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The F Statistic of the Skew Elliptical Variables

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Abstract

In this paper the properties of the F statistic of the elliptical variables are investigated and a version of the skew F distribution is introduced. Probability density function, distribution function, moment generating function and moments are obtained. The behavior of the F test for the skew normal population is investigated.

Keywords: Skew elliptical distribution, statistic, hypothesis testing.

AMS Subject Classification: 62H10.

§ 1. Introduction

The F statistic arises in many applications of the statistical science, mainly as the ratio of the independent estimators of the variances of the normal variables. Let $Q_i \sim \chi^2_{k_i}$, $i = 1, 2$, be two independent chi-square variables, then $F = Q_1/Q_2 \cdot k_2/k_1$ has the F distribution with k_1, k_2 degrees of freedom and its noncentral version is obtained by replacing chi-square variables with noncentral chi-square variables. The well known F test for the ratio of the variances of two normal populations and the F test for the linear hypothesis are based on the F statistic.

In recent years natural and useful extensions of the family of the normal distributions proceed along two connected directions, one of which is the symmetric extension including the elliptical distributions, see, for example, Fang et al. (1990) and another is the skew extension including the skew normal and skew elliptical distributions, see Azzalini and Dalla Valle (1996), Azzalini and Capitanio (1999), Branco and Dey (2001) and Fang (2003).

Definition 1 Let f be the density generator of an n-dimensional spherical distribution, satisfying $\int_{R^n} f(\mathbf{v}'\mathbf{v})\mathrm{d}\mathbf{v} = 1$, F_1 its one-dimensional marginal distribution function, $\lambda \in R$, $\alpha \in R^k$ and $\xi \in R^k$ be constant and Ω a $k \times k$ constant positive definite matrix, $k = n - 1$. Let $\mathbf{z} \in R^k$ be a random vector with probability density function

$$\int_{-\infty}^{\lambda+\alpha'(\mathbf{z}-\xi)} f(y_0^2 + (\mathbf{z}-\xi)'\Omega^{-1}(\mathbf{z}-\xi))\mathrm{d}y_0 |\Omega|^{-1/2} / F_1(\lambda/c_0), \qquad \mathbf{z} \in R^k,$$

where $c_0 = (1 + \alpha'\Omega\alpha)^{1/2}$. Then $\mathbf{z}$ is called to have the skew elliptical distribution and denoted by $\mathbf{z} \sim S_k(\xi, \Omega, \lambda, \alpha; f)$, see Fang (2003).

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Denote by $\phi_n(x)$ the normal density generator, i.e., $\phi_n(x) = \exp(-x/2)(2\pi)^{-n/2}$. Some special cases for the skew elliptical distributions are the elliptical distribution with $\alpha = 0$, the skew normal distribution with $f = \phi_n$ in Azzalini and Dalla Valle (1996), Azzalini and Capitanio (1999), Arnold and Beaver (2000), the skew elliptical distribution with $\lambda = 0$ in Branco and Dey (2001).

The parameters λ and α are called skewness parameters and the later shape parameter by different authors. They provide a flexible way to model data presenting skewness, which arise frequently in practical cases. In the univariate normal case with $\lambda = 0$, $\Omega = 1$, the index of skewness of $\mathbf{z}$ is equal to $(4/\pi - 1)(2/\pi)^{1/2}(1 - 2/\pi \cdot \alpha^2/c_0^2)^{-3/2} \cdot \alpha^3/c_0^3$ (Azzalini and Dalla Valle, 1996, p.716). Thus positive α leads to positive skewness and the opposite holds for negative α. The skew elliptical distribution can be generated via the mechanism called truncation, censoring, conditioning or screening by different authors. It can be shown that $\mathbf{z}$ in Definition 1 has the stochastic representation

$$\mathbf{z} \overset{\mathrm{d}}{=} \mathbf{y} | \lambda + \alpha'(\mathbf{y} - \xi) > y_0,$$

where $(y_0, \mathbf{y}')'$ is n-dimensional elliptical distributed with location $(0, \xi')'$, scale matrix in the block form $\mathrm{diag}(0, \Omega)$ and density generator f. That is to say, we retain the observation $\mathbf{y}$ if the condition $\lambda + \alpha'(\mathbf{y} - \xi) > y_0$ is satisfied. The variable y_0 is called hidden variable or screening variable, which is unknown. See Fang (2003) for this and some other stochastic representations of the skew elliptical distributions. While maintaining mathematical tractability of the symmetric (elliptical) distributions, the skew elliptical distributions have additional parameters to regulate the skewness and thus is useful in reducing unrealistic assumptions in real data fitting. For more explanation of the background of this area of research, the interested reader is referred to Arnold and Beaver (2000), Azzalini and Dalla Valle (1996) and Azzalini and Capitanio (1999) and references therein.

In this paper we investigate the properties of the F statistic of the skew elliptical variables with emphasis on the skew normal case. This research will provide an insight into the applications of the F statistic under the skew elliptical population and stimulate further research.

In Section 2 the basic properties are investigated. Formulas of the probability density function, distribution function, moment generating function and the mean are obtained and illustrated with numerical calculations. In Section 3 two examples of the F test are given. The robustness of the significance level against the normality and the unbiasedness of the test are investigated. The proofs of the propositions are collected in Section 4. Finally, a brief discussion is given in Section 5.

§ 2. Basic Properties

The F statistic is expressed as the ratio of two independent quadratic forms of the normal variables. For the extension of its distribution under the skew elliptical populations we shall first recall the extension of the chi-square distribution, the distribution of the quadratic forms under the skew elliptical populations in Fang (2004).

Definition 2 Assume $\mathbf{z} \sim S_k(\xi, I, \lambda, \alpha; f_n)$. Partition $\mathbf{z}$ into h parts as $\mathbf{z} = (\mathbf{z}_1', \cdots, \mathbf{z}_h')'$, where $\mathbf{z}_i$ is k_i-dimensional. Partition ξ and α in the same manner. Let $Q_i = \mathbf{z}_i'\mathbf{z}_i$, $Q = (Q_1, \cdots, Q_h)$, $\tau_{i,11} = \xi_i'\xi_i$, $\tau_{i,12} = \xi_i'\alpha_i$, $\tau_{i,22} = \alpha_i'\alpha_i$. The distribution of $Q = (Q_1, \cdots, Q_h)$ is called the noncentral generalized Dirichlet distribution with parameters $k_1/2, \cdots, k_h/2$, $\tau_{i,11}, \tau_{i,12}, \tau_{i,22}$, $i = 1, \cdots, h$, λ and density function generator f_n, where $\sum_{i=1}^{h} k_i = k$, $n = k+1$, $\tau_{i,12}^2 \le \tau_{i,11}\tau_{i,22}$, for $k_i \ge 2$ and $\tau_{i,12}^2 = \tau_{i,11}\tau_{i,22}$ for $k_i = 1$, $i = 1, \cdots, h$, and denoted by $NG_h(k_1/2, \cdots, k_h/2; \tau, \lambda; f_n)$.

Definition 2 extends the Dirichlet distribution in a way that if $\lambda = 0$ and $\xi = 0$, then Q is a scale mixture of the Dirichlet distribution (Fang, 2003). That the noncentral generalized Dirichlet distribution (NG) is natural extension of the classical chi-square distribution can be seen as follows. Let $f_n = \phi_n$, $h = 1$. If $\lambda = 0$, $\tau_{1,12} = 0$; or $\tau_{1,22} = 0$, then $Q_1 \sim \chi_{k_1}^2(\tau_{1,11})$, the noncentral chi-square distribution. Let $f_n = \phi_n$, $h = 2$. If $\lambda = 0$, $\tau_{i,12} = 0$, $i = 1, 2$; or $\tau_{i,22} = 0$, $i = 1, 2$, then $Q_i \sim \chi_{k_i}^2(\tau_{i,11})$, are two independent chi-square variables.

Definition 3 Under the assumption of Definition 2 with $h = 2$, the distribution of $F = Q_1/Q_2 \cdot k_2/k_1$ is called the skew F distribution with parameters k_1, k_2, $\tau_{i,11}, \tau_{i,12}, \tau_{i,22}$, $i = 1, 2$, λ and density function generator f_n, and denoted by $SF(k_1, k_2, \tau, \lambda; f_n)$.

The skew F distribution retains a basic property of the usual F distribution that its inverse also belongs to the family of the skew F distributions. In the notation of Definition 3 we have $(\mathbf{z}_2', \mathbf{z}_1')' \sim S_k((\xi_2', \xi_1')', I, \lambda, (\alpha_2', \alpha_1')'; f_n)$ as seen from its probability density function in Definition 1. Hence by definition 3, $1/F = Q_2/Q_1 \cdot k_1/k_2 = \mathbf{z}_2'\mathbf{z}_2/\mathbf{z}_1'\mathbf{z}_1 \cdot k_1/k_2 \sim SF(k_2, k_1, \tau^*, \lambda; f_n)$, where $\tau_{1,ij}^* = \tau_{2,ij}$, $\tau_{2,ij}^* = \tau_{1,ij}$, $i, j = 1, 2$.

Denote by Φ the distribution function of the standard normal distribution and ϕ its density function. Let $l_{i1} = \tau_{i,12}\tau_{i,22}^{-1/2}$, if $\tau_{i,22} \ne 0$ or 0, otherwise, $l_{i2} = (\tau_{i,11} - l_{i1}^2)^{1/2}$, if $k_i \ge 2$; $l_{i1} = \tau_{i,12}\tau_{i,22}^{-1/2}$, if $\tau_{i,22} \ne 0$, or $\tau_{i,11}^{1/2}$, otherwise, if $k_i = 1$. Let $c(k_i) = \pi^{k_i/2-1}/\Gamma(k_i/2 - 1)$ for $k_i \ge 3$ and 1 for $k_i = 1, 2$.

Theorem 1 Let $\mathbf{y}_i$ with components y_{ij} be of dimension $\min(k_i, 3)$, $b = I_{(0,x)}(\mathbf{y}_1'\mathbf{y}_1/\mathbf{y}_2'\mathbf{y}_2 \cdot k_2/k_1)|y_{13}|^{k_1-3}|y_{23}|^{k_2-3}$, where $|y_{i3}|^{k_i-3}$ vanishes if $k_i = 1, 2$. Denote $k_i' = \min(k_i, 2)$. Then the distribution function of $SF(k_1, k_2, \tau, \lambda; f_n)$ at $x > 0$ is given by

$$c(k_1)c(k_2)\int bI_{\left(-\infty,\, \lambda+\sum_{i=1}^{2}(\tau_{i,22}^{1/2}y_{i1}-\tau_{i,12})\right)}(y_0)$$
$$\times \quad f_n\Big(y_0^2 + \sum_{i=1}^{2}\mathbf{y}_i'\mathbf{y}_i - 2\sum_{i=1}^{2}\sum_{j=1}^{k_i'} l_{ij}y_{ij} + \sum_{i=1}^{2}\tau_{i,11}\Big)\mathrm{d}y_0\mathrm{d}\mathbf{y}_1\mathrm{d}\mathbf{y}_2/F_1(\lambda/c_0). \qquad (1)$$

The dimensions for the integration in Theorem 1 can be further reduced if some of the parameters vanish. The probability density function of the skew F distribution can be obtained from the distribution function by first making suitable transformation and then taking derivative. However, the expressions are of different forms according to $k_i = 1$, $k_i = 2$ or $k_i \ge 3$. We present one case in the following Theorem. Expressions in other cases can be obtained by similar argument.

Theorem 2 Denote by f_0 the probability density function of the usual central F distribution. The probability density function of $SF(k_1, k_2, \tau, \lambda; \phi_n)$ at x with $k_1 \ge 3$, $k_2 \ge 3$, $\tau_{i,11} = 0$,

$\tau_{i,12} = 0$ is given by

$$f_0(x)\frac{2^{-k/2}}{\pi\Gamma(k/2)\Phi(\lambda/c_0)}\prod_{i=1}^{2}\frac{\Gamma(k_i/2)}{\Gamma(k_i/2-1/2)}\int y^{k/2-1}\exp(-y/2)\prod_{i=1}^{2}(\sin\theta_i)^{k_i-2}$$
$$\times\quad \Phi(\lambda+y^{1/2}(1+xk_1/k_2)^{-1/2}[(x\tau_{1,22}k_1/k_2)^{1/2}\cos\theta_1+\tau_{2,22}^{1/2}\cos\theta_2])\mathrm{d}y\mathrm{d}\theta_1\mathrm{d}\theta_2, \qquad (2)$$

where the integration in taken for $0 < y < \infty$, $0 < \theta_i < \pi$, $i = 1, 2$.

Theorem 2 shows the probability density function of the skew F distribution is the probability density function of the usual central F distribution multiplied by a factor representing the effect of the skewness parameters λ, α and the location parameter ξ in the skew elliptical population. It can be checked that this factor equals to 1 if the skewness and location parameters vanish. Calculation shows the main shape of the F density function is retained if $f_n = \phi_n$.

In principle the method of Theorem 1 can be applied to obtain the moment generating function of the skew F distribution as integrals in the space of lower dimension. We omit these formulas and give the moment generating function and the mean of the skew F distribution under the restriction $f_n = \phi_n$, which is of more compact form than in the general case. We also adopt a simpler way of proof by a conditioning argument using the corresponding result on the NG distribution with normal density generator. Moments of higher order can be obtained by similar arguments.

Theorem 3 Use the notation of $\mathbf{y}_i$ and k_i' in Theorem 1 and let $b = I_{(0,1/2)}(t/\mathbf{y}_2'\mathbf{y}_2 \cdot k_2/k_1)|y_{23}|^{k_2-3}$, where $|y_{23}|^{k_2-3}$ vanishes if $k_2 = 1, 2$. The moment generating function of $SF(k_1, k_2, \tau, \lambda; \phi_n)$ at t is given by

$$c(k_2)\int b\Phi\Big(\frac{\lambda+\tau_{2,22}^{1/2}y_{21}-\tau_{2,12}+2\tau_{1,12}\widetilde{t}(1-2\widetilde{t})^{-1}}{(1+\tau_{1,22}(1-2\widetilde{t})^{-1})^{1/2}}\Big)$$
$$\times\quad \exp\Big(\tau_{1,11}\widetilde{t}(1-2\widetilde{t})^{-1}-\Big(\mathbf{y}_2'\mathbf{y}_2-2\sum_{j=1}^{k_2'}l_{2j}y_{2j}+\tau_{2,11}\Big)\Big/2\Big)$$
$$\times\quad (1-2\widetilde{t})^{-k_1/2}(2\pi)^{-k_2/2}\mathrm{d}\mathbf{y}_2/\Phi(\lambda/c_0), \qquad (3)$$

where $\widetilde{t} = t/\mathbf{y}_2'\mathbf{y}_2 \cdot k_2/k_1$.

Theorem 4 Suppose $k_2 \geq 3$ and $\xi_2 = 0$. The mean of $SF(k_1, k_2, \tau, \lambda; \phi_n)$ is given by

$$\frac{k_2B}{k_1(k_2-2)}+\frac{k_2}{k_1 2^{k_2/2}\pi^{1/2}\Gamma(k_2/2-0.5)}\Big[I_1A-I_3\Big(\frac{\tau_{1,22}}{1+\tau_{1,22}}+\frac{B}{k_2-2}\Big)\Big], \qquad (4)$$

where

$$I_1=\int_{\substack{r>0\\0<\theta<2\pi}}|\sin\theta|^{k_2-2}r^{k_2-3}\phi\Big(\frac{\lambda+\tau_{2,22}^{1/2}r\cos\theta}{(1+\tau_{1,22})^{1/2}}\Big)\exp\Big(-\frac{r^2}{2}\Big)\mathrm{d}r\mathrm{d}\theta/\Phi(\lambda/c_0),$$

$$I_3=\frac{\tau_{2,22}^{1/2}}{(1+\tau_{1,22})^{1/2}}\int_{\substack{r>0\\0<\theta<2\pi}}\cos\theta|\sin\theta|^{k_2-2}r^{k_2-2}\phi\Big(\frac{\lambda+\tau_{2,22}^{1/2}r\cos\theta}{(1+\tau_{1,22})^{1/2}}\Big)\exp\Big(-\frac{r^2}{2}\Big)\mathrm{d}r\mathrm{d}\theta/\Phi(\lambda/c_0),$$

$$A=\frac{2\tau_{1,12}}{(1+\tau_{1,22})^{1/2}}-\frac{\lambda\tau_{1,22}}{(1+\tau_{1,22})^{3/2}},\qquad B=\tau_{1,11}+k_1.$$

Theorem 4 shows the mean of the skew F distribution can be expressed by the sum of the mean of the usual noncentral F distribution and a term representing the effect of the skewness parameters. This additional term vanishes if $\lambda = 0$ and $\tau_{1,12} = 0$ or $\tau_{i,22} = 0$ so that $A = 0$ and $I_3 = 0$, conforming with the independent chi-square case explained in the remark below Definition 2. Table 1 gives numerical results of Theorem 4 for various cases with $k_1 = 10$, $k_2 = 10$, $\xi_2 = 0$ and τ as function of α and ξ, where $\alpha_i = (\alpha_{i1}, 0)'$, $\xi_1 = (\xi_{11}, 0)'$. The numbers in the second column ($\xi_1 = 0$) of the first four rows ($\lambda = 0$) are the means of the non-skew central F distribution. The numbers in the first and third columns ($\xi_{11} = -1$, 1) of the first row ($\lambda = 0$) are the means of the non-skew non-central F distribution. The numbers in the third row ($\lambda = 0$, $\tau_{i,12} = 0$) are the means of the non-skew F distribution. Comparing the first and the third columns with the second column, we can see the mean increases as the noncentral parameter $\tau_{1,11}$ increases. This is because B increases with $\tau_{1,11}$, see alternative formula (7) of the mean in the proof of Theorem 4. Comparing the fifth, second and eighth rows in the first and the third columns we can see that the mean decreases as λ increases if $\alpha_2 = 0$. This is mainly caused by A as the decreasing function of λ. Comparing the pairs of the sixth and seventh rows, the third and the fourth rows, the ninth and the tenth rows in the first column and the third column, we can see that the mean increases or decreases as $\tau_{1,22}$ increases if $\xi_{11} > 0$ or $\xi_{11} < 0$.

Table 1 Mean of SF

λ	α_{11}	$\alpha_{21}\backslash\xi_{11}$	-1	0	1
0	0	0	1.3750	1.2500	1.3750
	5	0	1.2722	1.2500	1.4728
0	0	1	1.3750	1.2500	1.3750
	5	1	1.2787	1.2500	1.4713
1	5	0	1.2698	1.2408	1.4617
1	0	1	1.3920	1.2654	1.3920
	5	1	1.2722	1.2417	1.4613
-1	5	0	1.2883	1.2592	1.4802
-1	0	1	1.3580	1.2346	1.3580
	5	1	1.2885	1.2581	1.4776

§ 3. Applications

In this section we give two examples to illustrate the application of the distribution theory derived in previous sections.

Example 1 Suppose $\mathbf{z}_i$ is the sample from $N(\xi_i, \sigma_i^2)$ of size k_i, $\mathbf{z}_1$ and $\mathbf{z}_2$ are independent. For testing the hypothesis $H: \sigma_1^2/\sigma_2^2 \le 1$, the usual F statistic is $V = s_1^2/s_2^2 \cdot (k_2 - 1)/(k_1 - 1)$, where $s_i^2 = \mathbf{z}_i' P_i \mathbf{z}_i$, $P_i = I - \mathbf{1}\mathbf{1}'/k_i$. In a more general setting we consider the data coming from the skew normal distribution and investigate the effect of the skewness parameters on the level of the F test. We thus assume $\mathbf{z} \sim S_k(\xi, \Omega, \lambda, \alpha; \phi_n)$, where $\mathbf{z} = (\mathbf{z}_1', \mathbf{z}_2')'$, $\xi = (\mathbf{1}'\xi_1, \mathbf{1}'\xi_2)'$, $\Omega = \mathrm{diag}(I\sigma_1^2, I\sigma_2^2)$, $\alpha = (\alpha_1', \alpha_2')'$, where the partition is according to $k = k_1 + k_2$, $\mathbf{1}$ is k_i-dimensional with all components being 1. The independent normal case is recovered if the skewness parameters vanish. Thus the skew normal case represents a departure from the independent normal case. Let Γ_i be

$k_i \times k_i$ orthogonal matrix with $\mathbf{1}k_i^{-1/2}$ as its first column and Γ_{i2} the rest $k_i - 1$ columns,

$$\mathbf{y} = \begin{pmatrix} y_{11} \\ \mathbf{y}_{12} \\ y_{21} \\ \mathbf{y}_{22} \end{pmatrix} = \begin{pmatrix} \Gamma_1' \mathbf{z}_1/\sigma_1 \\ \Gamma_2' \mathbf{z}_2/\sigma_2 \end{pmatrix},$$

where $\mathbf{y}_{i2}$ is $k_i - 1$ dimensional. Then calculation shows (see Fang 2003)

$$\begin{pmatrix} \mathbf{y}_{12} \\ \mathbf{y}_{22} \end{pmatrix} \sim S_{k-2}\left(0, I, c_1\lambda, c_1 \begin{pmatrix} \Gamma_{12}'\alpha_1\sigma_1 \\ \Gamma_{22}'\alpha_2\sigma_2 \end{pmatrix}; \phi_{k-1}\right),$$

where $c_1 = [1 + \alpha_1'(I - P_1)\alpha_1\sigma_1^2 + \alpha_2'(I - P_2)\alpha_2\sigma_2^2]^{-1/2}$. Hence

$$(s_1^2/\sigma_1^2, s_2^2/\sigma_2^2) = (\mathbf{y}_{12}'\mathbf{y}_{12}, \mathbf{y}_{22}'\mathbf{y}_{22}) \sim NG_2((k_1 - 1)/2, (k_2 - 1)/2; \widetilde{\tau}, c_1\lambda; \phi_{k-1})$$

by definition 2, where $\widetilde{\tau}_{i,11} = 0$, $\widetilde{\tau}_{i,12} = 0$, $\widetilde{\tau}_{i,22} = c_1^2\alpha_i' P_i \alpha_i \sigma_i^2$, $i = 1, 2$. Let $F = V/\Delta$, where $\Delta = \sigma_1^2/\sigma_2^2$. Then $F = s_1^2/s_2^2 \cdot \sigma_2^2/\sigma_1^2 \cdot (k_2 - 1)/(k_1 - 1) \sim SF(k_1 - 1, k_2 - 1, \widetilde{\tau}, c_1\lambda; \phi_{k-1})$.

If $\lambda = 0$, then the distribution of V does not depend on α (see Fang 2003) and the test can be performed as in the normal case. If $\lambda \neq 0$, then the true significance level will be different from the nominal level using the standard table and the unbiasedness of the F test can not be easily obtained from the expression $V = F\Delta$ as the distribution of F depends on σ_i. Let x be the p-th quantile of the usual central F distribution of $k_1 - 1$, $k_2 - 1$ degrees of freedom. Table 2 gives estimated $\mathsf{P}(V > x)$ for various σ_1 and α_1 with $p = 0.9$, $\alpha_2 = \alpha_1 = (\alpha_{11}, 0)'$, $\sigma_2 = 1$. If $k_1 = 11$, $k_2 = 11$, then $x = 2.3223$. If $k_1 = 2$, $k_2 = 6$, then $x = 4.0604$. For each combination of the parameters, 2×10^4 skew normal vectors are generated and the frequency that the statistic V is larger than x is used to estimate $\mathsf{P}(V > x)$. The row corresponding to $\alpha_{11} = 0$ is the power function of the usual (non-skew) F test.

Table 2 Power function

(k_1, k_2)	λ	$\alpha_{11}\backslash\sigma_1$	0.2	0.5	1	2	8
(11,11)	0	0	0	0.0008	0.1000	0.7976	1.0000
	1	0.5	0	0.0009	0.1008	0.7937	1.0000
		5	0	0.0004	0.0990	0.7966	1.0000
		10	0	0.0005	0.1015	0.7946	1.0000
	-1	0.5	0	0.0009	0.1014	0.8090	1.0000
		5	0	0.0007	0.1028	0.8049	1.0000
		10	0	0.0009	0.0999	0.8028	1.0000
(2,6)	0	0	0.0002	0.0100	0.1000	0.3599	0.8112
	1	0.5	0.0001	0.0092	0.0950	0.3428	0.8033
		5	0.0003	0.0120	0.0964	0.3545	0.8096
		10	0.0001	0.0097	0.0962	0.3518	0.8124
	-1	0.5	0.0001	0.0110	0.1095	0.4070	0.8202
		5	0.0002	0.0110	0.1018	0.3736	0.8141
		10	0.0002	0.0101	0.0988	0.3664	0.8090

It can be seen that the numbers in each row is increasing with σ_1 for fixed σ_2 and α. This implies that $\mathsf{P}(V > x)$ is increasing with Δ and the unbiasedness of the F test is maintained. The column corresponding to $\sigma_1 = 1$ gives the estimate of the true significance level. We can see that the departure of the true significance level from the nominal level is not severe for the values of the parameters taken in this example.

Example 2 For the investigation of the robustness of the test for the location parameters in the linear models as developed by normality theory, we suppose $\mathbf{z} \sim S_k(\xi, I\sigma^2, \lambda, \alpha; \phi_n)$. The testing problem is $H : \xi \in \omega_0$ versus $K : \xi \in \omega - \omega_0$, where ω is an r-dimensional linear subspace of R^k and ω_0 is a q-dimensional subspace of ω, $0 \le q < r$. The usual testing statistic is $F = \|\widehat{\xi} - \widehat{\xi}_0\|^2/\|\mathbf{z} - \widehat{\xi}\|^2 \cdot (k-r)/(r-q)$, where $\widehat{\xi} = P_\omega \mathbf{z}$, $\widehat{\xi}_0 = P_{\omega_0}\mathbf{z}$, P_ω and P_{ω_0} are the projection matrices on the spaces ω and ω_0 respectively. By the argument similar to Example 1, making transformation to obtain a canonical form of the statistic and taking the transformation of the skew parameters into account, we obtain $F \sim SF(r-q, k-r, \tau, c_1\lambda; \phi_{r+1})$, where $\tau_{1,11} = \xi'(P_\omega - P_{\omega_0})\xi/(\sigma^2)$, $\tau_{1,12} = c_1\xi'(P_\omega - P_{\omega_0})\alpha$, $\tau_{1,22} = c_1^2\alpha'(P_\omega - P_{\omega_0})\alpha\sigma^2$, $\tau_{2,11} = 0$, $\tau_{2,12} = 0$, $\tau_{2,22} = c_1^2\alpha'(I - P_\omega)\alpha\sigma^2$, $c_1 = (1 + \alpha' P_{\omega_0}\alpha\sigma^2)^{-1/2}$. It can be seen that $\tau_{1,11} = 0$ and $\tau_{1,12} = 0$ under H. If $\lambda = 0$, then F has the usual F distribution and the significance level of the test is guaranteed. However, the power function in this case will depend on τ and hence the parameters σ, λ and α. In the case that $\lambda \ne 0$, both the level and power will depend on the skew parameters. Numerical calculation can be made for the investigation of the departure from the normal model.

It should be noted that the F statistic for the skew normal samples must be used with care. In the case where invariance of the distribution is not obtained as shown in the examples, the usage of the F statistic should be avoided.

§ 4. Proofs of the Propositions

Proof of Theorem 1 Using the notation of Definition 3,

$$\mathsf{P}(F < x) = \int I_{(0,x)}(\mathbf{z}_1'\mathbf{z}_1/\mathbf{z}_2'\mathbf{z}_2 \cdot k_2/k_1)\Big[\int_{-\infty}^{\lambda+\alpha'(\mathbf{z}-\xi)} f_n(y_0^2 + (\mathbf{z}-\xi)'(\mathbf{z}-\xi))\mathrm{d}y_0\Big]\mathrm{d}\mathbf{z}/F_1(\lambda/c_0).$$

The case of $k_i = 1$ is easy to establish. For $k_i \ge 2$ and α_i, ξ_i are not all 0, make transformation $\mathbf{y}_i = \Gamma_i'\mathbf{z}_i$, where Γ_i is orthogonal $k_i \times k_i$ matrix satisfying conditions in the following three cases. If $\alpha_i \ne 0$ and ξ_i is not proportional to α_i so that $\tau_{i,22} \ne 0$ and $l_{i2} \ne 0$, let the first two columns of Γ_i be $\alpha_i/\|\alpha_i\|$ and $(\xi_i - l_{i1}\alpha_i/\|\alpha_i\|)/l_{i2}$. If $\alpha_i \ne 0$ and ξ_i is proportional to α_i, so that $\tau_{i,22} \ne 0$ and $l_{i2} = 0$, let the first column of Γ_i be $\alpha_i/\|\alpha_i\|$. If $\alpha_i = 0$ and $\xi_i \ne 0$, so that $\tau_{i,11} \ne 0$, $l_{i1} = 0$, let the second column of Γ_i be $\xi_i/\|\xi_i\|$. In all cases, we have $\Gamma_i'\alpha_i = (\tau_{i,22}^{1/2}, 0)'$, $\Gamma_i'\xi_i = (l_{i1}, l_{i2}, 0)'$. Hence

$$\begin{aligned}\mathsf{P}(F < x) &= \int I_{(0,x)}(\mathbf{y}_1'\mathbf{y}_1/\mathbf{y}_2'\mathbf{y}_2 \cdot k_2/k_1) I_{\left(-\infty,\, \lambda + \sum_{i=1}^{2}(\tau_{i,22}^{1/2}y_{i1} - \tau_{i,12})\right)}(y_0) \\ &\quad \times f_n\Big(y_0^2 + \sum_{i=1}^{2}\mathbf{y}_i'\mathbf{y}_i - 2\sum_{i=1}^{2}\sum_{j=1}^{k_i'} l_{ij}y_{ij} + \sum_{i=1}^{2}\tau_{i,11}\Big)\mathrm{d}y_0\mathrm{d}\mathbf{y}_1\mathrm{d}\mathbf{y}_2/F_1(\lambda/c_0),\end{aligned}$$

where $\mathbf{y}_i$ is of the same dimension as $\mathbf{z}_i$. If $k_i \geq 3$, by a formula in Fang et al. (1990, p.23), the integration with respect to $(y_{i3}, \cdots, y_{ik_i})$ can be reduced to one dimensional and we denote the new variable by y_{i3}. This leads to (1) in Theorem 1. #

Proof of Theorem 2 Making transformation $y_{i1} = r_i \cos\theta_{i1}$, $y_{i2} = r_i \sin\theta_{i1}\cos\theta_{i2}$, $y_{i3} = r_i \sin\theta_{i1}\sin\theta_{i2}$, in (1) and then taking derivative with respect to x, we obtain the p.d.f. of SF for $k_i \geq 3$, which can be expressed in the following form by making a further transformation $r_2^2(1 + xk_1/k2) = y$,

$$\begin{aligned}
& f_0(x)\frac{\pi^{k/2-2}(k_1-2)(k_2-2)}{16\Gamma(k/2)F_1(\lambda/c_0)}\int y^{k/2-1}\prod_{i=1}^{2}(\sin\theta_{i1})^{k_i-2}|\sin\theta_{i2}|^{k_i-3} \\
\times\ & I\Big\{y_0 < \lambda + y^{1/2}(1+xk_1/k_2)^{-1/2}[(x\tau_{1,22}k_1/k_2)^{1/2}\cos\theta_{11} + \tau_{2,22}^{1/2}\cos\theta_{21}] - \sum_{i=1}^{2}\tau_{i,12}\Big\} \\
\times\ & f_n\Big(y_0^2 + y - 2y^{1/2}(1+xk_1/k_2)^{-1/2}[(xk_1/k_2)^{1/2}(l_{11}\cos\theta_{11} + l_{12}\sin\theta_{11}\cos\theta_{12}) \\
+\ & l_{21}\cos\theta_{21} + l_{22}\sin\theta_{21}\cos\theta_{22}] + \sum_{i=1}^{2}\tau_{i,11}\Big)\mathrm{d}y_0\mathrm{d}y\mathrm{d}\theta_{11}\mathrm{d}\theta_{12}\mathrm{d}\theta_{21}\mathrm{d}\theta_{22}, \qquad (5)
\end{aligned}$$

where the integration in taken for $0 < y < \infty$, $0 < \theta_{i1} < \pi$, $0 < \theta_{i2} < 2\pi$, $i = 1, 2$. This is the p.d.f. for general density generator f_n with $k_i \geq 3$. Let $F_1 = \Phi$ and $l_{ij} = 0$, $\tau_{i,11} = 0$, $\tau_{i12} = 0$ by the assumption of Theorem 2. Integrating out θ_{i2} with

$$\int_0^{2\pi}|\sin\theta_{i2}|^{k_i-3} = 2\pi^{1/2}\Gamma(k_i/2-1)/\Gamma(k_i/2-1/2),$$

we obtain (2). #

Proof of Theorem 3 Under the notation of Definition 3, by Fang (2003), $\mathbf{z}_1|\mathbf{z}_2 \sim S_{k_1}(\xi_1, I, \lambda_1, \alpha_1; \phi_{k_1+1})$, $\mathbf{z}_2 \sim S_{k_2}(\xi_2, I, c_2\lambda, c_2\alpha_2; \phi_{k_2+1})$, where $\lambda_1 = \lambda + \alpha_2'(\mathbf{z}_2 - \xi_2)$, $c_2 = (1 + \alpha_1'\alpha_1)^{-1/2}$. Then $Q_1|\mathbf{z}_2 \sim NG_1(k_1/2; \{\tau_{1,11}, \tau_{1,12}, \tau_{1,22}\}, \lambda_1; \phi_{k_1+1})$. Note $c_2\lambda + c_2\alpha_2'(\mathbf{z}_2 - \xi_2) = \lambda_1/(1+\tau_{1,22})^{1/2}$ and $c_2\lambda/(1 + c_2^2\alpha_2'\alpha_2)^{1/2} = \lambda/c_0$. Using the moment generating function for NG_1 in Fang (2004), we obtain

$$\begin{aligned}
\mathsf{E}(\exp(tF)) &= \mathsf{E}_{\mathbf{z}_2}[\mathsf{E}(\exp(\widetilde{t}\mathbf{z}_1'\mathbf{z}_1))|\mathbf{z}_2] \\
&= \int I_{(0,1/2)}(\widetilde{t})\Phi\Big(\frac{\lambda_1 + 2\tau_{1,12}\widetilde{t}(1-2\widetilde{t})^{-1}}{(1+\tau_{1,22}(1-2\widetilde{t})^{-1})^{1/2}}\Big)\exp(\tau_{1,11}\widetilde{t}(1-2\widetilde{t})^{-1}) \\
&\quad \times(1-2\widetilde{t})^{-k_1/2}/\Phi(\lambda_1/(1+\tau_{1,22})^{1/2})\Phi(c_2\lambda + c_2\alpha_2'(\mathbf{z}_2 - \xi_2)) \\
&\quad \times\exp(-(\mathbf{z}_2-\xi_2)'(\mathbf{z}_2-\xi_2)/2)(2\pi)^{-k_2/2}/\Phi(c_2\lambda/(1+c_2^2\alpha_2'\alpha_2)^{1/2}) \\
&= \int I_{(0,1/2)}(\widetilde{t})\Phi\Big(\frac{\lambda_1 + 2\tau_{1,12}\widetilde{t}(1-2\widetilde{t})^{-1}}{(1+\tau_{1,22}(1-2\widetilde{t})^{-1})^{1/2}}\Big)\exp(\tau_{1,11}\widetilde{t}(1-2\widetilde{t})^{-1}) \\
&\quad \times(1-2\widetilde{t})^{-k_1/2}\exp(-(\mathbf{z}_2-\xi_2)'(\mathbf{z}_2-\xi_2)/2)(2\pi)^{-k_2/2}/\Phi(\lambda/c_0),
\end{aligned}$$

where $\widetilde{t} = t/\mathbf{z}_2'\mathbf{z}_2 \cdot k_2/k_1$ and $\mathsf{E}_{\mathbf{z}_2}$ denotes taking expectation with respect to the distribution of $\mathbf{z}_2$. Making transformations as in the proof of Theorem 1 on $\mathbf{z}_2$, we obtain (3) in Theorem 3. #

Proof of Theorem 4 As in the proof of Theorem 3, we use the distributions of $\mathbf{z}_1|\mathbf{z}_2$ and $\mathbf{z}_2$ and the formula of mean of the quadratic form in Fang (2004) to obtain

$$\begin{aligned} \mathsf{E}(F) &= \mathsf{E}_{\mathbf{z}_2}[\mathsf{E}(Q_1|\mathbf{z}_2)Q_2^{-1}]\cdot k_2/k_1 \\ &= \frac{k_2}{k_1}\int\Big[\zeta_1\Big(\frac{\lambda_1}{(1+\tau_{1,22})^{1/2}}\Big)\Big(\frac{2\tau_{1,12}}{(1+\tau_{1,22})^{1/2}}-\frac{\lambda_1\tau_{1,22}}{(1+\tau_{1,22})^{3/2}}\Big)+\tau_{1,11}+k_1\Big] \\ &\quad\times(\mathbf{z}_2\mathbf{z}_2')^{-1}\Phi(c_2\lambda+c_2\alpha_2'\mathbf{z}_2)\exp(-\mathbf{z}_2\mathbf{z}_2'/2)(2\pi)^{-k_2/2}\mathrm{d}\mathbf{z}_2/\Phi(\lambda/c_0) \\ &= \frac{k_2}{k_1}\int\Big\{\phi\Big(\frac{\lambda+\alpha_2'\mathbf{z}_2}{(1+\tau_{1,22})^{1/2}}\Big)\Big[A-\frac{\alpha_2'\mathbf{z}_2\tau_{1,22}}{(1+\tau_{1,22})^{3/2}}\Big]+\Phi\Big(\frac{\lambda+\alpha_2'\mathbf{z}_2}{(1+\tau_{1,22})^{1/2}}\Big)B\Big\} \\ &\quad\times(\mathbf{z}_2'\mathbf{z}_2)^{-1}\exp(-\mathbf{z}_2'\mathbf{z}_2/2)(2\pi)^{-k_2/2}\mathrm{d}\mathbf{z}_2/\Phi(\lambda/c_0), \end{aligned} \tag{6}$$

where $\zeta_1(x)=\phi(x)/\Phi(x)$. By a formula in Zhang and Fang (1982, p.468) (6) is equal to

$$\begin{aligned} &\frac{k_2}{k_1}(2\pi)^{-k_2/2}\frac{\pi^{k_2/2-0.5}}{\Gamma(k_2/2-0.5)}\int_{R^2}|\mathbf{y}_2|^{k_2-2}\Big\{\phi\Big(\frac{\lambda+\|\alpha_2\|y_1}{(1+\tau_{1,22})^{1/2}}\Big)\Big[A-\frac{\|\alpha_2\|y_1\tau_{1,22}}{(1+\tau_{1,22})^{3/2}}\Big] \\ +\;&\Phi\Big(\frac{\lambda+\|\alpha_2\|y_1}{(1+\tau_{1,22})^{1/2}}\Big)B\Big\}(\mathbf{y}'\mathbf{y})^{-1}\exp(-\mathbf{y}'\mathbf{y}/2)\mathrm{d}\mathbf{y}/\Phi(\lambda/c_0). \end{aligned}$$

Making transformation $y_1=r\cos\theta$, $y_2=r\sin\theta$, $r>0$, $0<\theta<2\pi$, we obtain

$$\mathsf{E}(F)=\frac{k_2}{k_1 2^{k_2/2}\pi^{1/2}\Gamma(k_2/2-0.5)}\Big[I_1A-I_3\Big(\frac{\tau_{1,22}}{1+\tau_{1,22}}\Big)+I_2B\Big], \tag{7}$$

where

$$I_2=\int|\sin\theta|^{k_2-2}r^{k_2-3}\Phi\Big(\frac{\lambda+\tau_{2,22}^{1/2}r\cos\theta}{(1+\tau_{1,22})^{1/2}}\Big)\exp\Big(-\frac{r^2}{2}\Big)\mathrm{d}r\mathrm{d}\theta/\Phi(\lambda/c_0). \tag{8}$$

Let $t=0$, $\tau_{2,11}=0$, $\tau_{2,11}=0$, in (3) and make transformation $y_{21}=r\cos\theta_1$, $y_{22}=r\sin\theta_1\cos\theta_2$, $y_{23}=r\sin\theta_1\sin\theta_2$, $r>0$, $0<\theta_1<\pi$, $0<\theta_2<2\pi$, we obtain

$$\begin{aligned} 1 &= c(k_2)\int\Phi\Big(\frac{\lambda+\tau_{2,22}^{1/2}y_{21}-\tau_{2,12}}{(1+\tau_{1,22})^{1/2}}\Big)\exp(-\mathbf{y}_2'\mathbf{y}_2/2)(2\pi)^{-k_2/2}\mathrm{d}\mathbf{y}_2/\Phi(\lambda/c_0) \\ &= c(k_2)(2\pi)^{-k_2/2}\int r^{k_2-1}|\sin\theta_1|^{k_2-2}|\sin\theta_2|^{k_2-3}\Phi\Big(\frac{\lambda+\tau_{2,22}^{1/2}r\cos\theta_1}{(1+\tau_{1,22})^{1/2}}\Big) \\ &\quad\times\exp\Big(-\frac{r^2}{2}\Big)\mathrm{d}r\mathrm{d}\theta_1 d\theta_2/\Phi(\lambda/c_0). \end{aligned} \tag{9}$$

Integrating out θ_2 and changing range of integration of θ_1, we obtain

$$\begin{aligned} 1 &= \frac{1}{\Gamma(k_2/2-0.5)\pi^{1/2}2^{k_2/2}}\int_{\substack{r>0\\0<\theta<2\pi}}r^{k_2-1}|\sin\theta|^{k_2-2}\Phi\Big(\frac{\lambda+\tau_{2,22}^{1/2}r\cos\theta}{(1+\tau_{1,22})^{1/2}}\Big) \\ &\quad\times\exp\Big(-\frac{r^2}{2}\Big)\mathrm{d}r\mathrm{d}\theta/\Phi(\lambda/c_0). \end{aligned} \tag{10}$$

Integrating by parts with respect to $\mathrm{d}r$ in (8) and applying (10) we have

$$I_2=-\frac{I_3}{k_2-2}+\frac{\Gamma(k_2/2-0.5)\pi^{1/2}2^{k_2/2}}{k_2-2}, \tag{11}$$

which combined with (7) establishes (4). #

§5. Discussion

The main properties of the F statistic of the skew elliptical variables are investigated and a version of the skew F distribution is introduced. This enlarge the family of the usual F distributions and provides a tool for the investigation of the robustness of the F test with departure represented by the skew normal distribution. However, the expressions are rather involved as the result of the presence of the skewness parameters, incurring difficulties for further derivation of the properties of the F test in theory. We give some numerical results to illustrate the situation. Invariance of the distribution of the F statistic is obtained for the central case with one of the skewness parameter $\lambda = 0$. The non-robustness of the F test for variances in the skew normal family with $\lambda \neq 0$ is not surprising. Similar conclusion for independent non-normal population holds, see Lehmann (1986). The study of the skew F distribution also provides a basis for the statistical inference of the skew elliptical populations.

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斜球变量的 F 统计量

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本文研究斜球变量的 F 统计量的性质并引进了一类斜 F 分布．我们得到了密度函数，分布函数，矩发生函数和矩，考察了 F 检验在斜正态分布族内的性质．

关 键 词：斜球分布，统计量，假设检验．
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