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Ruin Problems for a Sparre Andersen Risk Model*

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Abstract

In this paper, we consider a Sparre Andersen risk model in which the claim inter-arrival distribution is a mixture of an exponential distribution and an Erlang(n) distribution. We discuss the exact and the asymptotic behavior of the ruin probability under this risk model as the initial capital u tends to infinity.

Keywords: Sparre Andersen model, ruin probability, the class $\mathcal{S}$.

AMS Subject Classification: 91B30, 62P05.

§ 1. Introduction

Consider the Sparre Andersen risk model,

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Z_i, \tag{1.1}$$

where $u \geq 0$ is the initial surplus, c denotes the insurer's premium income per unit time. $\{Z_i, i = 1, 2, \cdots\}$ is a sequence of independent and identically distributed (i.i.d.) random variables, and Z_i denotes the ith claim amount. $\{T_i, i \geq 1\}$ is a sequence i.i.d. random variables, which denote the times between claims forms, and T_i has a density function $K(t)$,

$$K(t) = \beta_1 K_1(t) + \beta_2 K_n(t) \equiv \beta_1 \lambda e^{-\lambda t} + \beta_2 \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}, \qquad t \geq 0,$$

where $n \geq 1$ is a positive integer, $\lambda \geq 0$, $\beta_1, \beta_2 \geq 0$ and $\beta_1 + \beta_2 = 1$. $K_1(t)$ and $K_n(t)$ are the density functions of exponential distribution and Erlang(n, λ) distribution, respectively.

We denote the distribution of Z_i by $P(x)$, and the mean individual claim amount by $\mathsf{E}[Z] = \mu$. We assume that Z_i has a density function $p(x)$, and also assume that for all i, $c\mathsf{E}(T_i) > \mathsf{E}(Z_i)$.

Let T denote the time of ruin, so that $T = \inf\{t \geq 0, U(t) < 0\}$, and $T = \infty$ if $U(t) \geq 0$ for all $t > 0$.

Let $\omega(\cdot, \cdot)$ be a nonnegative function. We are interested in the quantity

$$W(u) = \mathsf{E}[e^{-\delta T} 1_{\{T<\infty\}} \omega(U(T-), |U(T)|) | U(0) = u], \qquad u \geq 0,$$

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where δ is a positive constant, $U(T-)$ is the surplus immediately before ruin, and $|U(T)|$ is the deficit at ruin. δ can be interpreted as a force of interest and ω as some kind of penalty when ruin occurs. The function $W(u)$ is the expectation of the discounted penalty which determines the joint and the marginal distributions of T, $U(T-)$, and $|U(T)|$.

We need some further notation. Write $\overline{F}=1-F$ for the tail of the distribution F. Let f^* denote the Laplace transform of the function f, i.e. $f^*(\alpha)=\int_0^\infty e^{-\alpha x}f(x)\mathrm{d}x$.

For two integrable functions $g_1(x)$ and $g_2(x)$ defined on $[0,\infty)$, let

$$(g_1*g_2)(u):=\int_0^u g_1(u-z)g_2(z)\mathrm{d}z \quad \text{and} \quad (g_1\star g_2)(u):=\int_0^u g_1(u-z)\mathrm{d}g_2(z).$$

In particular, we use the notation $g^{2,*}(u):=(g_1*g_2)(u)$ and $g^{2,\star}(u):=(g_1\star g_2)(u)$ when $g_1=g_2=g$. For any positive integer k, $g^{k,*}(u)$ and $g^{k,\star}(u)$ can be defined similarly. $g^{k,\star}(u)$ denotes the nth convolution of g with itself.

§ 2. An Integro-Differential Equation and Laplace Transform

In this section we present some results for the expectation of the discounted penalty $W(u)$. Since most of the proofs are similar to those in Dickson and Hipp (2000, 2001) or Cheng and Tang (2003), we just state the results without proofs.

Theorem 2.1 The function $W(u)$ satisfies the integro-differential equation

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}c^k(-\lambda-\delta)^{n-k}\frac{\mathrm{d}^kW(u)}{\mathrm{d}u^k}\\ =\ & A_1\int_0^\infty W(u-x)p(x)\mathrm{d}x\\ &-\beta_1\lambda\sum_{k=1}^{n-1}\binom{n-1}{k}c^k(-\lambda-\delta)^{n-1-k}\frac{\mathrm{d}^k}{\mathrm{d}u^k}\Big(\int_0^\infty W(u-x)p(x)\mathrm{d}x\Big),\end{aligned}\tag{2.1}$$

where $A_1=\beta_1(-\lambda)(-\lambda-\delta)^{n-1}+\beta_2(-\lambda)^n$.

Remark 2.1 Letting $\beta_1=0$, $\beta_2=1$, $n=2$ in (2.1), we can obtain the integro-differential equation for Erlang(2) risk model, which has been considered in Dickson and Hipp (2001).

Remark 2.2 Letting $n=1$ in (2.1), we obtain the integro-differential equation for the classical model. For detail, see Gerber and Shiu (1998).

Taking the Laplace transform on both sides of Eq.(2.1), we can get the Laplace transform of the function $W(u)$,

$$W^*(\alpha)=\frac{A_1\int_0^\infty e^{-\alpha u}\int_u^\infty\omega(u,x-u)p(x)\mathrm{d}x\mathrm{d}u+G_1(\alpha,\delta)+G_2(\alpha,\delta)-(\triangle W)^*(\alpha)}{(\alpha c-\lambda-\delta)^n-[A_1-\beta_1\lambda(\alpha c-\lambda-\delta)^{n-1}+\beta_1\lambda(-\lambda-\delta)^{n-1}]p^*(\alpha)},\tag{2.2}$$

where A_1 is the same as in Theorem 2.1 and

$$(\triangle W)^*(\alpha)=\beta_1\lambda\sum_{k=1}^{n-1}\binom{n-1}{k}c^k(-\lambda-\delta)^{n-1-k}\int_0^\infty e^{-\alpha u}\Big[\frac{\mathrm{d}^k}{\mathrm{d}u^k}\Big(\int_u^\infty\omega(u,x-u)p(x)\mathrm{d}x\Big)\Big]\mathrm{d}u,$$

$$G_1(\alpha,\delta)=\sum_{k=1}^{n}\sum_{j=0}^{k-1}\binom{n}{k}c^k(-\lambda-\delta)^{n-k}\alpha^{k-1-j}W^{(j)}(0),$$

$$G_2(\alpha,\delta)=\beta_1\lambda\sum_{k=2}^{n-1}c^k(-\lambda-\delta)^{n-1-k}\sum_{i=1}^{k-1}\alpha^{k-1-i}\sum_{l=0}^{i-1}W^{(l)}(0)p^{(i-1-l)}(0).$$

Lemma 2.1 Let δ be strictly positive and n is a positive integer, then the equation

$$(\alpha c-\lambda-\delta)^n=\{A_1-\beta_1\lambda[(\alpha c-\lambda-\delta)^{n-1}-(-\lambda-\delta)^{n-1}]\}p^*(\alpha) \tag{2.3}$$

has exact n roots $\alpha_l(\delta)$ with $\mathrm{Re}(\alpha_l(\delta))>0$ $(l=1,2,\cdots,n)$, where A_1 is defined in Theorem 2.1.

Proof When $\alpha=0$, we have

$$|[\beta_1(-\lambda)(-\lambda-\delta)^{n-1}+\beta_2(-\lambda)^n]p^*(0)|<|(-\lambda-\delta)^n|.$$

So for $\rho>0$ sufficiently big, the inequality

$$|\{A_1-\beta_1\lambda[(\alpha c-\lambda-\delta)^{n-1}-(-\lambda-\delta)^{n-1}]\}p^*(\alpha)|<|(\alpha c-\lambda-\delta)^n|$$

holds on the imaginary axis and on the semi-circle $\{\alpha\in\mathbb{C},\mathrm{Re}(\alpha)>0,|\alpha|=\rho\}$. By Rouches theorem the Eq.(2.3) has exact n roots on the right-half plane. #

§ 3. A Defective Renewal Equation

Let $\Psi(u)$ denote the probability of ultimate ruin from initial surplus u, namely

$$\Psi(u)=\mathsf{P}(T<\infty|U(0)=u).$$

$W(u)$ reduces to $\Psi(u)$ if $\delta\equiv 0$ and $\omega(x_1,x_2)\equiv 1$. From (2.2), we get

$$\Psi^*(\alpha)=\frac{D(\alpha)+h^*(\alpha)}{(\alpha c-\lambda)^n-[\beta_2(-\lambda)^n-\beta_1\lambda(\alpha c-\lambda)^{n-1}]p^*(\alpha)}, \tag{3.0}$$

where

$$\begin{aligned}D(\alpha)&=\sum_{k=1}^{n}\sum_{j=0}^{k-1}\binom{n}{k}c^k(-\lambda)^{n-k}\Psi^{(j)}(0)\alpha^{k-1-j}\\&\quad+\beta_1\lambda\sum_{k=2}^{n-1}\binom{n-1}{k}c^k(-\lambda)^{n-1-k}\sum_{i=0}^{k-1}\alpha^{k-1-i}\Big[\frac{\mathrm{d}^i[1-\mathsf{P}(u)]}{\mathrm{d}u^i}\Big]\Big|_{u=0},\\h(u)&=(-\lambda)^n[1-\mathsf{P}(u)]-\beta_1\lambda\sum_{k=1}^{n-1}\binom{n-1}{k}c^k(-\lambda)^{n-1-k}\frac{\mathrm{d}^k[1-\mathsf{P}(u)]}{\mathrm{d}u^k}.\end{aligned}$$

Let

$$L(\alpha)=(\alpha c-\lambda)^n-[\beta_2(-\lambda)^n-\beta_1\lambda(\alpha c-\lambda)^{n-1}]p^*(\alpha). \tag{3.1}$$

By Lemma 2.1, the equation $L(\alpha)=0$ has exact n roots α_j's located on the nonnegative half-plane. Further, $L(0)=0$, so 0 is a root of equation $L(\alpha)=0$. Without lose of generality, we assume that $0=\alpha_n<\mathrm{Re}(\alpha_{n-1})<\cdots<\mathrm{Re}(\alpha_1)$.

Following Dickson and Hipp (2001), we introduce the operator $T_rf(x)$ for an integrable function and a complex r, defined by $T_rf(x)=\int_x^\infty e^{-r(u-x)}f(u)\mathrm{d}u$. Then for $\forall$ real r, r_1, r_2, we have

$$T_{r_1}T_{r_2}f(x)=T_{r_2}T_{r_1}f(x)=\begin{cases}\dfrac{T_{r_1}f(x)-T_{r_2}f(x)}{r_2-r_1}, & r_1\neq r_2;\\ \displaystyle\int_x^\infty (u-x)e^{-r(u-x)}f(u)\mathrm{d}u, & r_1=r_2=r,\end{cases}\tag{3.2}$$

and

$$f^*(r_1)-f^*(r_2)=(r_2-r_1)T_{r_2}f^*(r_1),\tag{3.3}$$

for $r_1\neq r_2$ and if $x=0$, $T_rf(0)=f^*(r)$.

Lemma 3.1 Let $L(\alpha)$ be defined by (3.1), then it can be written as

$$\begin{aligned}L(\alpha) &= \prod_{i=1}^{n}(\alpha-\alpha_i)\cdot\Big[c^n-\lambda^n\cdot T_{\alpha_n}\cdots T_{\alpha_1}p^*(\alpha)\\ &\quad +\beta_1\lambda\sum_{i=1}^{n-1}\Big[\sum_{\mathcal{M}}\prod_{k=1}^{i+1}(c\alpha_k-\lambda)^{j_k}\Big]\cdot(-1)^{n-i}c^i\cdot T_{\alpha_n}\cdots T_{\alpha_{i+1}}p^*(\alpha)\Big],\end{aligned}$$

where $\mathcal{M}=\Big\{j_k\Big|\sum_{k=1}^{i+1}j_k=n-1-i, j_k\in\mathbb{N}_0\Big\}$. $\mathbb{N}_0$ denotes the set of nonnegative integers. If $\mathcal{M}$ be an empty set, we assume $\sum_{\mathcal{M}}\prod_{k=1}^{i+1}(c\alpha_k-\lambda)^{j_k}\equiv 1$.

Proof Since

$$\begin{aligned}L(\alpha) &= (\alpha c-\lambda)^n-\beta_2(-\lambda)^n p^*(\alpha)+\beta_1\lambda(\alpha c-\lambda)^{n-1}p^*(\alpha)\\ &\equiv V_{11}(\alpha)-V_{21}(\alpha)+V_{31}(\alpha),\end{aligned}$$

and $\alpha_n,\cdots,\alpha_1$ are zeros of $L(\alpha)$, $L(\alpha_i)=0$, $i=1,2,\cdots,n$. Hence

$$\begin{aligned}L(\alpha) &= L(\alpha)-L(\alpha_n)\\ &= [V_{11}(\alpha)-V_{11}(\alpha_n)]-[V_{21}(\alpha)-V_{21}(\alpha_n)]+[V_{31}(\alpha)-V_{31}(\alpha_n)].\end{aligned}$$

So,

$$V_{12}(\alpha)\equiv V_{11}(\alpha)-V_{11}(\alpha_n)=(\alpha-\alpha_n)\cdot c\cdot g(n,\alpha,\alpha_n),$$

where

$$g(n,\alpha,\alpha_n)=(\alpha c-\lambda)^{n-1}+(\alpha c-\lambda)^{n-2}(\alpha_n c-\lambda)+\cdots+(\alpha_n c-\lambda)^{n-1}.$$

Using (3.3) and getting

$$V_{22}(\alpha)\equiv V_{21}(\alpha)-V_{21}(\alpha_n)=\beta_2(-\lambda)^n[(-1)(\alpha-\alpha_n)\cdot T_{\alpha_n}p^*(\alpha)].$$

Similarly

$$\begin{aligned}V_{32}(\alpha) &\equiv V_{31}(\alpha)-V_{31}(\alpha_n)\\ &= \beta_1\lambda[(\alpha c-\lambda)^{n-1}p^*(\alpha)-(\alpha_n c-\lambda)^{n-1}p^*(\alpha_n)]\\ &= \beta_1\lambda(\alpha-\alpha_n)[c\cdot g(n-1,\alpha,\alpha_n)p^*(\alpha)-(\alpha_n c-\lambda)^{n-1}\cdot T_{\alpha_n}p^*(\alpha)].\end{aligned}$$

Hence

$$L(\alpha) = V_{12}(\alpha) - V_{22}(\alpha) + V_{32}(\alpha).$$

Using the zeros $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ and repeating the steps as above, we can obtain

$$V_{1\,n+1}(\alpha) \equiv V_{1\,n}(\alpha) - V_{1\,n}(\alpha_1) = \prod_{l=1}^{n}(\alpha - \alpha_l)\cdot c^n,$$

$$V_{2\,n+1}(\alpha) \equiv V_{2\,n}(\alpha) - V_{2\,n}(\alpha_1) = \prod_{l=1}^{n}(\alpha - \alpha_l)\cdot \beta_2\lambda^n \cdot T_{\alpha_n}\cdots T_{\alpha_1}p^*(\alpha),$$

$$V_{3\,n+1}(\alpha) \equiv V_{3\,n}(\alpha) - V_{3\,n}(\alpha_1) = \prod_{l=1}^{n}(\alpha - \alpha_l)\cdot \beta_1\lambda\sum_{i=0}^{n-1}(-1)^{n-i}c^i$$

$$\times T_{\alpha_n}\cdots T_{\alpha_{i+1}}p^*(\alpha)\cdot \sum_{\mathcal{M}}\prod_{k=1}^{i+1}(c\alpha_{n-k+1} - \lambda)^{j_k}.$$

So we get

$$L(\alpha) = V_{1\,n+1}(\alpha) - V_{2\,n+1}(\alpha) + V_{3\,n+1}(\alpha),$$

and the result follows. #

Letting $I(\alpha)$ denote the numerator of (3.0). Since $\alpha_1, \cdots, \alpha_n$ are zeros of the denominator of (3.0), it must also be a zero of the numerator. As Lemma 3.1, we get

Lemma 3.2 $I(\alpha)$ can be expressed as

$$I(\alpha) = (-1)^n\prod_{i=1}^{n}(\alpha - \alpha_i)\cdot T_{\alpha_n}\cdots T_{\alpha_1}h^*(\alpha).$$

Theorem 3.1 $\Psi^*(\alpha)$ can be written as

$$\Psi^*(\alpha) = \frac{(-1)^n\cdot T_{\alpha_n}\cdots T_{\alpha_1}h^*(\alpha)}{c^n - \lambda^n\cdot T_{\alpha_n}\cdots T_{\alpha_1}p^*(\alpha) + Q(p^*(\alpha))}, \tag{3.4}$$

where

$$Q(p^*(\alpha)) = \beta_1\lambda\sum_{i=1}^{n-1}\Big[\sum_{\mathcal{M}}\prod_{k=1}^{i+1}(c\alpha_k - \lambda)^{j_k}\Big]\cdot(-1)^{n-i}c^i\cdot T_{\alpha_n}\cdots T_{\alpha_{i+1}}p^*(\alpha).$$

$\mathcal{M}$ be the same as in Lemma 3.1.

Proof Using Lemma 3.1, Lemma 3.2 and (3.0), the result follows. #

Theorem 3.2 Let $\mu = \int_0^\infty [1 - \mathrm{P}(u)]du$, if $c > \lambda\mu$, then $\Psi(u)$ satisfies the defective renewal equation,

$$\Psi(u) = (\Psi * \gamma)(u) + \eta(u), \tag{3.5}$$

where

$$\gamma(u) = \frac{\lambda^n}{c^n}\cdot T_{\alpha_n}\cdots T_{\alpha_1}p(u) - \frac{1}{c^n}Q(p(u)),$$

$$\eta(u) = \frac{(-1)^n}{c^n}\cdot T_{\alpha_n}\cdots T_{\alpha_1}h(u).$$

Proof Equation (3.5) follows immediately from (3.4), since

$$\gamma^*(\alpha) = \frac{\lambda^n}{c^n}\cdot T_{\alpha_n}\cdots T_{\alpha_1}p^*(\alpha) - \frac{1}{c^n}Q(p^*(\alpha)),$$

$$\eta^*(u) = \frac{(-1)^n}{c^n}\cdot T_{\alpha_n}\cdots T_{\alpha_1}h^*(\alpha).$$

In order for the renewal equation to be defective, we require that $\gamma^*(0) < 1$. When $n = 1$, hence $\alpha_n = \alpha_1 = 0$,

$$\gamma(u) = \frac{\lambda}{c} \cdot T_{\alpha_1} p(u) = \frac{\lambda}{c} \cdot T_0 p(u),$$
$$\gamma^*(0) = T_0 \gamma(0) = \frac{\lambda}{c} \cdot T_{\alpha_1} T_0 p(0) = \frac{\lambda u}{c} < 1,$$

since $c > \lambda u$.

When $n = 2$, $\alpha_n = \alpha_2 = 0$,

$$\gamma(u) = \frac{1}{c^2}[(\lambda^2 - \beta_1 \lambda c \alpha_1) T_{\alpha_1} T_{\alpha_2} p(u) + \beta_1 \lambda c T_{\alpha_2} p(u)].$$

Using (3.2) and (3.3), we get

$$\begin{aligned}
\gamma^*(0) = T_0 \gamma(0) &= \frac{1}{c^2}[(\lambda^2 - \beta_1 \lambda c \alpha_1) T_{\alpha_1} T_0 T_0 p(0) + \beta_1 \lambda c T_0 T_0 p(0)] \\
&= \frac{\lambda^2 - \beta_1 \lambda c \alpha_1}{c^2} \cdot \frac{T_0 T_0 p(0) - T_{\alpha_1} T_0 p(0)}{\alpha_1} + \frac{\beta_1 \lambda c T_0 T_0 p(0)}{c^2} \\
&= \frac{\lambda^2 \mu}{c^2 \alpha_1} - \frac{\lambda^2 - \beta_1 \lambda c \alpha_1}{c^2 \alpha_1} \cdot \frac{T_0 p(0) - T_{\alpha_1} p(0)}{\alpha_1} \\
&= \frac{\lambda^2 \mu}{c^2 \alpha_1} - \frac{1}{c^2 \alpha_1}[(\lambda^2 - \beta_1 \lambda c \alpha_1)(1 - p^*(\alpha_1))].
\end{aligned}$$

Because $L(\alpha_1) = 0 = (c\alpha_1 - \lambda)^2 - (\lambda^2 - \beta_1 \lambda c \alpha_1) p^*(\alpha_1)$,

$$(\lambda^2 - \beta_1 \lambda c \alpha_1) p^*(\alpha_1) = (c\alpha_1 - \lambda)^2. \tag{3.6}$$

Substituting (3.6) into $\gamma^*(0)$, we get

$$\gamma^*(0) = \frac{\lambda^2 \mu}{c^2 \alpha_1} - \frac{1}{c^2 \alpha_1^2}[\lambda^2 - \beta_1 \lambda c \alpha_1 - (\alpha_1 c - \lambda)^2] = 1 - \frac{2\lambda c - \beta_1 \lambda c - \lambda^2 \mu}{\alpha_1 c^2}.$$

Since $c > \lambda \mu$ and $0 \le \beta_1 \le 1$, then

$$2\lambda c - \lambda^2 \mu - \beta_1 \lambda c > 0,$$

and $\gamma^*(0) < 1$, since $c^2 \alpha_1 > 0$.

When $n = 3$, $\alpha_n = \alpha_3 = 0$, we can obtain

$$\begin{aligned}
\gamma(u) &= \frac{\beta_1 \lambda c^2}{c^3} \cdot T_0 p(u) - \frac{\beta_1 \lambda c^2 (\alpha_1 + \alpha_2) - 2\beta_1 \lambda^2 c}{c^3} \cdot T_{\alpha_2} T_0 p(u) \\
&\quad + \frac{\lambda^3 + \beta_1 \lambda c^2 \alpha_1^2 - 2\beta_1 \lambda^2 c \alpha_1}{c^3} \cdot T_{\alpha_1} T_{\alpha_2} T_0 p(u).
\end{aligned}$$

Using (3.2), (3.3) and

$$0 = L(\alpha_i) = (\alpha_i c - \lambda)^3 - [(-\lambda)^3 - \beta_1 \lambda [(\alpha_i c - \lambda)^2 - (-\lambda)^2]] p^*(\alpha_i), \qquad i = 1, 2,$$

we can get

$$\begin{aligned}\gamma^*(0) &= T_0\gamma(0)\\ &= \frac{\beta_1\lambda c^2}{c^3}\cdot T_0T_0p(0) - \frac{\beta_1\lambda c^2(\alpha_1+\alpha_2)-2\beta_1\lambda^2 c}{c^3}\cdot T_{\alpha_2}T_0T_0p(0)\\ &\quad+\frac{\lambda^3+\beta_1\lambda c^2\alpha_1^2-2\beta_1\lambda^2 c\alpha_1}{c^3}\cdot T_{\alpha_1}T_{\alpha_2}T_0T_0p(0)\\ &= 1-\frac{3\lambda^2 c-\lambda^3\mu-2\beta_1\lambda^2 c}{c^3\alpha_1\alpha_2}.\end{aligned}$$

Since $c > \lambda\mu$ and $0 \le \beta_1 \le 1$, then

$$3\lambda^2 c-\lambda^3\mu-2\beta_1\lambda^2 c > 0.$$

Hence,

$$\gamma^*(0) < 1,$$

since $c^3\alpha_1\alpha_2 > 0$.

For a general value of n, we prove it by induction,

$$\gamma^*(0) = 1-\frac{n\lambda^{n-1}c-\lambda^n\mu-(n-1)\beta_1\lambda^{n-1}c}{c^n\prod\limits_{i=1}^{n-1}\alpha_i} < 1. \tag{3.7}$$

This proves the theorem. #

Corollary 3.1 If $c > \lambda\mu$, then $\int_0^\infty \gamma(z)\mathrm{d}z < 0$ and

$$\int_0^\infty \gamma(z)\mathrm{d}z = \begin{cases} 1-\dfrac{\lambda^{n-1}(\beta_1 c+\beta_2 nc-\lambda\mu)}{c^n\cdot\prod\limits_{i=1}^{n-1}\alpha_i}, & n\ge 2;\\ \dfrac{\lambda\mu}{c}, & n=1,\end{cases} \tag{3.8}$$

where $\gamma(u)$ is defined in Theorem 3.2.

§4. Exact and Asymptotic Behavior of the Ruin Probability

4.1 Exact Expression

Theorem 4.1 The solution of equation (3.5) can be expressed as

$$\Psi(u) = \sum_{k=0}^{\infty}(\eta*\gamma^{k,*})(u), \qquad u > 0, \tag{4.1}$$

where η, γ are defined in Theorem 3.2, and $\gamma^{0,*}$ is the Dirac function at 0. •

Proof It follows from (3.4), that for $\mathrm{Re}(\alpha) > 0$,

$$\Psi^*(\alpha) = \frac{\eta^*(\alpha)}{1-\gamma^*(\alpha)}.$$

Consequently,

$$\Psi^*(\alpha)=\eta^*(\alpha)\sum_{k=0}^{\infty}(\gamma^*(\alpha))^k. \tag{4.2}$$

By (3.7), $|\gamma^*(\alpha)|\le\gamma^*(0)<1$, Formula (4.1) can be obtained by inverting (4.2). #

Theorem 4.2 The ruin probability $\Psi(u)$ has the following expression

$$\Psi(u)=\Big(1-\int_0^\infty\gamma(z)\mathrm{d}z\Big)\sum_{k=0}^{\infty}\Big(\int_0^\infty\gamma(z)\mathrm{d}z\Big)^k\overline{\gamma_e^{k,*}}(u),\qquad u>0, \tag{4.3}$$

where $\gamma(u)$ is defined in Theorem 3.2 and

$$\gamma_e(u)=\Big(\int_0^\infty\gamma(x)\mathrm{d}x\Big)^{-1}\int_0^u\gamma(x)\mathrm{d}x.$$

Proof From Theorem 2.1, for $R=1-\Psi$ we have

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}c^k(-\lambda-\delta)^{n-k}\frac{\mathrm{d}^kR(u)}{\mathrm{d}u^k}\\ =\ &(-\lambda)^n\int_0^uR(u-x)p(x)\mathrm{d}x-\beta_1\lambda\sum_{k=1}^{n-1}\binom{n-1}{k}c^k(-\lambda)^{n-1-k}\frac{\mathrm{d}^k}{\mathrm{d}u^k}\int_0^uR(u-x)p(x)\mathrm{d}x.\end{aligned}$$

Similarly to (2.2), we have

$$R^*(\alpha)=\frac{G_1(\alpha,0)+G_2(\alpha,0)}{(\alpha c-\lambda)^n-[\beta_2(-\lambda)^n-\beta_1\lambda(\alpha c-\lambda)^{n-1}]p^*(\alpha)},$$

where

$$\begin{aligned}G_1(\alpha,0)&=\sum_{k=1}^{n}\sum_{j=0}^{k-1}\binom{n}{k}c^k(-\lambda)^{n-k}\alpha^{k-1-j}R^{(j)}(0),\\ G_2(\alpha,0)&=\beta_1\lambda\sum_{k=2}^{n-1}c^k(-\lambda)^{n-1-k}\sum_{i=1}^{k-1}\alpha^{k-1-i}\sum_{l=0}^{i-1}R^{(l)}(0)p^{(i-1-l)}(0).\end{aligned}$$

By the operator $T_rf(x)$, we simplify $R^*(\alpha)$ using the same method which was used in Theorem 3.1, and get

$$\begin{aligned}R^*(\alpha)&=\frac{(\alpha-\alpha_2)\cdots(\alpha-\alpha_n)\cdot c^n\cdot R(0)}{\alpha(\alpha-\alpha_2)\cdots(\alpha-\alpha_n)\cdot[c^n-\lambda^nT_{\alpha_n}\cdots T_{\alpha_1}p^*(\alpha)+Q(p^*(\alpha))]}\\ &=\frac{R(0)}{\alpha\Big(1-\Big(\dfrac{\lambda}{c}\Big)^n\cdot T_{\alpha_n}\cdots T_{\alpha_1}p^*(\alpha)+\dfrac{1}{c^n}Q(p^*(\alpha))\Big)},\end{aligned} \tag{4.4}$$

where $Q(p^*(\alpha))$ is defined in Theorem 3.1.

The final-value theorem of Laplace transforms giving

$$1=\lim_{u\to\infty}R(u)=\lim_{\alpha\to0}\alpha R^*(\alpha),$$

and hence

$$R(0)=1-\Big(\frac{\lambda}{c}\Big)^n\cdot T_{\alpha_n}\cdots T_{\alpha_0}p^*(0)+\frac{1}{c^n}\cdot Q(p^*(\alpha)).$$

It is easy to see that

$$R(0) = 1 - \int_0^\infty \gamma(z)\mathrm{d}z \quad \text{and} \quad \gamma^*(\alpha) = \left(\frac{\lambda}{c}\right)^n T_{\alpha_n}\cdots T_{\alpha_1} p^*(\alpha) - \frac{1}{c^n} Q(p^*(\alpha)).$$

Inverting (4.4) yields, for $u > 0$,

$$R(u) = R(0) \sum_{k=0}^{\infty} \left(\int_0^\infty \gamma(z)\mathrm{d}z\right)^k \gamma_e^{k,*}(u),$$

and the result (4.3) follows. #

4.2 Asymptotic Behavior

Throughout this section, $G(x)$ is a distribution function on $[0,\infty)$ such that $G(0) = 0, G(x) < 1$ for all $x \in (0,\infty)$ and $G(\infty) = 1$, and it has a mean τ. We note $G(u, u+z] = G(u+z) - G(u)$. In this section, the asymptotic formulaes for $\Psi(u)$ as $u \to \infty$ are obtained.

4.2.1 Exponential Behaviour

If $p(x)$ satisfies that there exists $\alpha_0 \in [-\infty, 0)$ such that $\lim_{\alpha \downarrow \alpha_0} p^*(\alpha) = \infty$. Then equation $L(\alpha) = 0$ has a negative root, which is denote as $-R$.

Theorem 4.3 Let $-R$ denote the negative root of equation $L(\alpha) = 0$. Then

$$\lim_{u\to\infty} e^{Ru}\Psi(u) = \frac{\prod_{k=1}^{n}(R+\alpha_k)\cdot T_{\alpha_n}\cdots T_{\alpha_1} h^*(-R)}{L'(-R)}. \tag{4.5}$$

Proof Since $0 = L(-R) = \prod_{i=1}^{n}(-R-\alpha_i)[c^n - \lambda^n \cdot T_{\alpha_n}\cdots T_{\alpha_1} p^*(-R) + Q(p^*(-R))]$,

$$c^n = \lambda^n \cdot T_{\alpha_n}\cdots T_{\alpha_1} p^*(-R) - Q(p^*(-R)). \tag{4.6}$$

Note that

$$\begin{aligned} \int_0^\infty e^{Ru}\gamma(u)\mathrm{d}u &= \frac{1}{c^n}\int_0^\infty e^{Ru}[\lambda^n \cdot T_{\alpha_n}\cdots T_{\alpha_1} p(u) - Q(p(u))]\mathrm{d}u \\ &= \frac{1}{c^n}[\lambda^n \cdot T_{\alpha_n}\cdots T_{\alpha_1} p^*(-R) - Q(p^*(-R))]. \end{aligned}$$

Consequently

$$\gamma^*(-R) = \int_0^\infty e^{Ru}\gamma(u)\mathrm{d}u = 1.$$

Because

$$\frac{\mathrm{d}\gamma^*(-\alpha)}{\mathrm{d}\alpha} = \int_0^\infty x e^{\alpha x}\gamma(x)\mathrm{d}x > 0,$$

the number R is unique. Multiplying (3.5) by e^{Ru},

$$e^{Ru}\Psi(u) = e^{Ru}(\Psi * \gamma)(u) + e^{Ru}\eta(u),$$

is a standard renewal equation. It follows from Smith's Key renewal theorem, that

$$\lim_{u\to\infty} e^{Ru}\Psi(u) = \frac{\displaystyle\int_0^\infty e^{Rx}\eta(x)\mathrm{d}x}{\displaystyle\int_0^\infty xe^{Rx}\gamma(x)\mathrm{d}x}. \tag{4.7}$$

It is easy to see that,

$$\int_0^\infty e^{Rx}\eta(x)\mathrm{d}x = \frac{(-1)^n}{c^n}\cdot T_{\alpha_n}\cdots T_{\alpha_1}h^*(-R), \tag{4.8}$$

$$\int_0^\infty xe^{Rx}\gamma(x)\mathrm{d}x = \frac{\mathrm{d}\gamma^*(\alpha)}{\mathrm{d}\alpha}\Big|_{\alpha=-R}. \tag{4.9}$$

By Lemma 3.1 and using (4.6),

$$\frac{\mathrm{d}\gamma^*(\alpha)}{\mathrm{d}\alpha}\Big|_{\alpha=-R} = -\frac{\partial}{\partial\alpha}\left(\frac{L(\alpha)}{c^n\prod\limits_{i=1}^n(\alpha-\alpha_i)}\right)\Bigg|_{\alpha=-R} = \frac{(-1)^nL'(-R)}{c^n\prod\limits_{i=1}^n(R+\alpha_i)}. \tag{4.10}$$

The result (4.5) follows from (4.7)-(4.10). #

4.2.2 Subexponential Behaviour

Definition 4.1 We say G belongs to the class $\mathcal{S}$ of subexponential distribution function if

$$\lim_{u\to\infty}\frac{\overline{G^{n,*}}(u)}{\overline{G}(u)} = n,$$

where $\overline{G}(u) = 1 - G(u)$.

Proposition 4.1 (Embrechts, et al. (1979)) If $G\in\mathcal{S}$, then for every $\varepsilon>0$ there exists some positive consants $K(\varepsilon)<\infty$ such that for all $n\in\mathbb{N}$ and $u>0$,

$$\frac{\overline{G^{n,*}}(u)}{\overline{G}(u)} \le K(\varepsilon)(1+\varepsilon)^n.$$

Lemma 4.1 If $\mathsf{P}_I\in\mathcal{S}$, then $\gamma_e\in\mathcal{S}$ and

$$\lim_{u\to\infty}\frac{1-\gamma_e(u)}{1-\mathsf{P}_I(u)} = \begin{cases}\dfrac{\mu\cdot\left(\lambda^n-\widehat{Q}\cdot\prod\limits_{i=1}^{n-1}\alpha_i\right)}{c^n\cdot\prod\limits_{i=1}^{n-1}\alpha_i-\lambda^{n-1}(\beta_1c+\beta_2nc-\lambda\mu)}, & n\ge 2;\\ 1, & n=1,\end{cases} \tag{4.11}$$

where

$$\widehat{Q} = \beta_1\lambda\sum_{i=1}^{n-1}(-1)^{n-i}c^i\Big[\sum_{\mathcal{M}}\prod_{k=1}^{i+1}(c\alpha_k-\lambda)^{j_k}\Big]\cdot\Big(\prod_{l=i+1}^{n-1}\alpha_l\Big)^{-1},$$

γ and γ_e be the same as Theorem 4.2 and $\mathcal{M}$ be the same as Lemma 3.1.

Proof When $n=1$, the proof is straightforward. For $n \geq 2$,

$$
\begin{aligned}
& 1-\gamma_e(u) \\
= & \int_u^\infty \gamma(z)\mathrm{d}z \Big/ \int_0^\infty \gamma(z)\mathrm{d}z = T_0\gamma(u) \Big/ \int_0^\infty \gamma(z)\mathrm{d}z \\
= & \frac{\lambda^n}{c^n}\cdot\Big[\Big(T_{\alpha_1}\cdots T_{\alpha_{n-1}}\int_u^\infty(1-\mathsf{P}(z))\mathrm{d}z\Big)\Big/\int_0^\infty\gamma(z)\mathrm{d}z\Big] \\
& -\frac{\beta_1\lambda}{c^n}\sum_{i=1}^{n-1}(-1)^{n-i}c^i\Big[\sum_{\mathcal{M}}\prod_{k=1}^{i+1}(c\alpha_k-\lambda)^{j_k}\Big]\cdot\Big[\Big(T_{\alpha_{i+1}}\cdots T_{\alpha_{n-1}}\int_u^\infty\overline{\mathsf{P}}(z)\mathrm{d}z\Big)\Big/\int_0^\infty\gamma(z)\mathrm{d}z\Big] \\
= & \frac{\lambda^n}{c^n}\cdot\Big[\Big(\int_0^\infty\cdots\int_0^\infty e^{-\alpha_1 z_1-\cdots-\alpha_{n-1}z_{n-1}}\int_{u+\sum_{i=1}^{n-1}z_i}^\infty\overline{\mathsf{P}}(z)\mathrm{d}z\mathrm{d}z_1\cdots\mathrm{d}z_{n-1}\Big)\Big/\int_0^\infty\gamma(z)\mathrm{d}z\Big] \\
& -\frac{\beta_1\lambda}{c^n}\sum_{i=1}^{n-1}(-1)^{n-i}c^i\Big[\sum_{\mathcal{M}}\prod_{k=1}^{i+1}(c\alpha_k-\lambda)^{j_k}\Big] \\
& \times\Big[\Big(\int_0^\infty\cdots\int_0^\infty e^{-\alpha_{i+1}z_{i+1}-\cdots-\alpha_{n-1}z_{n-1}}\int_{u+\sum_{k=i+1}^{n-1}z_i}^\infty\overline{\mathsf{P}}(z)\mathrm{d}z\mathrm{d}z_{i+1}\cdots\mathrm{d}z_{n-1}\Big)\Big/\int_0^\infty\gamma(z)\mathrm{d}z\Big],
\end{aligned}
$$

since $\alpha_n=0$.

It follows from the definition and Lebesgue's dominated convergence theorem,

$$
\lim_{u\to\infty}\frac{1-\gamma_e(u)}{1-\mathsf{P}_I(u)}=\frac{\mu\lambda^n}{c^n\prod_{k=1}^{n-1}\alpha_k\cdot\int_0^\infty\gamma(z)\mathrm{d}z}-\frac{\mu\widehat{Q}}{c^n\cdot\int_0^\infty\gamma(z)\mathrm{d}z}. \tag{4.12}
$$

So $\gamma_e\in\mathcal{S}$ by Theorem 2.7 in Teugels, J.L. (1975). Substituting (3.8) into (4.12), the result (4.11) be obtained. #

Theorem 4.4 Suppose $c>\lambda\mu$. If $\mathsf{P}_I\in\mathcal{S}$, then

$$
\lim_{u\to\infty}\frac{\Psi(u)}{\overline{\mathsf{P}_I}(u)}=\begin{cases}\dfrac{\mu\cdot\Big(\lambda^n-\widehat{Q}\cdot\prod_{i=1}^{n-1}\alpha_i\Big)}{\lambda^{n-1}(\beta_1 c+\beta_2 nc-\lambda\mu)}, & n\geq 2;\\[2ex] \dfrac{\lambda\mu}{c-\lambda\mu}, & n=1,\end{cases} \tag{4.13}
$$

where $\widehat{Q}$ be the same as in Lemma 4.1.

Proof Dividing both sides of (4.3) by $\overline{\mathsf{P}_I}(u)$, we can get

$$
\frac{\Psi(u)}{\overline{\mathsf{P}_I}(u)}=\frac{\overline{\gamma_e}(u)}{\overline{\mathsf{P}_I}(u)}\Big(1-\int_0^\infty\gamma(z)\mathrm{d}z\Big)\sum_{k=0}^\infty\Big(\int_0^\infty\gamma(z)\mathrm{d}z\Big)^k\cdot\frac{\overline{\gamma_e^{k,\star}}(u)}{\overline{\gamma_e}(u)}. \tag{4.14}
$$

Because $\mathsf{P}_I\in\mathcal{S}$, $\gamma_e\in\mathcal{S}$ by Lemma 4.1.

Choose $\varepsilon_0>0$ such that $(1+\varepsilon_0)\int_0^\infty\gamma(z)\mathrm{d}z<1$ since $\int_0^\infty\gamma(z)\mathrm{d}z<1$. By Proposition 4.1, there exists $K_1(\varepsilon_0)<\infty$ such that for all $u>0$ and all $k\geq 1$,

$$
\frac{\overline{\gamma_e^{k,\star}}(u)}{\overline{\gamma_e}(u)}\leq K_1(\varepsilon_0)(1+\varepsilon_0)^k.
$$

Hence, using Lebesgue's dominated theorem in (4.14), we have

$$\begin{aligned}\lim_{u\to\infty}\frac{\Psi(u)}{\overline{\gamma_e}(u)} &= \Big(\lim_{u\to\infty}\frac{\overline{\gamma_e}(u)}{\overline{\mathsf{P}_I}(u)}\Big)\cdot\Big(1-\int_0^\infty\gamma(z)\mathrm{d}z\Big)\cdot\sum_{k=0}^\infty k\Big(\int_0^\infty\gamma(z)\mathrm{d}z\Big)^k\\ &= \Big(\lim_{u\to\infty}\frac{\overline{\gamma_e}(u)}{\overline{\mathsf{P}_I}(u)}\Big)\cdot\Big[\int_0^\infty\gamma(z)\mathrm{d}z\Big/\Big(1-\int_0^\infty\gamma(z)\mathrm{d}z\Big)\Big],\end{aligned}\tag{4.15}$$

where we have used

$$\sum_{k=0}^\infty ky^{k-1}=\frac{1}{(1-y)^2},\qquad |y|<1.$$

Substituting (3.8) and (4.11) into (4.15), the result (4.13) can be obtained. #

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Sparre Andersen 风险模型的破产问题

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本文主要研究了一类 Sparre Andersen 模型，其索赔时间间隔的分布为指数分布与 Erlang(n) 分布的混合．得到了当初始资金 u 趋于无穷大时，破产概率 $\Psi(u)$ 的确切表达式和渐近表达式．

关 键 词: Sparre Andersen 风险模型，破产概率， S 族.

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