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带有不完全信息随机截尾试验下最大似然估计的相合性及渐近正态性 *

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摘　要

本文在条件 (Φ) 下，证明了带有不完全信息随机截尾试验的最大似然估计的相合性及渐近正态性，验证了 Weibull 分布、对数正态分布满足条件 (Φ).

关键词：带有不完全信息随机截尾试验，相合性，渐近正态性.

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§1. 引　言

设寿命变量 $X_1, X_2, \cdots$ 是概率空间 $(\Omega, \mathcal{F}, \mathrm{P}_\theta)$ ($\theta \in \Theta$, Θ 是 R^m 空间上的开集) 上的独立同分布随机变量序列，其分布函数为 $F(x,\theta)$, 密度函数为 $f(x,\theta)$. 又设截尾变量 $Y_1, Y_2, \cdots$ 是 $(\Omega, \mathcal{F}, \mathrm{P}_\theta)$ 上相互独立的正值随机变量序列，分布函数分别为 $G_1(t), G_2(t), \cdots$, 并且假定 $\{X_i\}$ 与 $\{Y_i\}$ 相互独立.

为估计参数 θ, 给出 n 个样品 $\{X_i, 1 \le i \le n\}$ 及观察数据 $\{Z_i, 1 \le i \le n\}$ 的取值情况如下:

(I) 当 $X_i < Y_i$, 表示产品在截尾前失效. 在通常的随机截尾试验下有 $Z_i = X_i$, 但在这里取值不同因为产品的失效状态还必须通过某种检测手段利用信号给予显示，所以，有两种可能情况发生：失效状态以概率 p ($0 < p \le 1$, p 是已知数，与 θ 无关) 被立即显示，此时 $Z_i = X_i$; 或以 $1-p$ 未被立即显示，直到截尾变量 Y_i 终止时才发现产品已失效，此时获得了不完全信息，即仅知道 $X_i \le Y_i$ 而不知道寿命 X_i 的准确值，故得 $Z_i = Y_i$, 称 p 为失效显示概率.

(II) 当 $X_i \ge Y_i$, 表示产品寿命已不小于截尾变量，故得 $Z_i = Y_i$.

若令

$$\alpha_i = \begin{cases} 1 & \text{若 } X_i < Y_i; \\ 0 & \text{若 } X_i \ge Y_i, \end{cases} \qquad \beta_i = \begin{cases} 0 & \text{若 } X_i < Y_i, \text{且失效未被显示}; \\ 1 & \text{其它}, \end{cases} \qquad (i = 1, 2, \cdots, n)$$

则 $\{(Z_i, \alpha_i, \beta_i), 1 \le i \le n\}$ 是带有不完全信息的随机截尾数据，且对每个 $i (1 \le i \le n)$ 有

$$Z_i = \begin{cases} X_i & \alpha_i = 1,\ \beta_i = 1, \\ Y_i & \alpha_i = 1,\ \beta_i = 0, \\ Y_i & \alpha_i = 0\ (\beta_i = 1), \end{cases} \qquad \mathrm{P}_\theta(\beta_i = 1 | \alpha_i = 1) = p.$$

由上易知，随机截尾试验模型是带有不完全信息的随机截尾试验模型的特例. 文献 [8] 中给出了随机截尾情形下 Weibull 分布参数的最大似然估计的相合性. 文献 [6] 给出了随机截尾情形下分布较一般且截尾变量独立不同分布的参数最大似然估计的相合性及渐近正态性. 随着可靠性理论的深入发展，Elperin & Gertsbakh (1988) 将随机截尾试验模型推广到带有不完全信息的随机截尾试验模型. 他们就单参数指数分布场合给出带有不完全信息随机截尾试验下平均寿命 $\theta = 1/\lambda$ 的 MLE 和区间估计的随机模拟计算 [1]. 尔后，文献 [2] 中证

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明了上述的MLE 具有强相合性. Elperin & Gertsbakh 又用 Bayes 方法研究了上述问题的区间估计[3]. [4,5] 又将带有不完全信息的随机截尾试验模型用于双参数寿命分布场合，给出了 Weibull 分布、对数正态分布和正态分布的 MLE 的存在唯一性定理. 在文献 [7] 中给出了带有不完全信息随机截尾试验下 Weibull 分布的相合性及渐近正态性. 据作者所知，对于分布较一般的带有不完全信息随机截尾试验下参数最大似然估计的相合性及渐近正态性还没有一个较好的结果. 鉴于此，本文做了初步探讨，给出文献 [6] 所提条件之下分布较一般的带有不完全信息随机截尾试验下 MLE 的强相合性及渐近正态性. 从而，就分布类型和实验模型这两个方面分别推广了 [7] 和 [6] 的结果. 本文的结构如下，除第一节引言之外，第二节为主要结果，第三节为若干引理，第四节为定理的证明.

§2. 主要结果

众所周知，基于 $\{(Z_i,\alpha_i,\beta_i),\ 1\le i\le n\}$ 的似然函数为 (见 [7]):

$$L(\theta)=\prod_{i=1}^{n} f(Z_i,\theta)^{\alpha_i\beta_i}F(Z_i,\theta)^{\alpha_i(1-\beta_i)}\overline{F}(Z_i,\theta)^{1-\alpha_i}, \tag{2.1}$$

其中: $\overline{F}=1-F$.

定义 1 (似然方程组) 称如下方程组为 (2.1) 的似然方程组,

$$\frac{\partial \ln L}{\partial\theta}=0, \tag{2.2}$$

其中:

$$\frac{\partial \ln L}{\partial\theta}=\Big(\frac{\partial \ln L}{\partial\theta_1},\frac{\partial \ln L}{\partial\theta_2},\cdots,\frac{\partial \ln L}{\partial\theta_m}\Big)^T.$$

以下 $\partial\ln f(x,\theta)/\partial\theta$, $\partial\ln F(x,\theta)/\partial\theta$, $\partial\ln\overline{F}(x,\theta)/\partial\theta$ 与之定义类同，均为向量.

定义 2 称似然函数正规，若对一切 $n\ge 2$ 只要 (Z_i,α_i,β_i), $i=1,2,\cdots,n$ 不全相等，似然方程组 (2.2) 有唯一解 $\widehat{\theta}^n=(\widehat{\theta}_1^n,\widehat{\theta}_2^n,\cdots,\widehat{\theta}_m^n)^T$, 其中 $\widehat{\theta}_s^n=\widehat{\theta}_s^n(Z_1,\cdots,Z_n,\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_n)$, $s=1,2,\cdots,m$.

当 $(Z_1,\alpha_1,\beta_1)=(Z_2,\alpha_2,\beta_2)=\cdots=(Z_n,\alpha_n,\beta_n)$ 时，令 $\widehat{\theta}_i^n=\theta_i^0$, $i=1,2,\cdots,m$, 其中 $(\theta_1^0,\theta_2^0,\cdots,\theta_m^0)$ 为 Θ 中一固定点. 因此 $\widehat{\theta}^n=(\widehat{\theta}_1^n,\widehat{\theta}_2^n,\cdots,\widehat{\theta}_m^n)^T$ 总有定义. 此时称 $\widehat{\theta}^n$ 是 θ 的最大似然估计.

其次，我们对 $f(x,\theta)$, $F(x,\theta)$, $G_i(x)$, $i\ge 1$ 施加如下条件，合称为条件 (Φ):

(1) $f(x,\theta)$ 为 $[0,+\infty)\times\Theta$ 上定义的正值函数, $f(x,\theta)$ 关于 x Borel 可测且 $\partial^2 f(x,\theta)/(\partial\theta_s\partial\theta_t)$, $\partial f(x,\theta)/\partial\theta_s$, $f(x,\theta)$ $(s,t=1,2,\cdots,m)$ 为 θ 的连续函数.

(2) 对于 $\forall\theta^0\in\Theta$, 存在 $\mu_{\theta^0}=\{\theta:\|\theta-\theta^0\|\le\eta_{\theta^0}\}\subset\Theta(\eta_{\theta^0}>0)$ 使得在 μ_{θ^0} 上，下式成立,

$$\begin{aligned}
&\Big|\frac{\partial f(x,\theta)}{\partial\theta_s}\Big|\le H_s(x), && \int_0^{\infty} H_s(x)\mathrm{d}x<+\infty,\\
&\Big|\frac{\partial^2 f(x,\theta)}{\partial\theta_s\partial\theta_t}\Big|\le \widetilde{H}_{st}(x), && \int_0^{\infty} \widetilde{H}_{st}(x)\mathrm{d}x<+\infty,\\
&\Big|\frac{\partial^2 \ln f(x,\theta)}{\partial\theta_s\partial\theta_t}\Big|\le \Phi_{st}(x), && \int_0^{\infty} \Phi_{st}^2(x)f(x,\theta^0)\mathrm{d}x<+\infty,\\
&\Big|\frac{\partial^2 \ln F(x,\theta)}{\partial\theta_s\partial\theta_t}\Big|\le \widehat{\Phi}_{st}(x), && \sup_{x\ge 0}\widehat{\Phi}_{st}^2(x)F(x,\theta^0)\le M,\\
&\Big|\frac{\partial^2 \ln \overline{F}(x,\theta)}{\partial\theta_s\partial\theta_t}\Big|\le \widetilde{\Phi}_{st}(x), && \sup_{x\ge 0}\widetilde{\Phi}_{st}^2(x)\overline{F}(x,\theta^0)\le M.
\end{aligned}$$

其中: M 与 x 无关，与 θ^0 有关.

(3)

$$\int_0^{\infty}\Big(\frac{\partial\ln f(x,\theta)}{\partial\theta_s}\Big)^4 f(x,\theta)\mathrm{d}x<+\infty,$$
$$\Big(\frac{\partial\ln\overline{F}(x,\theta)}{\partial\theta_s}\Big)^4\overline{F}(x,\theta)\to 0\qquad x\to\infty,$$
$$\Big(\frac{\partial\ln F(x,\theta)}{\partial\theta_s}\Big)^4 F(x,\theta)\to 0\qquad x\to 0.$$

(4) 似然函数正规 (见定义 2).

(5) 对 $\forall\theta\in\Theta$, $[\partial\ln f(x,\theta)/\partial\theta]\cdot[\partial\ln f(x,\theta)/\partial\theta]^T$ 或 $[\partial\ln F(x,\theta)/\partial\theta]\cdot[\partial\ln F(x,\theta)\partial\theta]^T$ 或 $[\partial\ln\overline{F}(x,\theta)/\partial\theta]\cdot[\partial\ln\overline{F}(x,\theta)/\partial\theta]^T$ 关于 x 正定.

(6) 存在分布函数 $G_0(x)$, 使得 $\lim\limits_{n\to\infty}(1/n)\sum\limits_{i=1}^{n}G_i(x)=G_0(x)$, 且至少存在一点 $x_0>0$, 使得 $G_0(x_0)<1$.

注 文献 [6] 中所提条件 (7) 为本文所提条件 (Φ) 中的 (5) 的一个充分条件.

易证 Weibull 分布、对数正态分布满足条件 (Φ) 中 (1)-(5).

我们的主要结论如下:

定理 1 设 $\{Z_i,\alpha_i,\beta_i\}$, $i=1,2,\cdots,n$ 满足条件 (Φ), $\widehat{\theta}^{(n)}$ 为 θ 的最大似然估计, 则有

$$\mathsf{P}_{\theta^0}\Big(\lim_{n\to\infty}\widehat{\theta}^{(n)}=\theta^0\Big)=1,\tag{2.3}$$

其中, $\theta^0=(\theta_1^0,\theta_2^0,\cdots,\theta_m^0)^T$ 为参数真值.

定理 2 设 $\{Z_i,\alpha_i,\beta_i\}$, $i=1,2,\cdots,n$ 满足条件 (Φ), $\widehat{\theta}^{(n)}$ 为 θ 的最大似然估计, 则有

$$\sqrt{n}(\widehat{\theta}^{(n)}-\theta^0)\xrightarrow{d}N(0,-G^{-1}(\theta^0)),\tag{2.4}$$

其中, $\theta^0=(\theta_1^0,\theta_2^0,\cdots,\theta_m^0)^T$ 为参数真值,

$$\begin{aligned}G(\theta)&=(g_{st}(\theta))_{m\times m},\\ g_{st}(\theta)&=-\Big(p\int_0^{\infty}\frac{\partial\ln f(x,\theta)}{\partial\theta_s}\frac{\partial\ln f(x,\theta)}{\partial\theta_t}\overline{G}_0(x)f(x,\theta^0)\mathrm{d}x\\&\quad+(1-p)\int_0^{\infty}\frac{\partial\ln F(x,\theta)}{\partial\theta_s}\frac{\partial\ln F(x,\theta)}{\partial\theta_t}F(x,\theta^0)\mathrm{d}G_0(x)\\&\quad+\int_0^{\infty}\frac{\partial\ln\overline{F}(x,\theta)}{\partial\theta_s}\frac{\partial\ln\overline{F}(x,\theta)}{\partial\theta_t}\overline{F}(x,\theta^0)\mathrm{d}G_0(x)\Big).\end{aligned}\tag{2.5}$$

§ 3. 若干引理

为给出定理 1 和定理 2 的证明, 我们需要如下一些引理.

引理 1([7]) 对任一给定的 $\theta^0\in\Theta$ 及任一 Borel 函数 $T(x)$, 若 $T(z_i)$ 关于概率测度 P_{θ^0} 可积, 则有,

$$\begin{aligned}\mathsf{E}_{\theta^0}[\alpha_i\beta_iT(z_i)]&=p\int_0^{\infty}T(x)\overline{G}_i(x)\mathrm{d}F(x,\theta^0),\\ \mathsf{E}_{\theta^0}[\alpha_i(1-\beta_i)T(z_i)]&=(1-p)\int_0^{\infty}T(x)F(x,\theta^0)\mathrm{d}G_i(x),\\ \mathsf{E}_{\theta^0}[(1-\alpha_i)T(z_i)]&=\int_0^{\infty}T(x)\overline{F}(x,\theta^0)\mathrm{d}G_i(x).\end{aligned}\tag{3.1}$$

其中: $\overline{G}_i(x)=1-G_i(x)$.

引理 2 在条件 (Φ) 下, 对每个 i $(i=1,2,\cdots,n)$ 有

$$\mathsf{E}_{\theta^0}\Big(\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_s}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_s}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_s}\Big)=0,\qquad s=1,2,\cdots,m.\tag{3.2}$$

证明: 首先由条件 (Φ) 中 (2) 知,

$$\begin{aligned}\int_0^\infty \Big|\frac{\partial f(x,\theta)}{\partial\theta_s}\Big|\overline{G}_i(x)\mathrm{d}x &\le \int_0^\infty H_s(x)\mathrm{d}x<+\infty, &(3.3)\\ \int_0^\infty \Big|\frac{\partial F(x,\theta^0)}{\partial\theta_s}\Big|\mathrm{d}G_i(x) &\le \int_0^\infty\int_0^x \Big|\frac{\partial f(y,\theta^0)}{\partial\theta_s}\Big|\mathrm{d}y\mathrm{d}G_i(x)\\ &\le \int_0^\infty\int_0^x H_s(y)\mathrm{d}y\mathrm{d}G_i(x)\\ &\le \int_0^\infty H_s(y)\mathrm{d}y<+\infty. &(3.4)\end{aligned}$$

所以, 把 $\partial\ln f(x,\theta^0)/\partial\theta_s$, $\partial\ln F(x,\theta^0)/\partial\theta_s$, $\partial\ln\overline{F}(x,\theta^0)/\partial\theta_s$ 分别当作 $T(x)$, 由引理 1 及 (3.3), (3.4) 知, 对每个 i $(i=1,2,\cdots,n)$ 有,

$$\begin{aligned}&\mathsf{E}_{\theta^0}\Big(\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_s}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_s}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_s}\Big)\\ =\;& p\int_0^\infty\frac{\partial\ln f(x,\theta^0)}{\partial\theta_s}\overline{G}_i(x)\mathrm{d}F(x,\theta^0)+(1-p)\int_0^\infty\frac{\partial\ln F(x,\theta^0)}{\partial\theta_s}F(x,\theta^0)\mathrm{d}G_i(x)\\ &+\int_0^\infty\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_s}\overline{F}(x,\theta^0)\mathrm{d}G_i(x)\\ =\;& p\int_0^\infty\frac{\partial f(x,\theta^0)}{\partial\theta_s}\overline{G}_i(x)\mathrm{d}x+(1-p)\int_0^\infty\frac{\partial F(x,\theta^0)}{\partial\theta_s}\mathrm{d}G_i(x)+\int_0^\infty\frac{\partial\overline{F}(x,\theta^0)}{\partial\theta_s}\mathrm{d}G_i(x)\\ =\;& \frac{\partial}{\partial\theta_s}\Big[p\int_0^\infty f(x,\theta^0)\overline{G}_i(x)\mathrm{d}x+(1-p)\int_0^\infty F(x,\theta^0)\mathrm{d}G_i(x)+\int_0^\infty\overline{F}(x,\theta^0)\mathrm{d}G_i(x)\Big]\\ =\;& 0.\end{aligned}$$

所以, (3.2) 成立. #

引理 3 在条件 (Φ) 下, 对 $\forall\varepsilon>0$, 当 $N\to+\infty$ 时, 下式成立,

$$\begin{aligned}&\mathsf{P}_{\theta^0}\Big\{\sup_{n\ge N}\Big|\frac{1}{n}\sum_{i=1}^n\Big[\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_s}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_s}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_s}\\ &\quad-\mathsf{E}_{\theta^0}\Big(\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_s}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_s}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_s}\Big)\Big]\Big|\ge\varepsilon\Big\}\\ &\longrightarrow 0. \qquad (3.5)\end{aligned}$$

证明: 首先由条件 (Φ) 中 (2), (3) 及 $F(x,\theta^0)\le 1$ 知,

$$\lim_{x\to 0}\Big[\frac{\partial\ln F(x,\theta^0)}{\partial\theta_s}\Big]^2F(x,\theta^0)\le\lim_{x\to 0}\sqrt{\Big[\frac{\partial\ln F(x,\theta^0)}{\partial\theta_s}\Big]^4F(x,\theta^0)}=0$$

和

$$\begin{aligned}\lim_{x\to\infty}\Big[\frac{\partial\ln F(x,\theta^0)}{\partial\theta_s}\Big]^2F(x,\theta^0) &= \lim_{x\to\infty}\frac{1}{F(x,\theta^0)}\Big(\frac{\partial F(x,\theta^0)}{\partial\theta_s}\Big)^2\\ &= \lim_{x\to\infty}\frac{1}{F(x,\theta^0)}\Big(\int_0^x\frac{\partial f(y,\theta^0)}{\partial\theta_s}\mathrm{d}y\Big)^2\\ &\le \Big(\int_0^\infty H_s(x)\mathrm{d}x\Big)^2<\infty.\end{aligned}$$

由上述二式及函数的连续性易知:

$$\sup_{x\ge 0}\Big[\frac{\partial\ln F(x,\theta^0)}{\partial\theta_s}\Big]^2F(x,\theta^0)<\infty. \qquad (3.6)$$

同理可证:

$$\sup_{x\geq 0}\Big[\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_s}\Big]^2\overline{F}(x,\theta^0)<\infty. \tag{3.7}$$

其次, 把 $(\partial\ln f(x,\theta^0)/\partial\theta_s)^2$, $(\partial\ln F(x,\theta^0)/\partial\theta_s)^2$, $(\partial\ln\overline{F}(x,\theta^0)/\partial\theta_s)^2$ 分别当作 $T(x)$, 并注意到 $\alpha_i=1$ 或 0, $\beta_i=1$ 或 0. 由引理 1 、引理 2 及 (3.6), (3.7), 条件 (Φ) 中 (3) 知, 存在某常数 $C>0$, 使

$$\begin{aligned}
&D_{\theta^0}\Big(\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_s}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_s}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_s}\Big)\\
=\ &\mathsf{E}_{\theta^0}\Big(\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_s}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_s}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_s}\Big)^2\\
=\ &\mathsf{E}_{\theta^0}\Big(\alpha_i^2\beta_i^2\Big(\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_s}\Big)^2+\alpha_i^2(1-\beta_i)^2\Big(\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_s}\Big)^2+(1-\alpha_i)^2\Big(\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_s}\Big)^2\Big)\\
=\ &\mathsf{E}_{\theta^0}\Big(\alpha_i\beta_i\Big(\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_s}\Big)^2+\alpha_i(1-\beta_i)\Big(\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_s}\Big)^2+(1-\alpha_i)\Big(\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_s}\Big)^2\Big)\\
=\ &p\int_0^\infty\Big[\frac{\partial\ln f(x,\theta^0)}{\partial\theta_s}\Big]^2\overline{G}_i(x)\mathrm{d}F(x,\theta^0)+(1-p)\int_0^\infty\Big[\frac{\partial\ln F(x,\theta^0)}{\partial\theta_s}\Big]^2F(x,\theta^0)\mathrm{d}G_i(x)\\
&+\int_0^\infty\Big[\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_s}\Big]^2\overline{F}(x,\theta^0)\mathrm{d}G_i(x)\\
\leq\ &p\int_0^\infty\Big[\frac{\partial\ln f(x,\theta^0)}{\partial\theta_s}\Big]^2f(x,\theta^0)\mathrm{d}x+(1-p)\sup_{x\geq 0}\Big[\frac{\partial\ln F(x,\theta^0)}{\partial\theta_s}\Big]^2F(x,\theta^0)\\
&+\sup_{x\geq 0}\Big[\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_s}\Big]^2\overline{F}(x,\theta^0)<C<\infty.
\end{aligned}$$

所以, 由柯尔莫哥洛夫强大数定律知, (3.5) 成立.　　#

引理 4　在条件 (Φ) 下, 当 $n\to\infty$ 时, 有

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n\mathsf{E}_{\theta^0}\Big(\alpha_i\beta_i\frac{\partial^2\ln f(Z_i,\theta^0)}{\partial\theta_s\partial\theta_t}+\alpha_i(1-\beta_i)\frac{\partial^2\ln F(Z_i,\theta^0)}{\partial\theta_s\partial\theta_t}+(1-\alpha_i)\frac{\partial^2\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_s\partial\theta_t}\Big)\\
\longrightarrow\ &-\Big[p\int_0^\infty\frac{\partial\ln f(x,\theta^0)}{\partial\theta_s}\frac{\partial\ln f(x,\theta^0)}{\partial\theta_t}\overline{G}_0(x)f(x,\theta^0)\mathrm{d}x\\
&+(1-p)\int_0^\infty\frac{\partial\ln F(x,\theta^0)}{\partial\theta_s}\frac{\partial\ln F(x,\theta^0)}{\partial\theta_t}F(x,\theta^0)\mathrm{d}G_0(x)\\
&+\int_0^\infty\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_s}\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_t}\overline{F}(x,\theta^0)\mathrm{d}G_0(x)\Big].
\end{aligned}\tag{3.8}$$

证明:　把 $\partial^2\ln f(x,\theta^0)/(\partial\theta_s\partial\theta_t)$, $\partial^2\ln F(x,\theta^0)/(\partial\theta_s\partial\theta_t)$, $\partial^2\ln\overline{F}(x,\theta^0)/(\partial\theta_s\partial\theta_t)$ 分别当作 $T(x)$, 由引理 1 、条件 (Φ) 中的 (2), (6) 及文献 [7] 中引理 3 知,

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n\mathsf{E}_{\theta^0}\Big(\alpha_i\beta_i\frac{\partial^2\ln f(Z_i,\theta^0)}{\partial\theta_s\partial\theta_t}+\alpha_i(1-\beta_i)\frac{\partial^2\ln F(Z_i,\theta^0)}{\partial\theta_s\partial\theta_t}+(1-\alpha_i)\frac{\partial^2\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_s\partial\theta_t}\Big)\\
=\ &\frac{1}{n}\sum_{i=1}^n\Big(p\int_0^\infty\frac{\partial^2\ln f(x,\theta^0)}{\partial\theta_s\partial\theta_t}\overline{G}_i(x)\mathrm{d}F(x,\theta^0)+(1-p)\int_0^\infty\frac{\partial^2\ln F(x,\theta^0)}{\partial\theta_s\partial\theta_t}F(x,\theta^0)\mathrm{d}G_i(x)\\
&+\int_0^\infty\frac{\partial^2\ln\overline{F}(x,\theta^0)}{\partial\theta_s\partial\theta_t}\overline{F}(x,\theta^0)\mathrm{d}G_i(x)\Big)\\
=\ &p\int_0^\infty\frac{\partial^2\ln f(x,\theta^0)}{\partial\theta_s\partial\theta_t}\Big(1-\frac{1}{n}\sum_{i=1}^nG_i(x)\Big)f(x,\theta^0)\mathrm{d}x+(1-p)\int_0^\infty\frac{\partial^2\ln F(x,\theta^0)}{\partial\theta_s\partial\theta_t}F(x,\theta^0)\mathrm{d}\Big(\frac{1}{n}\sum_{i=1}^nG_i(x)\Big)\\
&+\int_0^\infty\frac{\partial^2\ln\overline{F}(x,\theta^0)}{\partial\theta_s\partial\theta_t}\overline{F}(x,\theta^0)\mathrm{d}\Big(\frac{1}{n}\sum_{i=1}^nG_i(x)\Big)
\end{aligned}$$

$$
\begin{aligned}
\longrightarrow\ & p\int_0^\infty \frac{\partial^2\ln f(x,\theta^0)}{\partial\theta_s\partial\theta_t}(1-G_0(x))f(x,\theta^0)\mathrm{d}x+(1-p)\int_0^\infty \frac{\partial^2\ln F(x,\theta^0)}{\partial\theta_s\partial\theta_t}F(x,\theta^0)\mathrm{d}G_0(x)\\
&+\int_0^\infty \frac{\partial^2\ln \overline{F}(x,\theta^0)}{\partial\theta_s\partial\theta_t}\overline{F}(x,\theta^0)\mathrm{d}G_0(x)\\
=\ & -\Big[p\int_0^\infty \frac{\partial\ln f(x,\theta^0)}{\partial\theta_s}\frac{\partial\ln f(x,\theta^0)}{\partial\theta_t}\overline{G}_0(x)f(x,\theta^0)\mathrm{d}x\\
&+(1-p)\int_0^\infty \frac{\partial\ln F(x,\theta^0)}{\partial\theta_s}\frac{\partial\ln F(x,\theta^0)}{\partial\theta_t}F(x,\theta^0)\mathrm{d}G_0(x)\\
&+\int_0^\infty \frac{\partial\ln \overline{F}(x,\theta^0)}{\partial\theta_s}\frac{\partial\ln \overline{F}(x,\theta^0)}{\partial\theta_t}\overline{F}(x,\theta_0)\mathrm{d}G_0(x)\Big].
\end{aligned}
$$

上式最后一个等式根据文献 [7] 中引理 3 的证明而得．证毕． #

引理 5 在条件 (Φ) 下，当 $n\to\infty$ 时，有

$$
\mathsf{P}_{\theta^0}\{(Z_1,\alpha_1,\beta_1)=(Z_2,\alpha_2,\beta_2)=\cdots=(Z_n,\alpha_n,\beta_n)\}\longrightarrow 0. \tag{3.9}
$$

证明: 由于 $G_i(x)$ 为分布函数 $(i=1,2,\cdots,n)$, 故 $G_i(x)$ 的间断点最多为可数个，那么 $\{G_i(x)\}$ 的共同间断点也最多为可数个．不妨设 $\{G_i(x)\}$ 的共同间断点为 $\{x_q,\ q\in\widetilde{R},\ \widetilde{R}$ 为可数集$\}$, 则有

$$
\begin{aligned}
&\mathsf{P}_{\theta^0}\{(Z_1,\alpha_1,\beta_1)=(Z_2,\alpha_2,\beta_2)=\cdots=(Z_n,\alpha_n,\beta_n)\}\\
\le\ & \int\limits_{x_1=x_2=\cdots=x_n}\!\!\int\cdots\int p^n\prod_{i=1}^n(1-G_i(x_i))f(x_i,\theta^0)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n\\
&+\int\limits_{x_1=x_2=\cdots=x_n}\!\!\int\cdots\int(1-p)^n\prod_{i=1}^n F(x_i,\theta^0)\mathrm{d}G_1(x_1)\mathrm{d}G_2(x_2)\cdots\mathrm{d}G_n(x_n)\\
&+\int\limits_{x_1=x_2=\cdots=x_n}\!\!\int\cdots\int\prod_{i=1}^n(1-F(x_i,\theta^0))\mathrm{d}G_1(x_1)\mathrm{d}G_2(x_2)\cdots\mathrm{d}G_n(x_n)\\
\le\ & p^n\int\limits_{x_1=x_2=\cdots=x_n}\!\!\int\cdots\int\prod_{i=1}^n f(x_i,\theta^0)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n+(1-p)^n\int\limits_{x_1=x_2=\cdots=x_n}\!\!\int\cdots\int\mathrm{d}G_1(x_1)\mathrm{d}G_2(x_2)\cdots\mathrm{d}G_n(x_n)\\
&+\sum_{l\in\widetilde{R}}(\overline{F}(x_l,\theta^0))^n\prod_{i=1}^n G_i(x_l+0)-G_i(x_l-0)\\
\le\ & (1-p)^n\sum_{l\in\widetilde{R}}\prod_{i=1}^n G_i(x_l+0)-G_i(x_l-0)+\sum_{l\in\widetilde{R}}[\overline{F}(x^*,\theta^0)]^n\prod_{i=1}^n G_i(x_l+0)-G_i(x_l-0)\\
\le\ & (1-p)^n+[\overline{F}(x^*,\theta^0)]^n\longrightarrow 0,
\end{aligned}
$$

其中: $x^*=\min\{x_l,l\in\widetilde{R}\}$. 所以由上述知 (3.9) 成立． #

引理 6 在条件 (Φ) 下，对 $\forall\varepsilon>0$, 当 $N\to\infty$ 时，下式成立

$$
\begin{aligned}
&\mathsf{P}_{\theta^0}\Big\{\sup_{n\ge N}\sup_{\theta\in\mu_{\theta^0}}\Big|\frac1n\sum_{i=1}^n\Big(\alpha_i\beta_i\frac{\partial^2\ln f(Z_i,\theta)}{\partial\theta_s\partial\theta_t}+\alpha_i(1-\beta_i)\frac{\partial^2\ln F(Z_i,\theta)}{\partial\theta_s\partial\theta_t}+(1-\alpha_i)\frac{\partial^2\ln \overline{F}(Z_i,\theta)}{\partial\theta_s\partial\theta_t}\Big)\\
&\qquad-\frac1n\sum_{i=1}^n\mathsf{E}_{\theta^0}\Big(\alpha_i\beta_i\frac{\partial^2\ln f(Z_i,\theta)}{\partial\theta_s\partial\theta_t}+\alpha_i(1-\beta_i)\frac{\partial^2\ln F(Z_i,\theta)}{\partial\theta_s\partial\theta_t}+(1-\alpha_i)\frac{\partial^2\ln \overline{F}(Z_i,\theta)}{\partial\theta_s\partial\theta_t}\Big)\Big|\ge\varepsilon\Big\}\\
&\longrightarrow 0.
\end{aligned}
$$

证明过程请参见文献 [6] 中引理 4 的证明．

§4. 定理的证明

定理 1 的证明: 为证明定理 1, 我们需把 $(1/n)\ln[L(\theta)/L(\theta^0)]$ 表示成积分型余项的 Taylor 展开形式,

其如下

$$\begin{aligned}
\frac{1}{n}\ln\frac{L(\theta)}{L(\theta^0)} &= \frac{1}{n}\Big[\sum_{i=1}^n \alpha_i\beta_i \ln\frac{f(Z_i,\theta)}{f(Z_i,\theta^0)} + \alpha_i(1-\beta_i)\ln\frac{F(Z_i,\theta)}{F(Z_i,\theta^0)} + (1-\alpha_i)\ln\frac{\overline{F}(Z_i,\theta)}{\overline{F}(Z_i,\theta^0)}\Big] \\
&= \frac{1}{n}\sum_{i=1}^n \alpha_i\beta_i\Big(\sum_{s=1}^m(\theta_s-\theta_s^0)\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_s} \\
&\quad +\frac{1}{2}\sum_{s,t=1}^m(\theta_s-\theta_s^0)(\theta_t-\theta_t^0)2\int_0^1(1-u)\frac{\partial^2\ln f(Z_i,\theta^*)}{\partial\theta_s\partial\theta_t}\Big|_{\theta^*=\theta^0+u(\theta-\theta^0)}\mathrm{d}u\Big) \\
&\quad +\frac{1}{n}\sum_{i=1}^n \alpha_i(1-\beta_i)\Big(\sum_{s=1}^m(\theta_s-\theta_s^0)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_s} \\
&\quad +\frac{1}{2}\sum_{s,t=1}^m(\theta_s-\theta_s^0)(\theta_t-\theta_t^0)2\int_0^1(1-u)\frac{\partial^2\ln F(Z_i,\theta^*)}{\partial\theta_s\partial\theta_t}\Big|_{\theta^*=\theta^0+u(\theta-\theta^0)}\mathrm{d}u\Big) \\
&\quad +\frac{1}{n}\sum_{i=1}^n (1-\alpha_i)\Big(\sum_{s=1}^m(\theta_s-\theta_s^0)\frac{\partial\ln \overline{F}(Z_i,\theta^0)}{\partial\theta_s} \\
&\quad +\frac{1}{2}\sum_{s,t=1}^m(\theta_s-\theta_s^0)(\theta_t-\theta_t^0)2\int_0^1(1-u)\frac{\partial^2\ln \overline{F}(Z_i,\theta^*)}{\partial\theta_s\partial\theta_t}\Big|_{\theta^*=\theta^0+u(\theta-\theta^0)}\mathrm{d}u\Big) \\
&= \sum_{s=1}^m(\theta_s-\theta_s^0)\Big(\frac{1}{n}\sum_{i=1}^n\alpha_i\beta_i\frac{\ln f(Z_i,\theta^0)}{\partial\theta_s}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_s}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_s}\Big) \\
&\quad +\frac{1}{2}\sum_{s,t=1}^m(\theta_s-\theta_s^0)(\theta_t-\theta_t^0)2\int_0^1(1-u)\frac{1}{n}\sum_{i=1}^n\Big(\alpha_i\beta_i\frac{\partial^2\ln f(Z_i,\theta^*)}{\partial\theta_s\partial\theta_t} \\
&\quad +\alpha_i(1-\beta_i)\frac{\partial^2\ln F(Z_i,\theta^*)}{\partial\theta_s\partial\theta_t}+(1-\alpha_i)\frac{\partial^2\ln\overline{F}(Z_i,\theta^*)}{\partial\theta_s\partial\theta_t}\Big)\Big|_{\theta^*=\theta^0+u(\theta-\theta^*)}\mathrm{d}u \\
&= (k_1^{(n)},k_2^{(n)},\cdots,k_m^{(n)})(\theta-\theta^0)+\frac{1}{2}(\theta-\theta^0)^TG_n(\theta)(\theta-\theta^0) \\
&= (k_1^{(n)},k_2^{(n)},\cdots,k_m^{(n)})(\theta-\theta^0)+\frac{1}{2}(\theta-\theta^0)^T(G_n(\theta)-G(\theta))(\theta-\theta^0) \\
&\quad +\frac{1}{2}(\theta-\theta^0)^T(G(\theta)-G(\theta^0))(\theta-\theta^0)-\frac{1}{2}(\theta-\theta^0)^T(-G(\theta^0))(\theta-\theta^0).
\end{aligned}\tag{4.1}$$

其中,

$$\begin{aligned}
k_s^{(n)} &= \frac{1}{n}\sum_{i=1}^n\Big(\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_s}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_s}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_s}\Big), \\
G_n(\theta) &= (g_{st}^{(n)}(\theta))_{m\times m}, \\
g_{st}^{(n)}(\theta) &= 2\int_0^1(1-u)\frac{1}{n}\sum_{i=1}^n\Big(\alpha_i\beta_i\frac{\partial^2\ln f(Z_i,\theta^*)}{\partial\theta_s\partial\theta_t}+\alpha_i(1-\beta_i)\frac{\partial^2\ln F(Z_i,\theta^*)}{\partial\theta_s\partial\theta_t} \\
&\quad +(1-\alpha_i)\frac{\partial^2\ln\overline{F}(Z_i,\theta^*)}{\partial\theta_s\partial\theta_t}\Big)\Big|_{\theta^*=\theta^0+u(\theta-\theta^0)}\mathrm{d}u,\qquad s=1,2,\cdots,m,\ t=1,2,\cdots,m,
\end{aligned}\tag{4.2}$$

关于 $G(\theta)$ 的定义见 (2.5).

现在我们进行定理 1 的证明. 首先, 令

$$\begin{aligned}
C_n &= \{\omega:(Z_1,\alpha_1,\beta_1)=(Z_2,\alpha_2,\beta_2)=\cdots=(Z_n,\alpha_n,\beta_n)\}, \\
K_N &= \Big\{\omega:\max_{1\le s,t\le m}\sup_{n\ge N}\sup_{\theta\in\mu_{\theta^0}}\Big|\frac{1}{n}\sum_{i=1}^n\Big(\alpha_i\beta_i\frac{\partial^2\ln f(Z_i,\theta)}{\partial\theta_s\partial\theta_t}+\alpha_i(1-\beta_i)\frac{\partial^2\ln F(Z_i,\theta)}{\partial\theta_s\partial\theta_t}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta)}{\partial\theta_s\partial\theta_t}\Big) \\
&\quad -\frac{1}{n}\sum_{i=1}^n\mathsf{E}\Big(\alpha_i\beta_i\frac{\partial^2\ln f(Z_i,\theta)}{\partial\theta_s\partial\theta_t}+\alpha_i(1-\beta_i)\frac{\partial^2\ln F(Z_i,\theta)}{\partial\theta_s\partial\theta_t}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta)}{\partial\theta_s\partial\theta_t}\Big)\Big|<\varepsilon\Big\}.
\end{aligned}$$

则由引理 5 及引理 6 知

$$\lim_{n\to\infty}\mathsf{P}_{\theta^0}(C_n)=0,\qquad \lim_{N\to\infty}\mathsf{P}_{\theta^0}(K_N)=1.\tag{4.3}$$

所以，对 $\forall\varepsilon>0$. 当 $\omega\in K_N\cap C_N^c$ 且当 N 充分大时，有

$$\|G_n(\theta)-G(\theta)\|\le\varepsilon. \tag{4.4}$$

又因为 $G(\theta)$ 为 θ 的连续函数，故对 $\forall\varepsilon>0$, 总存在 $\zeta=\zeta(\varepsilon)>0$ 使得当 $\|\theta-\theta^0\|\le\zeta$ $(\zeta\le\eta_{\theta^0})$ 时，有

$$\|G(\theta)-G(\theta^0)\|\le\varepsilon. \tag{4.5}$$

其次，我们来证明 $-G(\theta^0)$ 的正定性．由条件 (Φ) 中 (5) 知，对 $\forall X=(x_1,x_2,\cdots,x_m)^T\neq 0$, 有

$$\begin{aligned}X^T(-G(\theta^0))X &= p\sum_{s,t=1}^m x_sx_t\int_0^\infty\frac{\partial\ln f(x,\theta^0)}{\partial\theta_s}\frac{\partial\ln f(x,\theta^0)}{\partial\theta_t}\overline{G}_0(x)f(x,\theta^0)\mathrm{d}x\\&\quad+(1-p)\sum_{s,t=1}^m x_sx_t\int_0^\infty\frac{\partial\ln F(x,\theta^0)}{\partial\theta_s}\frac{\partial\ln F(x,\theta^0)}{\partial\theta_t}F(x,\theta^0)\mathrm{d}G_0(x)\\&\quad+\sum_{s,t=1}^m x_sx_t\int_0^\infty\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_s}\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_t}\overline{F}(x,\theta^0)\mathrm{d}G_0(x)\\&= p\int_0^\infty\Big(\sum_{s=1}^m x_s\frac{\partial\ln f(x,\theta^0)}{\partial\theta_s}\Big)^2\overline{G}_0(x)f(x,\theta^0)\mathrm{d}x\\&\quad+(1-p)\int_0^\infty\Big(\sum_{s=1}^m x_s\frac{\partial\ln F(x,\theta^0)}{\partial\theta_s}\Big)^2F(x,\theta^0)\mathrm{d}G_0(x)\\&\quad+\int_0^\infty\Big(\sum_{s=1}^m x_s\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_s}\Big)^2\overline{F}(x,\theta^0)\mathrm{d}G_0(x)\\&> 0,\end{aligned} \tag{4.6}$$

于是， $-G(\theta^0)$ 的正定性得证．另外，由引理 3 不难看出，对 $\forall\delta\in(0,\zeta)$, 有

$$\lim_{N\to\infty}\mathsf{P}_{\theta^0}(\Delta_N)=1,\qquad \Delta_N=\Big\{\omega:\max_{1\le s\le m}\sup_{n\ge N}|k_s^{(n)}|<\frac{\lambda_{\min}\delta}{8}\Big\}, \tag{4.7}$$

这儿 $\lambda_{\min}$ 为 $-G(\theta^0)$ 的最小特征值．由 (4.6) 知 $\lambda_{\min}>0$. 最后，由 (4.1), (4.3), (4.4), (4.5), (4.7) 知，取 $\varepsilon<\lambda_{\min}/4$, 对 $\forall\delta\in(0,\zeta)$, 当 $\omega\in C_N^c\cap K_N\cap\Delta_N$, 且 N 充分大， $\theta\in\{\theta:\|\theta-\theta^0\|=\delta\}$ 时，有

$$\begin{aligned}\frac{1}{n}\ln\frac{L(\theta)}{L(\theta^0)} &\le \|\theta-\theta^0\|\max_{1\le s\le m}|k_s^{(n)}|+\frac12\|\theta-\theta^0\|^2\|G_n(\theta)-G(\theta)\|\\&\quad+\frac12\|\theta-\theta^0\|^2\|G(\theta)-G(\theta^0)\|-\frac12\lambda_{\min}\|\theta-\theta^0\|^2\\&\le \frac{\lambda_{\min}\delta^2}{8}+\frac12\delta^2\varepsilon+\frac12\delta^2\varepsilon-\frac12\delta^2\lambda_{\min}\le-\frac{\lambda_{min}\delta^2}{8}<0,\end{aligned} \tag{4.8}$$

需要特别指出的是， (4.8) 对 $\forall\delta\in(0,\zeta(\varepsilon))$ 成立．上式 (4.8) 表明当 $\theta\in\{\theta:\|\theta-\theta^0\|=\delta\}$ 且 N 充分大时，有 $(1/n)\ln[L(\theta)/L(\theta^0)]<0$. 又因为 $(1/n)\ln[L(\theta)/L(\theta^0)]$ 为 θ 的连续函数且 $(1/n)\ln[L(\theta^0)/L(\theta^0)]=0$, 所以 $(1/n)\ln[L(\theta)/L(\theta^0)]$ 在 $\{\theta:\|\theta-\theta^0\|\le\delta\}$ 上有极大值点 $\widetilde{\theta}^{(n)}$ 且 $\|\widetilde{\theta}^{(n)}-\theta^0\|<\delta$. 但是 $\widehat{\theta}^{(n)}$ 为分布参数的 MLE, 即： $\ln L(\widehat{\theta}^{(n)})$ 为极大值，又由似然函数正规 (见定义 2) 知，必有 $\widehat{\theta}^{(n)}=\widetilde{\theta}^{(n)}$. 所以 $\|\widehat{\theta}^{(n)}-\theta^0\|<\delta$ 成立．即当 $\omega\in C_N^c\cap R_N\cap\Delta_N$ 且 N 充分大时，

$$\sup_{n\ge N}\|\widehat{\theta}^{(n)}-\theta^0\|<\delta.$$

由 δ 的任意性 (取 $\delta\to0$) 及 (4.3), (4.7) 知， $\mathsf{P}_{\theta^0}\Big\{\omega:\lim_{n\to\infty}\widehat{\theta}^{(n)}=\theta^0\Big\}=1$. 证毕． #

定理 2 的证明: 为证定理 2 需首先证明下式，

$$\frac{1}{\sqrt{n}}\frac{\partial\ln L(\theta)}{\partial\theta}\Big|_{\theta=\theta^0}\xrightarrow{d}N(0,-G(\theta^0)). \tag{4.9}$$

要证 (4.9), 只需证对任意非零向量 $C=(C_1,C_2,\cdots,C_m)^T$, 有

$$C^T\frac{1}{\sqrt{n}}\frac{\partial\ln L(\theta)}{\partial\theta}\Big|_{\theta=\theta^0}\xrightarrow{d}N(0,C^T(-G(\theta^0))C). \tag{4.10}$$

现开始证明 (4.10) 成立. 记

$$a_i=\sum_{k=1}^m C_k\Big[\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_k}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_k}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_k}\Big],$$

则由 (2.1) 可知,

$$C^T\frac{1}{\sqrt{n}}\frac{\partial\ln L(\theta)}{\partial\theta}\Big|_{\theta=\theta^0}=\frac{1}{\sqrt{n}}\sum_{i=1}^n a_i.$$

首先, 把 $\frac{\partial\ln f(x,\theta^0)}{\partial\theta_k}\frac{\partial\ln f(x,\theta^0)}{\partial\theta_l}$, $\frac{\partial\ln F(x,\theta^0)}{\partial\theta_k}\frac{\partial\ln F(x,\theta^0)}{\partial\theta_l}$, $\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_k}\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_l}$ 分别当作 $T(x)$, 由引理 1 及引理 2 可知,

$$\begin{aligned}
B_n^2 &\triangleq D\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^n a_i\Big)=\frac{1}{n}\sum_{i=1}^n D(a_i)\\
&=\frac{1}{n}\sum_{i=1}^n \mathsf{E}\Big(\sum_{k=1}^m C_k\Big[\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_k}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_k}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_k}\Big]\Big)^2\\
&=\frac{1}{n}\sum_{i=1}^n \mathsf{E}\sum_{k,l=1}^m C_kC_l\Big[\alpha_i^2\beta_i^2\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_k}\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_l}\\
&\quad+\alpha_i^2(1-\beta_i)^2\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_k}\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_l}+(1-\alpha_i)^2\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_k}\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_l}\Big]\\
&=\frac{1}{n}\sum_{i=1}^n\sum_{k,l=1}^m C_kC_l\Big[p\int_0^\infty\frac{\partial\ln f(x,\theta^0)}{\partial\theta_k}\frac{\partial\ln f(x,\theta^0)}{\partial\theta_l}\overline{G}_i(x)\mathrm{d}F(x,\theta^0)\\
&\quad+(1-p)\int_0^\infty\frac{\partial\ln F(x,\theta^0)}{\partial\theta_k}\frac{\partial\ln F(x,\theta^0)}{\partial\theta_l}F(x,\theta^0)\mathrm{d}G_i(x)\\
&\quad+\int_0^\infty\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_k}\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_l}\overline{F}(x,\theta^0)\mathrm{d}G_i(x)\Big]\\
&=\sum_{k,l=1}^m C_kC_l\Big[p\int_0^\infty\frac{\partial\ln f(x,\theta^0)}{\partial\theta_k}\frac{\partial\ln f(x,\theta^0)}{\partial\theta_l}f(x,\theta^0)\Big(1-\frac{1}{n}\sum_{i=1}^n G_i(x)\Big)\mathrm{d}x\\
&\quad+(1-p)\int_0^\infty\frac{\partial\ln F(x,\theta^0)}{\partial\theta_k}\frac{\partial\ln F(x,\theta^0)}{\partial\theta_l}F(x,\theta^0)\mathrm{d}\Big(\frac{1}{n}\sum_{i=1}^n G_i(x)\Big)\\
&\quad+\int_0^\infty\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_k}\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_l}\overline{F}(x,\theta^0)\mathrm{d}\Big(\frac{1}{n}\sum_{i=1}^n G_i(x)\Big)\Big]\\
&=C^TC^{(n)}C,
\end{aligned} \tag{4.11}$$

其中, $C^{(n)}=(C_{st}^{(n)})_{m\times m}$,

$$\begin{aligned}
C_{st}^{(n)} &= p\int_0^\infty\frac{\partial\ln f(x,\theta^0)}{\partial\theta_s}\frac{\partial\ln f(x,\theta^0)}{\partial\theta_t}f(x,\theta^0)\Big(1-\frac{1}{n}\sum_{i=1}^n G_i(x)\Big)\mathrm{d}x\\
&\quad+(1-p)\int_0^\infty\frac{\partial\ln F(x,\theta^0)}{\partial\theta_s}\frac{\partial\ln F(x,\theta^0)}{\partial\theta_t}F(x,\theta^0)\mathrm{d}\Big(\frac{1}{n}\sum_{i=1}^n G_i(x)\Big)\\
&\quad+\int_0^\infty\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_s}\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_t}\overline{F}(x,\theta^0)\mathrm{d}\Big(\frac{1}{n}\sum_{i=1}^n G_i(x)\Big).
\end{aligned}$$

又由条件 (Φ) 中 (2), (3), (6) 知, $\lim\limits_{n\to\infty}C^{(n)}=-G(\theta^0)$, 所以由 (4.11) 知,

$$\lim_{n\to\infty}B_n^2=C^T(-G(\theta^0))C>0. \tag{4.12}$$

其次，由条件 (Φ) 中的 (3) 知，$\sup\limits_{x\ge0}(\partial\ln\overline{F}(x,\theta^0)/\partial\theta_k)^4\overline{F}(x,\theta^0)$, $\sup\limits_{x\ge0}(\partial\ln F(x,\theta^0)/\partial\theta_k)^4F(x,\theta^0)$ 有界，故

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^n\mathsf{E}|a_i|^4\\
\le\ &\max_{1\le k\le m}|C_k|^4\frac{1}{n^2}\sum_{i=1}^n\mathsf{E}\Big|\sum_{k=1}^m\Big(\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_k}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_k}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_k}\Big)\Big|^4\\
\le\ &\max_{1\le k\le m}|C_k|^4\frac{m^3}{n^2}\sum_{i=1}^n\sum_{k=1}^m\mathsf{E}\Big(\alpha_i\beta_i\frac{\partial\ln f(Z_i,\theta^0)}{\partial\theta_k}+\alpha_i(1-\beta_i)\frac{\partial\ln F(Z_i,\theta^0)}{\partial\theta_k}+(1-\alpha_i)\frac{\partial\ln\overline{F}(Z_i,\theta^0)}{\partial\theta_k}\Big)^4\\
=\ &m^3\max_{1\le k\le m}|C_k|^4\frac{1}{n^2}\sum_{i=1}^n\sum_{k=1}^m\Big[p\int_0^\infty\Big(\frac{\partial\ln f(x,\theta^0)}{\partial\theta_k}\Big)^4f(x,\theta^0)\overline{G}_i(x)\mathrm{d}x\\
&+(1-p)\int_0^\infty\Big(\frac{\partial\ln F(x,\theta^0)}{\partial\theta_k}\Big)^4F(x,\theta^0)\mathrm{d}G_i(x)+\int_0^\infty\Big(\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_k}\Big)^4\overline{F}(x,\theta^0)\mathrm{d}G_i(x)\Big]\\
=\ &\frac{m^3}{n}\max_{1\le k\le m}|C_k|^4\sum_{k=1}^m\Big[p\int_0^\infty\Big(\frac{\partial\ln f(x,\theta^0)}{\partial\theta_k}\Big)^4f(x,\theta^0)\frac{1}{n}\sum_{i=1}^n\overline{G}_i(x)\mathrm{d}x\\
&+(1-p)\int_0^\infty\Big(\frac{\partial\ln F(x,\theta^0)}{\partial\theta_k}\Big)^4F(x,\theta^0)\mathrm{d}\Big(\frac{1}{n}\sum_{i=1}^nG_i(x)\Big)\\
&+\int_0^\infty\Big(\frac{\partial\ln\overline{F}(x,\theta^0)}{\partial\theta_k}\Big)^4\overline{F}(x,\theta^0)\mathrm{d}\Big(\frac{1}{n}\sum_{i=1}^nG_i(x)\Big)\Big]\\
\stackrel{n\to\infty}{\longrightarrow}\ &0.
\end{aligned}$$

所以由上式和 (4.12) 知，

$$\frac{1}{n^2B_n^4}\sum_{i=1}^n\mathsf{E}|a_i|^4\stackrel{n\to\infty}{\longrightarrow}0.\tag{4.13}$$

因此，由 (4.12), (4.13) 根据李雅普洛夫定理知，(4.10) 成立，即 (4.9) 成立．

接下来，我们开始证明定理 2 的结果，(2.4) 成立．显然，有

$$\frac{\partial\ln L(\theta)}{\partial\theta}=\frac{\partial\ln L(\theta)}{\partial\theta}\Big|_{\theta=\theta^0}+\int_0^1\frac{\mathrm{d}}{\mathrm{d}u}\Big(\frac{\partial\ln L(\theta^0+u(\theta-\theta^0))}{\partial\theta}\Big)\mathrm{d}u,$$

在上式中，以 $\widehat{\theta}^{(n)}$ 代替 θ 得

$$\begin{aligned}
\frac{\partial\ln L(\theta)}{\partial\theta}\Big|_{\theta=\theta^0}&=\frac{\partial\ln L(\theta)}{\partial\theta}\Big|_{\theta=\widehat{\theta}^{(n)}}-\int_0^1\frac{\mathrm{d}}{\mathrm{d}u}\Big(\frac{\partial\ln L(\theta^0+u(\widehat{\theta}^{(n)}-\theta^0))}{\partial\theta}\Big)\mathrm{d}u\\
&=-\int_0^1\frac{\mathrm{d}}{\mathrm{d}u}\Big(\frac{\partial\ln L(\theta^0+u(\widehat{\theta}^{(n)}-\theta^0))}{\partial\theta}\Big)\mathrm{d}u\\
&=n\cdot D_n\cdot(\widehat{\theta}^{(n)}-\theta^0).
\end{aligned}\tag{4.14}$$

其中：$D_n=(d_{st}^{(n)})_{m\times m}$,

$$\begin{aligned}
d_{st}^{(n)}=&-\int_0^1\frac{1}{n}\sum_{i=1}^n\Big(\alpha_i\beta_i\frac{\partial^2\ln f(Z_i,\theta^0+u(\widehat{\theta}^{(n)}-\theta^0))}{\partial\theta_s\partial\theta_t}+\alpha_i(1-\beta_i)\frac{\partial^2\ln F(Z_i,\theta^0+u(\widehat{\theta}^{(n)}-\theta^0))}{\partial\theta_s\partial\theta_t}\\
&+(1-\alpha_i)\frac{\partial^2\ln\overline{F}(Z_i,\theta^0+u(\widehat{\theta}^{(n)}-\theta^0))}{\partial\theta_s\partial\theta_t}\Big)\mathrm{d}u.
\end{aligned}$$

在 (4.14) 两边同乘 $1/\sqrt{n}$ 得

$$\frac{1}{\sqrt{n}}\frac{\partial\ln L(\theta)}{\partial\theta}\Big|_{\theta=\theta^0}=\sqrt{n}D_n(\widehat{\theta}^{(n)}-\theta^0).\tag{4.15}$$

另外，又由引理 4 及引理 6 可类知，

$$\sup_{\theta\in\mu_{\theta^0}}\Big|\frac{1}{n}\sum_{i=1}^{n}\alpha_i\beta_i\frac{\partial^2\ln f(Z_i,\theta)}{\partial\theta_s\partial\theta_t}+\alpha_i(1-\beta_i)\frac{\partial^2\ln F(Z_i,\theta)}{\partial\theta_s\partial\theta_t}+(1-\alpha_i)\frac{\partial^2\ln\overline{F}(Z_i,\theta)}{\partial\theta_s\partial\theta_t} -\frac{1}{n}\sum_{i=1}^{n}\mathsf{E}\Big(\alpha_i\beta_i\frac{\partial^2\ln f(Z_i,\theta)}{\partial\theta_s\partial\theta_t}+\alpha_i(1-\beta_i)\frac{\partial^2\ln F(Z_i,\theta)}{\partial\theta_s\partial\theta_t}+(1-\alpha_i)\frac{\partial^2\ln\overline{F}(Z_i,\theta)}{\partial\theta_s\partial\theta_t}\Big)\Big| \xrightarrow{\text{a.e.}} 0. \tag{4.16}$$

$$\frac{1}{n}\sum_{i=1}^{n}\mathsf{E}\Big(\alpha_i\beta_i\frac{\partial^2\ln f(Z_i,\theta)}{\partial\theta_s\partial\theta_t}+\alpha_i(1-\beta_i)\frac{\partial^2\ln F(Z_i,\theta)}{\partial\theta_s\partial\theta_t}+(1-\alpha_i)\frac{\partial^2\ln\overline{F}(Z_i,\theta)}{\partial\theta_s\partial\theta_t}\Big)\overset{n\to\infty}{\longrightarrow} g_{st}(\theta). \tag{4.17}$$

所以由 (4.16), (4.17) 及定理 1 知，当 $n\to\infty$ 时，

$$D_n \overset{\text{a.e.}}{\to} -G(\theta^0). \tag{4.18}$$

最后，由 (4.9), (4.15) 和 (4.18) 我们马上就得到，

$$\sqrt{n}(\widehat{\theta}^{(n)}-\theta^0)=D_n^{-1}\Big(\frac{1}{\sqrt{n}}\frac{\partial\ln L(\theta)}{\partial\theta}\Big|_{\theta=\theta^0}\Big)\xrightarrow{d} N(0,-G^{-1}(\theta^0)).$$

证毕．　#

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Consistency and Asymptotic Normality of MLE for Random Censoring Model with Incomplete Information

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In this paper, We prove that MLE for life distributed parameter is strongly consistent and asymptotically normally distributed. At the same time, we verify that Weibull distribution and lognormal distribution are statified with conditions (Φ) proposed here, showing that conditions (Φ) are widely applicable.