应用概率统计 第二十卷
第三期 2004 年 8 月

Chinese Journal of Applied Probability
and Statistics Vol. 20 No. 3 Aug. 2004

Some Properties of Spectral Decomposition Estimate of Variance Components*

SHI JIANHONG WANG SONGGUI

(*College of Applied Sciences, Beijing University of Technology, Beijing, 100022*)

Abstract

Although there exist many estimates of variance components in linear mixed effects models, only the spectral decomposition estimate (SDE) and analysis of variance estimate (ANOVAE) have closed form in the general case. In this paper we compare the SDE with the ANOVAE in the linear mixed model with two variance components. Our results show that these two estimates of the variance components have the equal variance under some conditions. Thus the SDE shares some optimalities of the ANOVAE. Two examples are given to illustrate our theoretical results.

Keywords: Spectral decomposition estimate, ANOVAE, variance component, linear mixed model.

AMS Subject Classification: 62J05.

§ 1. Introduction

The estimation in linear mixed model has attracted a considerable interest of researchers, in particular, for the variance components, several estimation methods have been proposed. They are analysis of variance estimate (ANOVAE), maximum likelihood estimate (MLE), restricted maximum likelihood estimate (RMLE) and minimum norm quadratic unbiased estimate (MINQUE), see, for example, Wang and Chow (1994)[1]. These estimates have some shortcomings in different extent, for example, ANOVAE and MINQUE can not guarantee the nonnegativity of estimates, see, for example, Kelly and Mathew (1993)[2], however MINQUE, MLE and RMLE need to solve a system of non-linear equations, which usually do not have explicit solution and an iterative procedure is necessary, and MINQUE depends strongly on the initial guesses of the variance components, which has certain subjectivity, see Rao (1971)[3]. About statistical properties of these estimates, there are a few results in the literature up to now, so it is better to consider them as algorithms to produce some estimates.

Wang and Yin (2002)[4] proposed a new method of simultaneously estimating fixed effects and variance components. The corresponding estimates are called as spectral decomposition estimate (SDE). Both the ANOVAE and SDE have their closed forms in all cases which can bring some convenience in further statistical analysis. The purpose of this paper is to compare the SDE with the ANOVAE in the following model

$$y = X\beta + U\xi + \epsilon, \tag{1.1}$$

where y is a $N \times 1$ vector of observations, X is a known $N \times s$ matrix, U is a known $N \times m$ matrix, β is a $s \times 1$ vector of unknown parameters, ξ is a $m \times 1$ normally distributed random vector with mean vector 0 and covariance matrix $\sigma_1^2 I_m$, ϵ is an $N \times 1$ normally distributed random error vector with mean vector 0 and covariance matrix $\sigma^2 I_N$, and the vectors ξ and ϵ are independently distributed. β is the vector of fixed effects in the model and ξ represents random effects. This model can also be rewritten as

$$\mathsf{E}(y) = X\beta, \qquad \mathsf{Cov}\,(y) = \sigma^2 I_N + \sigma_1^2 V, \tag{1.2}$$

*The work was supported by the National Natural Science Foundation of China (10271010) and the Natural Science Foundation of Beijing (1032001).

Received 2002. 12. 2. Revised 2003. 9. 4.

where $V = UU'$. The one-way random model and the two-way mixed model without interaction are obviously special case of such models.

It is well known that for some models the ANOVAE is uniformly minimum variance estimate. Our results obtained in this paper show that for model (1.1) the SDE and ANOVAE of variance components have the equal variance under some conditions. Thus the SDE shares some optimalities of the ANOVAE.

The structure of this paper is follow. In the next section, we introduce the ANOVAE and SDE of variance components in model (1.1). In Section 3 we compare variances of theses two estimates and obtain some results about their variances to be equal. Finally, two examples are given in Section 4 to illustrate our theoretical results.

§ 2. Two Estimates of Variance Components

In what follows, A', $\mathrm{tr}(A)$, $\mathcal{M}(A)$, $\mathrm{rk}(A)$and A^- stand for the transpose, trace, column space, rank and a generalized inverse of A, respectively. Further, denote $P_A = A(A'A)^-A'$, which is the orthogonal projector onto $\mathcal{M}(A)$.

Let $p = \mathrm{rk}(X)$ and $q = N - p$. Suppose that Z is an $N \times q$ matrix satisfying $Z'X = 0$ and $Z'Z = I_q$. If $u = Z'Y$, we then get

$$\mathsf{E}(u) = 0, \qquad \mathsf{Cov}\,(u) = \sigma^2 I_q + \sigma_1^2 V_1, \tag{2.1}$$

where $V_1 = Z'VZ$. Let $s_1 = \mathrm{rk}(V_1)$. Consider the spectral decomposition of V_1

$$V_1 = \sum_{j=1}^{g} \tau_j A_j, \tag{2.2}$$

where $\tau_1 > \tau_2 > \cdots > \tau_g$ denote the distinct non-zero eigenvalues of V_1 with multiplicities $a_1, a_2, \cdots, a_g$, and A_js denote the corresponding projection matrix respectively. Clearly, $\mathrm{rk}(A_j) = a_j$ and $\sum\limits_{j=1}^{g} a_j = s_1$. Let $A = I - \sum\limits_{j=1}^{g} A_j$, then $\mathrm{rk}(A) = q - s_1$. It is easy to verify that the ANOVAE's of σ^2 and σ_1^2 are respectively given by

$$\widehat{\sigma}^2 = \frac{u'Au}{\mathrm{rk}(A)}, \tag{2.3}$$

$$\widehat{\sigma}_1^2 = \frac{1}{\mathrm{tr}V_1}\Big(\sum_{j=1}^{g} u'A_j u - \frac{\mathrm{rk}(V_1)}{\mathrm{rk}(A)} u'Au\Big). \tag{2.4}$$

It is well known that $\widehat{\sigma}^2$ and $\widehat{\sigma}_1^2$ are the unbiased estimation of σ^2 and σ_1^2 respectively and have many good statistical properties in some cases.

The SDE proposed by Wang and Yin (2002)[4] based on the spectral decomposition of the covariance matrix, and then by using some appropriate linear transformation to obtain several new singular linear models. The feature of these models is that they have the same fixed effects as the original model, but their covariances matrices do not involve any unknown variance component except a factor (this factor is one of eigenvalues of the covariance matrix of original model). Using the unified theory of least squares (see, for example, Wang and Chow (1994)[1] and Rao (1973)[5]) for every new model, we obtain estimates for fixed effects and the eigenvalues. The eigenvalues of the covariance matrix of original model are linear functions of variance components, so by solving a system of linear equations, we can obtain the estimate of the variance components. The prominent feature of the new method is that for the fixed effects we can obtain several spectral decomposition estimates, they all have some good statistical properties, so we can make use of them to do further statistical inference such as testing of hypothesis, interval estimate and model diagnosis, and so on.

Follow the method described above, the SDE of σ^2 and σ_1^2 can be obtained as follows. Let $s_2 = \mathrm{rk}(V)$. Consider the spectral decomposition of V

$$V = \sum_{i=1}^{k} \lambda_i M_i, \tag{2.5}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ denote the non-zero eigenvalues of V with multiplicities, say, $m_1, m_2, \cdots, m_k$, and M_is denote the corresponding projections respectively. Clearly, $\mathrm{rk}(M_i) = m_i$, and $\sum_{i=1}^{k} m_i = s_2$. Let $M = I - \sum_{i=1}^{k} M_i$, then $\mathrm{rk}(M) = N - s_2$.

Pre-multiplying (1.1) by M_i yields the following $k+1$ new linear models

$$y^{(i)} = X_i\beta + \varepsilon_i, \qquad \varepsilon_i \sim N(0, (\sigma^2 + \lambda_i\sigma_1^2)M_i), \tag{2.6}$$

where $y^{(i)} = M_i y$, $X_i = M_i X$ and $\varepsilon_i = M_i U\xi + M_i\epsilon$. The model (2.6) is singular and the estimate of parameters can be obtained by using the unified theory of least squares. we can first get the estimate of $\sigma^2 + \lambda_i\sigma_1^2$, say, α_i^*

$$\alpha_i^* = \frac{1}{r_i} y'(M_i - M_iX(X'M_iX)^- X'M_i)y, \tag{2.7}$$

where $r_i = \mathrm{rk}(M_i) - \mathrm{rk}(M_iX)$. When $k = 1$, (i.e. $V = \lambda_1 M_1$ and $M = I - M_1$), the SDE of σ^2 and σ_1^2, say $\tilde{\sigma}^2$ and $\tilde{\sigma}_1^2$ is respectively given by

$$\tilde{\sigma}^2 = \frac{1}{r} y'[M - MX(X'MX)^- X'M]y, \tag{2.8}$$

$$\tilde{\sigma}_1^2 = \frac{1}{\lambda_1}\Big[\frac{1}{r_1} y'(M_1 - P_{M_1X})y - \frac{1}{r} y'(M - P_{MX})y\Big], \tag{2.9}$$

where $r = \mathrm{rk}(M) - \mathrm{rk}(MX)$. Clearly, in the case $k = 1$, the SDE is unique and unbiased. Thus in the following section we will compare them with the ANOVAE only in terms of variances.

§ 3. Comparison of the SDE and ANOVAE

At first we give some lemmas which will play key role in the proof of the following theorems.

Lemma 3.1 Suppose $Y \sim N_p(\mu, V)$ and V is a $p \times p$ nonsingular. Then

$$\mathrm{Var}\,(Y'BY) = 2\,\mathrm{tr}[(BV)^2] + 4\mu' BVB\mu.$$

Lemma 3.2 Let A is a $m \times n_1$ matrix, B is a $m \times n_2$ matrix. Then for any A^- and B^-, we have

$$\mathrm{rk}(A \,\vdots\, B) = \mathrm{rk}(A) + \mathrm{rk}((I - AA^-)B) = \mathrm{rk}((I - BB^-)A) + \mathrm{rk}(B).$$

The proofs of the above lemmas can be found in Searle (1971)[6] and Wang and Jia (1994)[7] respectively.

In what follows we will compare the SDE and ANOVAE for the case $k = 1$ in decomposition (2.5) which implies that $V = U'U = \lambda_1 M_1$, that is, scalar multiple of a projection matrix. By Lemma 3.1, it is readily to get the variances of $\tilde{\sigma}^2$, $\hat{\sigma}^2$, $\tilde{\sigma}_1^2$ and $\hat{\sigma}_1^2$ as follows:

$$\mathrm{Var}\,(\hat{\sigma}^2) = \frac{2\sigma^4}{\mathrm{rk}(A)}, \tag{3.1}$$

$$\mathrm{Var}\,(\tilde{\sigma}^2) = \frac{2\sigma^4}{r}, \tag{3.2}$$

$$\mathrm{Var}\,(\hat{\sigma}_1^2) = \frac{2}{(\mathrm{tr}V_1)^2}\Big[\mathrm{rk}(V_1) + \frac{(\mathrm{rk}(V_1))^2}{\mathrm{rk}(A)}\Big]\sigma^4 + \frac{2\,\mathrm{tr}V_1^2}{(\mathrm{tr}V_1)^2}\sigma_1^4 + \frac{4}{\mathrm{tr}V_1}\sigma^2\sigma_1^2, \tag{3.3}$$

$$\mathrm{Var}\,(\tilde{\sigma}_1^2) = \frac{2}{\lambda_1^2}\Big(\frac{1}{r_1} + \frac{1}{r}\Big)\sigma^4 + \frac{2}{r_1}\sigma_1^4 + \frac{4}{\lambda_1 r_1}\sigma_1^2\sigma^2, \tag{3.4}$$

where A is defined above of (1.5). We are now in the position to prove Theorem 3.1.

Theorem 3.1 $\mathrm{Var}(\tilde{\sigma}^2) = \mathrm{Var}(\hat{\sigma}^2)$.

Proof By Lemma 3.2, we have

$$\begin{aligned} \mathrm{rk}(V \vdots X) &= \mathrm{rk}(V) + \mathrm{rk}((I - VV^{+})X) \\ &= \mathrm{rk}(V) + \mathrm{rk}((I - M_1)X) = \mathrm{rk}(V) + \mathrm{rk}(MX) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \mathrm{rk}(V \vdots X) &= \mathrm{rk}(X) + \mathrm{rk}((I - XX^{-})V) \\ &= \mathrm{rk}(X) + \mathrm{rk}((I - P_X)V) = \mathrm{rk}(X) + \mathrm{rk}(P_Z V) \\ &= \mathrm{rk}(X) + \mathrm{rk}(ZZ'V) = \mathrm{rk}(X) + \mathrm{rk}(Z'V). \end{aligned} \tag{3.6}$$

Since matrix V is positive semidefinite, thus there must exist a matrix Q with full row rank such that $V = Q'Q$. using this fact and some properties of matrix rank we can obtain

$$\mathrm{rk}(Z'V) = \mathrm{rk}(Z'Q'Q) = \mathrm{rk}(Z'Q') = \mathrm{rk}(Z'Q'QZ) = \mathrm{rk}(V_1),$$

that is,

$$\mathrm{rk}(V \vdots X) = \mathrm{rk}(X) + \mathrm{rk}(V_1). \tag{3.7}$$

By using (3.5) and (3.7), we get

$$\mathrm{rk}(X) + \mathrm{rk}(V_1) = \mathrm{rk}(V) + \mathrm{rk}(MX), \tag{3.8}$$

where $M = I - M_1$ is defined below (2.7). Note that

$$\begin{aligned} \mathrm{rk}(M) - \mathrm{rk}(MX) &= N - \mathrm{rk}(V) - \mathrm{rk}(MX) \\ &= N - (\mathrm{rk}(V) + \mathrm{rk}(MX)) \end{aligned}$$

and

$$\mathrm{rk}(A) = N - \mathrm{rk}(X) - \mathrm{rk}(V_1) = N - (\mathrm{rk}(X) + \mathrm{rk}(V_1))$$

and using (3.8) yields

$$r = \mathrm{rk}(M) - \mathrm{rk}(MX) = \mathrm{rk}(A), \tag{3.9}$$

which completes the proof of the theorem. #

Next we will study the SDE of σ_1^2. In some special cases the SDE of σ_1^2 have the same variance as the ANOVAE of σ_1^2 and have simple expression. Suppose that $\mathcal{M}(X) \subseteq \mathcal{M}(M_1)$, $\tilde{\sigma}_1^2$ can be rewritten as

$$\tilde{\sigma}_1^2 = \frac{1}{\lambda_1}\Big[\frac{1}{s_2 - p} y'(M_1 - P_X)y - \frac{1}{N - s_2} y'My\Big].$$

However when $\mathcal{M}(X) \subseteq \mathcal{M}(M))$, $\tilde{\sigma}_1^2$ can be rewritten as

$$\tilde{\sigma}_1^2 = \frac{1}{\lambda_1}\Big[\frac{1}{s_2} y'My - \frac{1}{q - s_2} y'(M - P_X)y\Big].$$

The following two theorems show that under some condition the variance of the SDE and ANOVAE are equal.

Theorem 3.2 Suppose that $\mathcal{M}(X) \subseteq \mathcal{M}(M_1)$, then $\mathrm{Var}(\tilde{\sigma}_1^2) = \mathrm{Var}(\hat{\sigma}_1^2)$.

Proof Recalling that $Z'X = 0$, $\text{rk}(Z) = N - \text{rk}(X)$ and $Z'Z = I$, hence $P_X + ZZ' = I$. Using this fact and the assumption $\mathcal{M}(X) \subset \mathcal{M}(M_1)$, it is readily verified that

$$\begin{aligned} \text{tr}(V_1) &= \text{tr}(Z'VZ) = \text{tr}(ZZ'VZZ') \\ &= \text{tr}(I - P_X)V(I - P_X) = \lambda_1 \text{tr}(M_1 - P_X) \\ &= \lambda_1(\text{tr}M_1 - \text{tr}((X'X)^- X'X)) \\ &= \lambda_1(\text{rk}(M_1) - \text{rk}(X)) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \text{tr}V_1^2 &= \text{tr}(ZZ'VZZ'VZZ') \\ &= \text{tr}(I - P_X)VZZ'V(I - P_X) \\ &= \lambda_1^2 \text{tr}(M_1 ZZ' M_1) = \lambda_1^2 \text{tr}(M_1 - P_X) \\ &= \lambda_1^2(\text{rk}(M_1) - \text{rk}(X)) \end{aligned} \tag{3.11}$$

and also

$$\begin{aligned} \text{rk}(V_1) &= \text{rk}(ZZ'VZZ') = \text{rk}(M_1 - P_X) \\ &= \text{rk}(M_1) - \text{rk}(X). \end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we get that

$$\frac{\text{rk}(V_1)}{\text{tr}V_1^2} = \frac{1}{\lambda_1^2(\text{rk}(M_1) - \text{rk}(X))}, \qquad \frac{\text{rk}(V_1)^2}{\text{tr}V_1^2} = \frac{1}{\lambda_1^2}. \tag{3.13}$$

In terms of (3.10) and (3.11), it is easily to verify that the corresponding coefficients of σ_1^4 and $\sigma^2\sigma_1^2$ in (3.3) and (3.4) are equal. Combining (3.13) with (3.9), we know that the coefficients of σ^4 are also equal. The proof is completed. #

Theorem 3.3 Suppose that $\mathcal{M}(X) \subseteq \mathcal{M}(M)$, then $\text{Var}(\tilde{\sigma}_1^2) = \text{Var}(\hat{\sigma}_1^2)$.

Proof When $\mathcal{M}(X) \subseteq \mathcal{M}(M)$ by using Lemma 3.1, the variance of $\tilde{\sigma}_1^2$ is given by

$$\text{Var}(\tilde{\sigma}_1^2) = \frac{2}{(\text{tr}V)^2}\Big(\text{rk}(M_1) + \frac{(\text{rk}(M_1))^2}{r}\Big)\sigma^4 + \frac{2\,\text{tr}V^2}{(\text{tr}V)^2}\sigma_1^4 + \frac{4}{\text{tr}V}\sigma_1^2\sigma^2. \tag{3.14}$$

By Lemma 3.2 and the condition $\mathcal{M}(X) \subseteq \mathcal{M}(M)$, it is obvious that

$$\text{tr}V_1 = \text{tr}(ZZ'VZZ') = \text{tr}(I - P_X)V(I - P_X) = \text{tr}V, \tag{3.15}$$

$$\begin{aligned} \text{tr}V_1^2 &= \text{tr}(Z'VZZ'VZ) \\ &= \text{tr}(I - P_X)VZZ'V(I - P_X) = \text{tr}VZZ'V \\ &= \text{tr}(I - P_X)VV(I - P_X) = \text{tr}V^2, \end{aligned} \tag{3.16}$$

$$\text{rk}(V_1) = \text{rk}(I - P_X)V(I - P_X) = \text{rk}(M_1). \tag{3.17}$$

By (3.15), (3.17) and (3.9), it is easy to see that the coefficients of σ^4 in (3.3) and (3.14) are same. From (3.16) and (3.17), we can see that corresponding coefficients of σ_1^4 and $\sigma_1^2\sigma^2$ in (3.3) and (3.14) are the same. The proof is completed. #

The following theorem show that two estimates have same probability of non-negativity under some conditions.

Theorem 3.4 Suppose that $Z'M_1Z$ is an idempotent symmetric matrix, then

(1) $\mathsf{Var}(\widetilde{\sigma}_1^2) = \mathsf{Var}(\widehat{\sigma}_1^2)$;

(2) $\mathsf{P}(\widetilde{\sigma}_1^2 < 0) = \mathsf{P}(\widehat{\sigma}_1^2 < 0)$.

Proof (1) Since $Z'M_1Z$ is an idempotent symmetric matrix, (i.e., $g = 1$ and $A_1 = Z'M_1Z$), the variance of $\widetilde{\sigma}_1^2$ can be simplified as

$$\mathsf{Var}(\widetilde{\sigma}_1^2) = \frac{2}{\lambda_1^2}\Big(\frac{1}{\mathrm{rk}(Z'M_1Z)} + \frac{1}{\mathrm{rk}(A)}\Big)\sigma^4 + \frac{2\sigma_1^4}{\mathrm{rk}(Z'M_1Z)} + \frac{4\sigma^2\sigma_1^2}{\lambda_1(\mathrm{rk}(Z'M_1Z))}. \tag{3.18}$$

Comparing (3.4) with (3.18), we know to prove (1) it is sufficient to verify that $\mathrm{rk}(Z'M_1Z) = \mathrm{rk}(M_1) - \mathrm{rk}(M_1X)$. In fact, since $Z'M_1Z$ is an idempotent symmetric matrix, thus

$$Z'M_1Z = Z'M_1ZZ'M_1Z = Z'M_1Z - Z'M_1X(X'X)^-X'M_1Z,$$

which implies that $Z'M_1X = 0$. By using this fact, we have

$$\begin{aligned}
\mathrm{rk}(Z'M_1Z) &= \mathrm{rk}(Z'M_1Z - Z'M_1X(X'M_1X)^-M_1XZ)\\
&= \mathrm{rk}(Z'(M_1 - P_{M_1X})Z)\\
&= \mathrm{rk}(ZZ'(M_1 - P_{M_1X})ZZ')\\
&= \mathrm{rk}(I - P_X)(M_1 - P_{M_1X})(I - P_X)\\
&= \mathrm{rk}(M_1 - P_{M_1X}) = \mathrm{rk}(M_1) - \mathrm{rk}(M_1X).
\end{aligned}$$

Thus statement (1) is proved.

(2) Since $y \sim N(X\beta, \sigma^2I + \sigma_1^2V)$, thus

$$\frac{r_1}{\sigma^2 + \lambda_1\sigma_1^2}\alpha_1^* = \frac{y'(M_1 - P_{M_1X})y}{\sigma^2 + \lambda_1\sigma_1^2} \sim \chi^2_{r_1},$$
$$\frac{r}{\sigma^2}\alpha_0^* = \frac{y'(M - P_{MX})y}{\sigma^2} \sim \chi^2_r$$

and

$$\begin{aligned}
\mathsf{P}(\widetilde{\sigma}_1^2 < 0) &= \mathsf{P}\Big(\frac{\alpha_1^*}{\alpha_0^*} < 1\Big) = \mathsf{P}\Big(\frac{\alpha_1^*}{\alpha_0^*}\frac{\sigma^2}{\sigma^2 + \lambda_1\sigma_1^2} < \frac{\sigma^2}{\sigma^2 + \lambda_1\sigma_1^2}\Big)\\
&= \mathsf{P}\Big(F_{r_1,r} < \frac{\sigma^2}{\sigma^2 + \lambda_1\sigma_1^2}\Big),
\end{aligned}$$

where $\alpha_0^* = (1/r)y'(M - P_{MX})y$. χ_n^2 and $F_{m,n}$ denote χ^2 variable with degree of freedom n and F variable with degree of freedom m and n.

On the other hand, when $Z'M_1Z$ is an idempotent symmetric matrix, the ANOVAE $\widehat{\sigma}_1^2$ can be rewritten as

$$\widehat{\sigma}_1^2 = \frac{1}{\lambda_1}\Big(\frac{u'Z'M_1Zu}{\mathrm{rk}(Z'M_1Z)} - \frac{u'Au}{\mathrm{rk}(A)}\Big)$$

and since $u \sim N(0, \sigma^2I + \sigma_1^2\lambda_1Z'M_1Z)$, thus

$$\frac{u'Z'M_1Zu}{\sigma^2 + \sigma_1^2\lambda_1} \sim \chi^2_{\mathrm{rk}(Z'M_1Z)}, \qquad \frac{u'Eu}{\sigma^2} \sim \chi^2_{\mathrm{rk}(A)},$$

$$\begin{aligned}
\mathsf{P}(\widehat{\sigma}_1^2 < 0) &= \mathsf{P}\Big(\frac{u'Z'M_1Zu}{u'Au}\frac{\mathrm{rk}(A)}{\mathrm{rk}(Z'M_1Z)} < 1\Big)\\
&= \mathsf{P}\Big(\frac{u'Z'M_1Zu}{u'Au}\frac{\mathrm{rk}(A)}{\mathrm{rk}(Z'M_1Z)}\frac{\sigma^2}{\sigma^2 + \sigma_1^2\lambda_1} < \frac{\sigma^2}{\sigma^2 + \sigma_1^2\lambda_1}\Big)\\
&= \mathsf{P}\Big(F_{(\mathrm{rk}(Z'M_1Z),\mathrm{rk}(A))} < \frac{\sigma^2}{\sigma^2 + \lambda_1\sigma_1^2}\Big).
\end{aligned}$$

Since $\mathrm{rk}(Z'M_1Z) = \mathrm{rk}(M_1) - \mathrm{rk}(M_1X)$, $\mathrm{rk}(A) = \mathrm{rk}(M) - \mathrm{rk}(MX)$, thus part (2) is proved. #

§ 4. Two Examples

In this section, two examples are given to illustrate our theoretical results obtained in the previous sections.

Example 4.1 One-way random effects model

Consider the following one-way random effects model

$$y = \alpha\iota_{ab} + (I_a \otimes \iota_b)\xi + \epsilon,$$

where $y = (y_{11}, \cdots, y_{1b}, \cdots, y_{a1}, \cdots, y_{ab})'$, ι_{ab} and ι_b are respectively vectors of ones of dimension ab and a . $\otimes$ denotes the Kroneker product. $\xi = (\xi_1, \cdots, \xi_a)'$ represents random effects with normal distribution $N(0, \sigma_1^2 I_a)$, ϵ represents error vector with normal distribution $N(0, \sigma^2 I_{ab})$, and the vector ξ and ϵ are independently distributed.

For this model, the notations introduced in the previous sections are respectively

$$X = \iota_{ab}, \qquad V = I_a \otimes J_b = b(I_a \otimes \overline{J}_b),$$
$$M_1 = I_a \otimes \overline{J}_b, \qquad M = I - (I_a \otimes \overline{J}_b) = I_a \otimes (I_b - \overline{J}_b),$$
$$\lambda_1 = b, \qquad s_2 = a, \qquad p = 1, \qquad r = a(b-1),$$

where J_b is a matrix of ones of dimension b, $\overline{J}_b = J_b/b$.

Since $X = \iota_{ab} = (I_a \otimes \overline{J}_b)\iota_{ab} = M_1\iota_{ab}$, thus the assumption of Theorem 3.2 for the present case.

The ANOVAE of σ^2 and σ_1^2, are respectively given by (see, for example, Wang and Chow (1994)[1])

$$\widehat{\sigma}^2 = Q_2, \qquad \widehat{\sigma}_1^2 = \frac{1}{b}(Q_1 - Q_2),$$

where

$$Q_1 = \frac{1}{a-1}\sum_{i=1}^{a}\sum_{j=1}^{b}(\overline{y}_{i.} - \overline{y}_{..})^2,$$
$$Q_2 = \frac{1}{a(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \overline{y}_{i.})^2.$$

It is easy to verify that they are also the SDE of σ^2 and σ_1^2.

Example 4.2 Consider the following model

$$y = X\beta + Z_1\zeta + \epsilon,$$

where y, X, β, ϵ are the same as in (1.1), and Z_1 is a matrix composed of the first q_1 $(q_1 < q)$ column vectors of Z defined in Section 2, ζ is a $q_1 \times 1$ normally distributed random vector with mean vector 0 and covariance matrix $\sigma_1^2 I_{q_1}$, and the vectors ζ and ϵ are independently distributed.

For this example, the notations introduced in the previous sections are respectively

$$V = M_1 = Z_1 Z_1', \qquad M = I - M_1 = Z_2 Z_2' + P_X,$$
$$A_1 = V_1 = Z'VZ = \begin{pmatrix} I_{q_1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 \\ 0 & I_{q_2} \end{pmatrix},$$
$$\lambda_1 = 1, \qquad s_2 = q_1, \qquad r = q_2,$$

where Z_2 is a matrix of the last q_2 $(= q - q_1)$ column of Z.

Since $X = (I - Z_1 Z_1')X = MX$, so the example satisfies the assumptions of Theorem 3.3. By (2.3) and (2.4) it follows that the ANOVAE of σ^2 and σ_1^2 are given by

$$\widehat{\sigma}^2 = \frac{u'Au}{\mathrm{rk}(A)} = \frac{y'ZAZ'y}{\mathrm{rk}(A)} = \frac{y'Z_2Z_2'y}{q_2},$$
$$\widehat{\sigma}_1^2 = \frac{1}{\mathrm{tr}V_1}\Big(\sum_{j=1}^{g} u'A_ju - \frac{\mathrm{rk}(V_1)}{\mathrm{rk}(A)}u'Au\Big) = \frac{1}{q_1}\Big(y'Z_1Z_1'y - \frac{q_1}{q_2}y'Z_2Z_2'y\Big).$$

By using (2.8) and (2.9) we can show that these are also the SDE of σ^2 and σ_1^2.

Acknowledgements The authors gratefully acknowledge the editors and referees for their helpful comments and suggestions.

References

[1] Wang, S.G. and Chow, S.C., *Advanced Linear Models*, Marcel Dekker Inc., New York, 1994.

[2] Kelly, R.J. and Mathew, T., Improved estimators of variance components with smaller probability of negativity, *Journal of the Royal Statistical Society (Series B)*, **55**(1993), 897–911.

[3] Rao, C.R., Minimum variance and covariance components-MINQUE theory, *Journal of Multivariate Analysis*, **1**(1971), 257–275.

[4] Wang, S.G. and Yin, S.J., A new estimate of the parameters in linear mixed models, *Science in China (Series A)*, **45**(2002), 1301–1311.

[5] Rao, C.R., *Linear Statistical Inferences and Its Applications*, John Wiley, New York, 1973.

[6] Searle, S.R., *Linear Models*, John Wiley, New York, 1971.

[7] Wang, S.G. and Jia, Z.Z., *Matrix Inequalities*, Anhui Education Press, Hefei, 1994.

方差分量谱分解估计的几个性质

史建红 王松桂

(北京工业大学应用数理学院，北京 100022)

对于线性混合模型中方差分量的估计，虽有多种方法，但一般情况下只有方差分析估计和谱分解估计有显式解．本文就线性混合模型中含两个方差分量的情形，对方差分析估计和谱分解估计进行了比较，证明了在一些条件下两个估计的方差相等，由此推出谱分解估计也具有方差分析估计的某些优良性．文末用实例进一步说明了文中的结果．

关 键 词：谱分解估计，方差分析估计，方差分量，线性混合模型．

学科分类号：O212.1.