应用概率统计 第十九卷
第一期 2003 年 2 月

Chinese Journal of Applied Probability
and Statistics Vol. 19 No. 1 Feb. 2003

Convergence Theorems for Set-valued Conditional Expectations

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Abstract

In this paper, a representation theorem is given for the weak limit of a sequence of sets in a Banach space. We obtain Fatou's lemmas and dominated convergence theorems for set-valued conditional expectations with an increasing sequence of σ-fields.

Keywords: Measurable multifunction, set-valued conditional expectation, weak limit, Fatou's lemma.

AMS Subject Classification: 60G05.

§ 1. Introduction

Since the concept of set-valued conditional expectation was introduced by Hiai F. and Umegak H. in 1977 [1], various convergence results have been studied by many authors [1-4]. In 1984, Pucci P. and Vitillaro G. [5] defined a new type of weak lower and upper limit of a sequence of nonempty subsets in Banach space. They give two Fatou's lemmas and a dominated convergence theorem for Aumman integrals. Later Wang Jianhua [6] obtained some convergence results for integrable selections in the sense of this weak limit. In [7], we give a representation theorem for the set-valued conditional expectation. With this result, some Fatou's lemmas and dominated convergence theorems were obtained. All these results were limited to the case of set-valued conditional expectation with a fixed σ-field.

In this paper, we continue the work of this field. After giving a representation theorem for the weak limit of a sequence of subsets in Banach space, we obtain new versions of Fatou's lemmas and dominated convergence theorems for set-valued conditional expectations with an increasing sequence of σ-fields. Finally, we give a convergence result for integrable selections. These theorems extend the earlier results.

Throughout this paper, let X be a real separable Banach space with dual X^*, D^* is the countably dense subset of X^* with respect to the Mackey topology $M(X^*, X)$. We will be using the following notations:

$\mathsf{P}_0(X) = \{A \subseteq X : A \text{ is a nonempty subset}\}$;

$\mathsf{P}_{(b)f(c)}(X) = \{A \subseteq X : A \text{ is a nonempty (bounded) closed (convex) subset}\}$;

$\mathsf{P}_{wk(c)}(X) = \{A \subseteq X : A \text{ is a nonempty weak compact (convex) subset}\}$.

For $A \in \mathsf{P}_f(X)$, the support function of A is defined by

$$S(x^*, A) = \sup\{(x^*, x) : x \in A\}, \qquad x^* \in X^*.$$

The norm of A is: $|A| = \sup\{\|x\| : x \in A\}$.

Received 2000. 12. 14. Revised 2001. 11. 19.

§2. Weak Limit

Let $\{A_n\}_{n\geq 1}\subset \mathsf{P}_0(X)$, following Pucci [5], we define:

$$\underline{\lim_n} A_n = \{x\in X : (x^*,x)\leq \underline{\lim_n} S(x^*,A_n),\ \forall\, x^*\in X^*\},$$
$$\overline{\lim_n} A_n = \{x\in X : (x^*,x)\leq \overline{\lim_n} S(x^*,A_n),\ \forall\, x^*\in X^*\}.$$

It is clear that $\underline{\lim}_n A_n$ and $\overline{\lim}_n A_n$ are two closed convex subsets of X, and in general $\underline{\lim}_n A_n\subseteq \overline{\lim}_n A_n$. We say that $\{A_n\}_{n\geq 1}$ is weakly convergent to A, denoted by $\lim_n A_n = A$, if and only if $\underline{\lim}_n A_n = \overline{\lim}_n A_n = A$.

The following lemma is well known which is useful in this paper.

Lemma 2.1 Let $\{a_n\}_{n\geq 1}$ be a bounded real sequence, then $\underline{\lim}_n a_n = \inf_H \sup_{m\in H} a_m$, where the infimum is taken over all cofinal subsets H of $\{1,2,\cdots,n,\cdots\}$.

Theorem 2.2 Let $\{A_n\}_{n\geq 1}\subset \mathsf{P}_0(X)$, we have

(1) $\overline{\lim}_n A_n = \bigcap_{n=1}^{\infty}\Big(\overline{co}\bigcup_{n=m}^{\infty} A_m\Big)$;

(2) If $\{A_n\}_{n\geq 1}$ is uniformly bounded, i.e. there exists $A\in \mathsf{P}_b(X)$ such that $A_n\subseteq A$, $\forall\ n\geq 1$, then $\underline{\lim}_n A_n = \bigcap_H\Big(\overline{co}\bigcup_{m\in H} A_m\Big)$, where H is the same as in Lemma 2.1.

Proof (1) See [7].

(2) Let $x\in \underline{\lim}_n A_n$, for every $x^*\in X^*$ and $H=\{n_1,n_2,\cdots\}$, we have

$$\begin{aligned}(x^*,x) &\leq \underline{\lim_n} S(x^*,A_n)\leq \underline{\lim_i} S(x^*,A_{n_i})\leq \sup_i S(x^*,A_{n_i})\\ &= \sup_{m\in H} S(x^*,A_m) = S\Big(x^*,\overline{co}\bigcup_{m\in H}A_m\Big).\end{aligned}$$

Hence it follows by the separation theorem that $x\in \overline{co}\bigcup_{m\in H}A_m$, and so

$$x\in\bigcap_H\Big(\overline{co}\bigcup_{m\in H}A_m\Big). \tag{2.1}$$

Conversely, let $x\in\bigcap_H\Big(\overline{co}\bigcup_{m\in H}A_m\Big)$, then $x\in\overline{co}\bigcup_{m\in H}A_m$, for all H, so

$$(x^*,x)\leq S\Big(x^*,\overline{co}\bigcup_{m\in H}A_m\Big) = \sup_{m\in H} S(x^*,A_m),\qquad x^*\in X^*. \tag{2.2}$$

It follows from the arbitrariness of H and Lemma 2.1 that

$$(x^*,x)\leq \inf_H\sup_{m\in H} S(x^*,A_m) = \underline{\lim_m} S(x^*,A_m),\qquad x^*\in X^*. \tag{2.3}$$

Therefore

$$x\in \underline{\lim_n} A_n. \tag{2.4}$$

By (2.1) and (2.4), we conclude that

$$\underline{\lim_n} A_n = \bigcap_H\Big(\overline{co}\bigcup_{m\in H}A_m\Big).\qquad \#$$

Remark 2.3 In [7, Theorem 2], we proved that if $\{A_n\}_{n\geq 1}\subset \mathsf{P}_0(X)$ and $A_n\subseteq G$, $n\geq 1$, with $G\in \mathsf{P}_{wkc}(X)$, then

$$\overline{\lim_n} S(x^*,A_n) = S\big(x^*,\overline{\lim_n} A_n\big),\qquad x^*\in X^*. \tag{2.5}$$

Under the same assumptions and using the same method of the proof of the theorem, we can prove that

$$\underline{\lim_n} S(x^*,A_n) = S\big(x^*,\underline{\lim_n} A_n\big),\qquad x^*\in X^*. \tag{2.6}$$

§ 3. Convergence Theorems

Let $(\Omega, \Sigma, \mathsf{P})$ be a complete finite measure space, $\{\Sigma_n\}_{n\geq 1}$ an increasing sequence of complete sub-σ-fields of Σ. We denote by $\Sigma_\infty = \sigma\Big(\bigvee\limits_{n=1}^{\infty} \Sigma_n\Big)$.

The mapping $F : \Omega \to \mathsf{P}_f(X)$ is called a Σ-measurable multifunction, or shortly a random set, if for any open subset $O \subset X$, $\{\omega \in \Omega : F(\omega) \cap O \neq \phi\} \in \Sigma$. For details we refer the reader to [8].

Lemma 3.1 If $F_n : \Omega \to \mathsf{P}_f(X)$ is a sequence of random sets, then

(1) $\overline{\lim\limits_n} F_n$ is measurable.

(2) If we assume further that $F_n \subset G$, $n \geq 1$, where G is a random set taking the values in $\mathsf{P}_{wkc}(X)$, then $\underline{\lim\limits_n} F_n$ is measurable.

Proof (1) By Theorem 2.2 (1), we have $\overline{\lim\limits_n} F_n = \bigcap\limits_{n=1}^{\infty} \Big(\overline{co} \bigcup\limits_{m=n}^{\infty} F_m\Big)$. Thus the measurability of $\overline{\lim\limits_n} F_n$ was followed immediately from [8, Theorem 2.1.11, Corollary 2.1.3 and Theorem 2.1.13].

(2) We know from [8, Corollary 2.1.1] that $S(x^*, F_n)$ is measurable, then we have by (2.6) that $S\big(x^*, \underline{\lim\limits_n} F_n\big)$ is measurable. Notice that $\underline{\lim\limits_n} F_n$ takes values in $\mathsf{P}_{wkc}(X)$, we can conclude from [8, Theorem 2.1.19] that $\underline{\lim\limits_n} F_n$ is measurable. #

Lemma 3.2 If $\{x_n\}_{n\geq 1}$ is a sequence of random variables whose absolute values are dominated by an integrable r.v. y almost surely, i.e., $|x_n| \leq y$, a.s., then

(1) $\overline{\lim\limits_n} \mathsf{E}(x_n|\Sigma_n) \leq \mathsf{E}\big(\overline{\lim\limits_n} x_n|\Sigma_\infty\big)$, a.s..

(2) $\underline{\lim\limits_n} \mathsf{E}(x_n|\Sigma_n) \geq \mathsf{E}\big(\underline{\lim\limits_n} x_n|\Sigma_\infty\big)$, a.s..

(3) If $\lim\limits_n x_n = x$, then $\lim\limits_n \mathsf{E}(x_n|\Sigma_n) = \mathsf{E}(x|\Sigma_\infty)$, a.s..

Proof (1) Let $u_n(\omega) = \sup\limits_{m\geq n} x_m(\omega)$, it is easy to see that $|u_n| \leq g$, a.s., and $u_n \to \overline{\lim\limits_n} x_n$, a.s., as $n \to \infty$, of course, $|\overline{\lim\limits_n} x_n| \leq g$, a.s..

For any $n \geq m$, it is evidently that $\mathsf{E}(x_n|\Sigma_n) \leq \mathsf{E}(u_m|\Sigma_n)$. Since $\{\mathsf{E}(u_m|\Sigma_n)\}_{n\geq 1}$ is a martingale right closed by the integrable r.v. u_m, by [9, Corollary 2.19], we have

$$\overline{\lim_n} \mathsf{E}(x_n|\Sigma_n) \leq \lim_n \mathsf{E}(u_m|\Sigma_n) = \mathsf{E}(u_m|\Sigma_\infty), \qquad \forall\, m \geq 1.$$

So using Fatou's lemma for the real valued conditional expectation, we conclude that

$$\overline{\lim_n} \mathsf{E}(x_n|\Sigma_n) \leq \overline{\lim_m} \mathsf{E}(u_m|\Sigma_\infty) \leq \mathsf{E}\big(\overline{\lim_m} u_m|\Sigma_\infty\big) = \mathsf{E}\big(\overline{\lim_n} x_n|\Sigma_\infty\big).$$

(2) It is similar to the proof of (1).

(3) It follows immediately from (1) and (2). #

Lemma 3.3 Let $F_n : \Omega \to \mathsf{P}_f(X)$, $n \geq 1$, be a sequence of random sets. If there exists an integrably bounded random set $G : \Omega \to \mathsf{P}_{wkc}(X)$ such that $F_n \subseteq G$, a.s., then

(1) $\overline{\lim\limits_n} \mathsf{E}(F_n|\Sigma_n) \subseteq \mathsf{E}(G|\Sigma_\infty)$, a.s..

(2) $\underline{\lim\limits_n} \mathsf{E}(F_n|\Sigma_n) \subseteq \mathsf{E}(G|\Sigma_\infty)$, a.s..

Proof (1) By the martingale convergence theorem and [8, theorem 2.4.18], we have

$$\begin{aligned}
\overline{\lim_n} S(x^*, \mathsf{E}(F_n|\Sigma_n)) &\leq \lim_n S(x^*, \mathsf{E}(G|\Sigma_n)) \\
&= \lim_n \mathsf{E}(S(x^*, G)|\Sigma_n) \\
&= \mathsf{E}(S(x^*, G)|\Sigma_\infty), \qquad \text{a.s.}.
\end{aligned}$$

From the representation theorem of multivalued conditional expectation [7, Theorem 4], we have

$$\begin{aligned}\overline{\lim_n}\mathsf{E}(F_n|\Sigma_n) &= \bigcap_{k=1}^{\infty}\{x\in X:(x^*,x)\le\overline{\lim_n}S(x_k^*,\mathsf{E}(F_n|\Sigma_n))\}\\ &\subseteq \bigcap_{k=1}^{\infty}\{x\in X:(x^*,x)\le\mathsf{E}(S(x_k^*,G)|\Sigma_\infty)\}=\mathsf{E}(G|\Sigma_\infty), \qquad \text{a.s..}\end{aligned}$$

(2) It is similar to the proof of (1). #

Theorem 3.4 Let $F_n:\Omega\to\mathsf{P}_f(X)$, $n\ge 1$, be a sequence of random sets. If $F_n\subseteq G$, a.s., where $G:\Omega\to\mathsf{P}_{wkc}(X)$ is an integrably bounded random set, then

(1) $\overline{\lim_n}\mathsf{E}(F_n|\Sigma_n)\subseteq\mathsf{E}(\overline{\lim_n}F_n|\Sigma_\infty)$, a.s..

(2) $\underline{\lim_n}\mathsf{E}(F_n|\Sigma_n)\supseteq\mathsf{E}(\underline{\lim_n}F_n|\Sigma_\infty)$, a.s..

Proof We know from Lemma 3.1 that $\overline{\lim_n}F_n$ and $\underline{\lim_n}F_n$ are measurable.

(1) For any $x^*\in X^*$, using (2.5), Lemma 3.2 and [8, Theorem 2.4.18], we have

$$\begin{aligned}S(x^*,\overline{\lim_n}\mathsf{E}(F_n|\Sigma_n)) &= \overline{\lim_n}S(x^*,\mathsf{E}(F_n|\Sigma_n))\\ &= \overline{\lim_n}\mathsf{E}(S(x^*,F_n)|\Sigma_n)\\ &\le \mathsf{E}(\overline{\lim_n}S(x^*,F_n)|\Sigma_\infty)\\ &= \mathsf{E}(S(x^*,\overline{\lim_n}F_n)|\Sigma_\infty)\\ &= S(x^*,\mathsf{E}(\overline{\lim_n}F_n|\Sigma_\infty)), \qquad \text{a.s..}\end{aligned}$$

There exists $N_1\in\Sigma$ with $\mathsf{P}(N_1)=0$ such that the above inequality holds for all $x_k^*\in D^*$ and all $\omega\in\Omega\backslash N_1$. On the other hand, there exists $N_2\in\Sigma$ with $\mathsf{P}(N_2)=0$ such that for all $\omega\in\Omega\backslash N_2$, $\overline{\lim_n}\mathsf{E}(F_n|\Sigma_n)(\omega)\subseteq\mathsf{E}(G|\Sigma_\infty)(\omega)$, which implies that

$$\overline{\lim_n}\mathsf{E}(F_n|\Sigma_n)(\omega)\in\mathsf{P}_{wkc}(X), \qquad \omega\in\Omega\backslash N_2.$$

It is easy to see that $\mathsf{E}(\overline{\lim_n}F_n|\Sigma_\infty)(\omega)\subseteq\mathsf{E}(G|\Sigma_\infty)(\omega)$, a.s., then there exists $N_3\in\Sigma$ with $\mathsf{P}(N_3)=0$ such that $\mathsf{E}(\overline{\lim_n}F_n|\Sigma_\infty)(\omega)\in\mathsf{P}_{wkc}(X)$ for all $\omega\in\Omega\backslash N_3$. Let $N=N_1\cup N_2\cup N_3$, evidently, $\mathsf{P}(N)=0$, then it follows from [7, Lemma 2] that

$$S(x^*,\overline{\lim_n}\mathsf{E}(F_n|\Sigma_n)(\omega))\le S(x^*,\mathsf{E}(\overline{\lim_n}F_n|\Sigma_n)(\omega)), \qquad x^*\in X^*,\ \omega\in\Omega\backslash N.$$

Thus $\overline{\lim_n}\mathsf{E}(F_n|\Sigma_n)\subseteq\mathsf{E}(\overline{\lim_n}F_n|\Sigma_\infty)$, a.s..

(2) It is similar to the proof of (1). #

Theorem 3.5 Suppose that the assumptions of Theorem 3.4 hold. If we assume further that $\lim_n F_n=F$, a.s., then

$$\lim_n\mathsf{E}(F_n|\Sigma_n)=\mathsf{E}(F|\Sigma_\infty), \qquad \text{a.s..}$$

Proof It follows immediately from Theorem 3.4. #

Remark 3.6 If $X=R^1$ and F_n in Theorem 3.4 and Theorem 3.5 are real random variables, then the conclusions of Lemma 3.2 can be reduced from the two theorems immediately.

Theorem 3.7 Under the assumptions of Theorem 3.5, we have

$$S^1_{\mathsf{E}(F_n|\Sigma_n)}\xrightarrow{W}S^1_{\mathsf{E}(F|\Sigma)},$$

where the $W-\lim$ is the pointwise limit of the support function.

Proof By Theorem 3.5 we know that $\lim_n \mathsf{E}(F_n|\Sigma_n) = \mathsf{E}(F|\Sigma_\infty)$, a.s.. It follows from [7, Theorem 2] that $\mathsf{E}(F_n|\Sigma_n) \xrightarrow{w} \mathsf{E}(F|\Sigma_\infty)$, a.s.. Since

$$|\mathsf{E}(F_n|\Sigma_n)(\omega)| \le |\mathsf{E}(G|\Sigma_n)(\omega)| \le \mathsf{E}(|G||\Sigma_n)(\omega), \qquad \text{a.s..}$$

$\{\mathsf{E}(F_n|\Sigma_n)\}_{n\ge1}$ is uniformly integrable because of [9, Theorem 1.8, Theorem 1.9]. Thus we can conclude from [7, Lemma 4] that $S^1_{\mathsf{E}(F_n|\Sigma_n)} \xrightarrow{W} S^1_{\mathsf{E}(F|\Sigma)}$. #

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集值条件期望的收敛定理

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本文给出了集合序列弱极限的表示定理，得到了随机集列关于 σ- 域流的条件期望序列在弱收敛意义下的 Fatou 型引理和控制收敛定理.

关 键 词: 集值可测函数，集值条件期望，弱极限， Fatou 引理.

学 科 分 类 号: O211.4.