N}^*C_z(\lambda)$ and $\lambda \mapsto (I - \mathscr{N}^*C_z(\lambda))^{-1}$. □

5.2 On the Regularity of the Factors $I + \mathscr{N}^*B_z^*(\lambda)$ and $I + \mathscr{P}C_z(\lambda)$ on $D_{\rho,\theta}$ for $\lambda \in \mathbb{R}^*$

In this section we demonstrate Theorem 4 which study the regularity of the factors $I + \mathscr{N}^*B_z^*(\lambda)$ and $I + \mathscr{P}C_z(\lambda)$ on $D_{\rho,\theta}$ for $\lambda \in \mathbb{R}^*$.

Proof of Theorem 4

1. First case: when $z \in D_{\rho,\theta} \setminus K(\delta_1, \delta_2)$ and $\lambda \in \mathbb{R}^*$, this is a direct consequence of Corollary 24.

   Second case: when $z \in K(\delta_1, 0)$, by Corollary 22, we have

   $$(I - \mathscr{N}^*C_z(\lambda))^{-1} = I + \mathscr{N}^*B_z(\lambda), \qquad \operatorname{Re}\lambda \geqslant 0, \tag{44}$$

   $$(I - \mathscr{P}B_z^*(\lambda))^{-1} = I + \mathscr{P}C_z(\lambda), \qquad \operatorname{Re}\lambda \leqslant 0. \tag{45}$$

   Now, by Proposition 25, the quantities of left hand-side of the above formula are proven to be analytic with respect to $z \in K^*(\delta_1, \delta_2)$ for some $\delta_2 > 0$ small enough and for $z \neq 1/k(\lambda)$. Recall that $z \neq 1/k(\lambda) \Leftrightarrow \lambda \neq \lambda_\pm(z)$. We hence obtain the expected result, using the fact that $D_{\rho,\theta} \subset (D_{\rho,\theta} \setminus K(\delta_1, \delta_2)) \cup K^*(\delta_1, \delta_2)$.

2. The equalities (6) and (7) are direct consequences of the following proposition due to M.S. Bratiichuk ([6; Theorem 3 and Corollary 1]):

   Proposition 26 There exist $\delta_1 > 0$, for $z \in \overline{K}(\delta_1, 0)$, and any $\varepsilon > 0$ such that Theorem 16 is satisfied, one gets

   (a) for $\operatorname{Re}\lambda < \lambda_+(z,\varepsilon)$ with $\lambda \neq \lambda_+(z)$,

   $$\begin{aligned}(I - \mathscr{P}B_z^*(\lambda))^{-1} &= I + \mathscr{P}C_z(\lambda) \\ &= I - \frac{[I - \mathscr{N}^*C_z(\lambda_+(z))]\Pi_+(z)}{(\lambda_+(z) - \lambda)\beta_+(z)} + \int_0^{+\infty} \mathrm{e}^{\lambda x} k_+(z, \mathrm{d}x);\end{aligned} \tag{46}$$

   (b) for $\operatorname{Re}\lambda > \lambda_-(z,\varepsilon)$ with $\lambda \neq \lambda_-(z)$,

   $$\begin{aligned}(I - \mathscr{N}^*C_z(\lambda))^{-1} &= I + \mathscr{N}^*B_z^*(\lambda) \\ &= I - \frac{\Pi_-(z)[I - \mathscr{P}B_z^*(\lambda_-(z))]}{(\lambda_-(z) - \lambda)\beta_-(z)} + \int_{-\infty}^{0} \mathrm{e}^{\lambda x} k_-(z, \mathrm{d}x),\end{aligned} \tag{47}$$

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where $k_+(z,\cdot)$ (resp. $k_-(z,\cdot)$) is a measure on $[0,+\infty[$ (resp. $]-\infty,0]$) taking values in the vector space $M_{N\times N}(\mathbb{C})$ of $N\times N$ complex matrices, such that for $z\in\overline{K}(\delta,0)$, one gets

$$\|k_+(z,x)\|\leqslant C\mathrm{e}^{-(\lambda_+(z)+\varepsilon)x}\quad\text{for}\quad x>0, \tag{48}$$

$$\|k_-(z,x)\|\leqslant C\mathrm{e}^{-(\lambda_-(z)-\varepsilon)x}\quad\text{for}\quad x<0, \tag{49}$$

where $k_+(z,x)=k_+(z,]x,+\infty[)$ for $x>0$ and $k_-(z,x)=k_-(z,]-\infty,x[)$ for $x<0$.

Furthermore, the following limits exist:

$$\lim_{|z|\uparrow 1}\frac{(I-\mathscr{N}^*C_z(\lambda_+(z)))\Pi_+(z)}{\beta_+(z)}=A_+\quad\text{and}\quad\lim_{|z|\uparrow 1}\frac{\Pi_-(z)(I-\mathscr{P}B_z^*(\lambda_-(z)))}{\beta_-(z)}=A_-, \tag{50}$$

where A_+ (resp. A_-) is a $N\times N$ matrix with non positive (resp. non negative) coefficients.

Indeed, according to this proposition, one gets

$$\lim_{z\to 1}\lambda_+(z)(I-\mathscr{P}B_z^*(0))^{-1}=-A_+, \tag{51}$$

$$\lim_{z\to 1}\lambda_-(z)(I-\mathscr{N}^*C_z(0))^{-1}=-A_-. \tag{52}$$

On the other hand, for $q<z<1$, one gets

$$(1-z)(I-zP(0))^{-1}=[\sqrt{1-z}\,(I-\mathscr{N}^*C_z(0))^{-1}][\sqrt{1-z}\,(I-\mathscr{P}B_z^*(0))^{-1}],$$

with $(I-zP(0))^{-1}=z\Pi(0)/(1-z)+\sum\limits_{n=0}^{+\infty}z^nR^n(0)$; so

$$\lim_{z\to 1}[\sqrt{1-z}\,(I-\mathscr{N}^*C_z(0))^{-1}][\sqrt{1-z}\,(I-\mathscr{P}B_z^*(0))^{-1}]=\Pi(0).$$

Since $\lim\limits_{z\to 1}\sqrt{1-z}/\lambda_-(z)=-\sqrt{k''(0)/2}$ and $\lim\limits_{z\to 1}\sqrt{1-z}/\lambda_+(z)=\sqrt{k''(0)/2}$ (see (28)), we hence obtain

$$-\frac{k''(0)}{2}A_-A_+=\Pi(0)=\begin{pmatrix}\nu_1&\nu_2&\cdots&\nu_N\\ \nu_1&\nu_2&\cdots&\nu_N\\ \vdots&\vdots&\vdots&\vdots\\ \nu_1&\nu_2&\cdots&\nu_N\end{pmatrix},$$

which yields the result. □

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5.3 On the Regularity of the Factors $I+\mathscr{N}^*B_z^*(0)$ and $I+\mathscr{P}C_z(0)$ on $D_{\rho,\theta}$

We will give in this section the proof of Theorem 5, which focus on the regularity of the factors $I+\mathscr{N}^*B_z^*(0)$ and $I+\mathscr{P}C_z(0)$ on $D_{\rho,\theta}$.

Proof of Theorem 5 First case: when $z\in D_{\rho,\theta}\setminus K(\delta_1,\delta_2)$, the analysis of $z\mapsto\sqrt{1-z}\,(I+\mathscr{P}C_z(0))$ (resp. $z\mapsto\sqrt{1-z}\,(I+\mathscr{N}^*B_z^*(0))$) is derived from Corollary 24 and the fact that $z\mapsto(I-zP(0))^{-1}$ is analytic in $D_{\rho,\theta}\setminus K(\delta_1,\delta_2)$.

Second case: the map $z\mapsto\sqrt{1-z}\,(I-\mathscr{P}B_z^*(0))^{-1}$ is the analytic expansion on $K^*(\delta_1,\delta_2)$ of $z\mapsto\sqrt{1-z}\,(I+\mathscr{P}C_z(0))$, and, by (43), one gets

$$\sqrt{1-z}\,(I-\mathscr{P}B_z^*(0))^{-1}=\sqrt{1-z}\,B_+^{-1}(z,0)F_+(z,0).$$

By Proposition 25, the map $z\mapsto B_+^{-1}(z,0)$ is analytic on $K^*(\delta_1,\delta_2)$ and one gets

$$\sqrt{1-z}\,F_+(z,0)=\sqrt{1-z}\,\Big(I-\frac{\lambda_+(z)-a_+}{\lambda_+(z)}\Pi_+(z)\Big).$$

So that $\lim\limits_{z\to1}\sqrt{1-z}\,F_+(z,0)$ exists, since $\sqrt{1-z}/\lambda_+(z)\to\sqrt{k''(0)/2}$ as $z\to1$, which follows that the function $z\mapsto\sqrt{1-z}\,(I+\mathscr{P}C_z(0))$ is analytic on $D_{\rho,\theta}$. □

§6. Proofs of Lemma 7 and Lemma 8

Proof of Lemma 7 Fix $x>0$ such that $h_{i,j}(\cdot)$ is discontinuous at x. The map $x\mapsto h_{i,j}(x)$ is increasing and right-continuous on $\mathbb{R}_+^*$, the set of its points of discontinuity is countable and there thus exists a sequence $(\varepsilon_k)_{k\geqslant1}$ of non negative reals converging towards 0 and such that $h_{i,j}$ is continuous at $x+\varepsilon_k$ for any $k\geqslant1$; consequently, for any $k\geqslant1$ one gets

$$\sqrt{n}\,\mathsf{P}_i(m_n\geqslant-x,X_n=j)\leqslant\sqrt{n}\,\mathsf{P}_i(m_n\geqslant-x-\varepsilon_k,X_n=j)$$

and so

$$\limsup_{n\to+\infty}\sqrt{n}\,\mathsf{P}_i(m_n\geqslant-x,X_n=j)\leqslant h_{i,j}(x+\varepsilon_k).$$

The map $h_{i,j}$ being right continuous, tending $\varepsilon_k\to0$ one obtains the inequality (14).

The proof of (15) is a little bit complicated. For any $0<\delta<1$, we have

$$\mathsf{P}_i(S_{\tau_l}\geqslant-x,\tau_l\leqslant n,\tau_{l+1}>n,X_n=j)\geqslant\mathsf{P}_i(S_{\tau_l}\geqslant-x,\tau_l\leqslant\delta n,\tau_{l+1}>n,X_n=j).\quad(53)$$

From Markov property,

$$\begin{aligned}&\mathsf{P}_i(S_{\tau_l} \geqslant -x, \tau_l \leqslant \delta n, \tau_{l+1} > n, X_n = j)\\ &= \sum_{k\in E} \mathsf{E}_i[(S_{\tau_l} \geqslant -x, \tau_l \leqslant \delta n, X_{\tau_l} = k)\mathsf{P}_k(\tau_1 > n, X_{n-\tau_l} = j)]\\ &= \sum_{\substack{k\in E\\ 0\leqslant p\leqslant \delta n}} \mathsf{E}_i[(S_p \geqslant -x, \tau_l = p, X_p = k)\mathsf{P}_k(\tau_1 > n, X_{n-p} = j)]. \qquad (54)\end{aligned}$$

In addition,

$$\begin{aligned}&\sqrt{n}\,\mathsf{P}_k(\tau_1 > n, X_{n-p} = j)\\ &= \sqrt{n}\,\mathsf{P}_k(\tau_1 > n-p, X_{n-p} = j) - \sqrt{n}\,\mathsf{P}_k(n-p < \tau_1 \leqslant n, X_{n-p} = j). \qquad (55)\end{aligned}$$

Writing

$$\mathsf{P}_k(n-p < \tau_1 \leqslant n, X_{n-p} = j) \leqslant \mathsf{P}_k(n-p < \tau_1 \leqslant n) = \mathsf{P}_k(\tau_1 > n-p) - \mathsf{P}_k(\tau_1 > n),$$

by (17), the last term in the right-hand side of (55) converges to 0 and the first term converges to $-\sqrt{k''(0)/(2\pi)}\,(A_+)_{k,j}$, as $n \to +\infty$. We thus obtain that

$$\lim_{n\to+\infty} \sqrt{n}\,\mathsf{P}_k(\tau_1 > n, X_{n-p} = j) = -\sqrt{\frac{k''(0)}{2\pi}}\,(A_+)_{k,j}.$$

Using Fatou's lemma and the inequality (53) and (54), one concludes

$$\begin{aligned}&\liminf_{n\to+\infty} \sqrt{n}\,\mathsf{P}_i(S_{\tau_l} \geqslant -x, \tau_l \leqslant n, \tau_{l+1} > n, X_n = j)\\ &\geqslant \liminf_{n\to+\infty} \sqrt{n}\,\mathsf{P}_i(S_{\tau_l} \geqslant -x, \tau_l \leqslant \delta n, \tau_{l+1} > n, X_n = j)\\ &\geqslant -\sqrt{\frac{k''(0)}{2\pi}} \sum_{k\in E} \mathsf{P}_i(S_{\tau_l} \geqslant -x, X_{\tau_l} = k)(A_+)_{k,j} = -\sqrt{\frac{k''(0)}{2\pi}} B_l(x)_{i,j}. \qquad \square\end{aligned}$$

Proof of Lemma 8 We will use the two following facts, whose proofs may be found in [13]:

Fact 27 ([13]) Let $c, \nu \in \mathbb{R}_+^*$ and $(a_n)_{n\geqslant 0}$ be a monotone sequence of non negative reals such that $\sum_{n=0}^{+\infty} a_n s^n \leqslant c(1-s)^{-\nu}$ for any $s \in [0,1[$. Then

$$\forall\, n \geqslant 2, \qquad a_n \leqslant c\,\mathrm{e}\,(1-\mathrm{e}^{-1})^{-\nu} 2^{1+\nu} n^{\nu-1}.$$

Fact 28 ([13]) Let H be a non-decreasing function on $\mathbb{R}^+$ such that $H(0) = 0$ and the integral $\widetilde{H}(\lambda) := \int_0^{+\infty} \mathrm{e}^{-\lambda x}\mathrm{d}H(x)$ does exist for any $\lambda > 0$. If there exist $\delta, \gamma > 0$ such that

$$\forall\, \lambda \in\,]0, \delta], \qquad \widetilde{H}(\lambda) \leqslant c\lambda^{-\gamma},$$

then, for all $x \geqslant \delta^{-1}$, one gets $H(x) \leqslant c\,\mathrm{e}\,x^{\gamma}$.

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Taking into account (12), we get for any $i \in E$,

$$\lim_{\lambda \to 0} \lambda \sum_{j \in E} H_{i,j}(\lambda) = \sqrt{\frac{2}{k''(0)}} > 0,$$

which implies that there exist two constants $\delta, c > 0$ such that for any $\lambda \in]0, \delta]$ and $s \in]0, 1[$,

$$\sup_{i \in E} \sum_{n=0}^{+\infty} s^n \mathsf{E}_i(\mathrm{e}^{\lambda m_n}) \leqslant c\lambda^{-1}(1-s)^{-1/2}.$$

For $\lambda > 0$, the sequence $(\mathsf{E}(\mathrm{e}^{\lambda m_n}))_{n \geqslant 0}$ is decreasing with respect to n and Fact 27 with $\nu = 1/2$ leads to

$$\sqrt{n}\, \mathsf{E}_i(\mathrm{e}^{\lambda m_n}) \leqslant c\, \mathrm{e}\, (1 - \mathrm{e}^{-1})^{-1/2} 2^{3/2} \lambda^{-1}, \qquad \text{for all } i \in E,\ n \geqslant 2 \text{ and } \lambda \in]0, \delta].$$

Applying now Fact 28 with $\gamma = 1$, we get, for all $x \geqslant \delta^{-1} > 0$, $n \geqslant 2$ and $i, j \in E$,

$$\sqrt{n}\, \mathsf{P}_i(m_n \geqslant -x, X_n = j) \leqslant \sqrt{n}\, \mathsf{P}_i(m_n \geqslant -x) \leqslant c_1 x,$$

where $c_1 = c\, \mathrm{e}^2\, (1 - \mathrm{e}^{-1})^{-1/2} 2^{3/2}$.

On the other hand, for $0 \leqslant x < \delta^{-1}$, one gets

$$\sqrt{n}\, \mathsf{P}_i(m_n \geqslant -x, X_n = j) \leqslant \sqrt{n}\, \mathsf{P}_i(m_n \geqslant -\delta^{-1}, X_n = j) \overset{n \to +\infty}{\longrightarrow} h_{i,j}(\delta^{-1})$$

and one thus may write, for any $i, j \in E$ and $x \geqslant 0$,

$$\sqrt{n}\, \mathsf{P}_i(m_n \geqslant -x, X_n = j) \leqslant c_1 x + c_2,$$

where $c_2 = \sup\limits_{\substack{n \geqslant 1 \\ i,j \in E}} \mathsf{P}_i(m_n \geqslant -\delta^{-1}, X_n = j)$. $\square$

§7. Significations of Some Symbols

- For any row vector $\nu = (\nu_1, \ldots, \nu_N)$, we denote by ν^t or ${}^t\nu$ the column vector.
- For a subset A of a set X, we denote the indicator function by $1_A : X \to \{0, 1\}$, defined by

$$1_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

- μ^{*n} signifies the n-fold convolution of a probability measure μ.

- For matrices $M=(m_{i,j}(\mathrm{d}x))_{i,j}$ and $N=(n_{i,j}(\mathrm{d}x))_{i,j}$, where $m_{i,j}$ and $n_{i,j}$ are positive measures on $\mathbb{R}$. Then $M\bullet N$ contains the convolution of the elements of M and N, i.e.

$$M\bullet N=\Big(\sum_k m_{i,k}(\mathrm{d}x)*n_{k,j}(\mathrm{d}x)\Big)_{i,j}. \tag{56}$$

Moreover, $M^{\bullet k}$ means k times product of M by itself in the sense of (56).

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具有马尔可夫增量的随机游动的最小值的局部极限定理

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摘 要: 本文考虑具有马尔可夫增量的随机游动(半马尔可夫链)的最小值, 提出一种Presman因式分解的方法, 对其分布的渐进行为进行研究并给出了局部极限定理. 该结果可以应用于对马尔可夫随机环境下的分支过程的生存概率的渐进行为进行估计.

关键词: 局部极限定理; 马尔可夫链; 半马尔可夫链; 随机游动

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