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# A Class of Integral Equations of Erlang(2) Risk Process under Interest Force*

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**Abstract**

In this paper, we consider an Erlang(2) risk model with a constant interest force for an insurance portfolio. By using the techniques of Sundt & Teugels (1995) and Yang & Zhang (2001a, 2001b and 2001c), the integral equation and exponential integral equation satisfied by survival probability have been obtained. Then we have investigated the two-order differential equation satisfied by the Laplace-Stieltjes transform of survival probability.

**Keywords**: Erlang(2) process, Survival probability, Interest force, Laplace-Stieltjes transform.
**AMS Subject Classification**: 60K05, 62P05, 90A46.

## § 1. Introduction

In the classical risk theory, in which claims occur as a Poisson process, it is often assumed that there is no investment income. However, as we know, a large portion of the surplus of the insurance companies comes from investment income. In recent years risk models with interest rate have recently received a remarkable amount of attention and there have been some papers which incorporate deterministic interest rate models in the risk theory. Sundt and Teugels (1995) considered a compound Poisson model with a constant interest force. By using similar techniques to the classical model, upper and lower bounds for the ruin probability were obtained. Some related problems were discussed in Boogaert and Crijns (1987). Yang (1999) considered a discrete time risk model with a constant interest force and by using martingale inequalities, both Lundberg-type inequality and non-exponential upper bounds for ruin probabilities were obtained. Paulsen and Gjessing (1997) considered a diffusion perturbed classical risk model. Under the assumption of stochastic investment income, but a constant interest rate, a Lundberg-type inequality was obtained. Paulsen (1998) provided a very good survey on this subject. Yang and Zhang (2001a, 2001b and 2001c) showed that the techniques of Sundt and Teugels (1995) can be used to deal with the distributions of surplus immediately after ruin and before ruin.

Recently, there is also considerable interest in the development of Erlang(2) no-interest risk model, which assumes that the waiting times between claims are Erlang(2) distributions (claims arrival thus forming an Erlang process). Dickson (1998) considered the problem of finding the survival probability for Erlang risk processes and then derived expressions for the probability and severity of ruin and for the probability of absorption by an upper barrier. Dickson and Hipp (1998) considered a risk process in which claim inter-arrival times have an Erlang(2) distribution. They considered the infinite time survival probability as a compound geometric random variable and gave expressions from which both the survival probability from initial surplus zero and the ladder height distribution can be calculated.

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Apart from purely mathematical breakthrough, Erlang(2) risk models allow us to assume contagion between claims, i.e., to deal with non-Poissonian claims' arrivals. In fact, renewal non-Poissonian risk models do not look like a mere analytical over-complication, modern mass media and telecommunication networks could introduce substantial and sometimes unpredictable dependence into behaviour of insured persons which eventually could make an assumption on the Poissonian origin of claims' arrival suspicious [see Malinovskii (1998)].

The purpose of this paper is to consider the survival probability of Erlang(2) process under interest force. We have obtained the integral equation and exponential integral equation satisfied by the survival probability. Also we present a two-order differential equation satisfied by the Laplace-Stieltjes transform of survival probability.

## § 2. The Model

In this paper we shall consider a special Sparre Andersen risk model for the risk process, in which claims occur as an Erlang(2) process. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables, where $T_1$ denotes the time until the first claim and for $i > 1$, $T_i$ denotes the time between the $(i-1)$-th and $i$-th claims. We assume that $T_i$ has an Erlang(2,$\beta$) distribution with density function

$$k(t) = \beta^2 t e^{-\beta t}, \qquad t > 0,\ \beta > 0.$$

We also assume that the premium which the insurance company receives is paid continuously with a constant rate $p$. In addition to the premium income, the company also receives interest on its reserves with a constant force of $\delta$. Let $U_\delta(t)$ denote the value of the reserve at time $t$. From the above assumption, it follows that

$$\mathrm{d}U_\delta(t) \doteq p\,\mathrm{d}t + U_\delta(t)\cdot\delta\,\mathrm{d}t - \mathrm{d}X(t), \tag{2.1}$$

where $X(t) = \sum_{j=1}^{N(t)} Y_j$. $N(t)$ denotes the number of claims occurring in an insurance portfolio in the time interval $(0, t]$ while $Y_i$ denotes the amount of the $i$th claim. We assume that $\{N(t), t \geq 0\}$ is a Eralng(2) renewal process with waiting times $T_i$, $i \geq 1$, i.e.,

$$N(t) = \max\{n : T_1 + T_2 + \cdots + T_n \leq t\}.$$

We also assume that the claim amounts are independent of the claim number process, positive and mutually independent and identically distributed with the common distribution $F$. $F$ satisfies $F(0) = 0$ and $\mu = \int_0^\infty x\mathrm{d}F(x)$.

From Sundt and Teugels (1995) and (2.1), we know that

$$U_\delta(t) = u\,e^{\delta t} + p\bar{s}_{\overline{t}|}^{(\delta)} - \int_0^t e^{\delta(t-v)}\mathrm{d}X(v), \tag{2.2}$$

where $u = U_\delta(0) \geq 0$ is the initial surplus of an insurance company and

$$\bar{s}_{\overline{t}|}^{(\delta)} = \int_0^t e^{\delta v}\mathrm{d}v = \begin{cases} t, & \text{if } \delta = 0, \\ \dfrac{e^{\delta t} - 1}{\delta}, & \text{if } \delta > 0. \end{cases}$$

Let $\psi_\delta(u)$ denote the ultimate ruin probability with initial reserve $u$. That is

$$\psi_\delta(u) = \mathsf{P}\Big\{ \bigcup_{t\geq 0} (U_\delta(t) < 0) \,\Big|\, U_\delta(0) = u \Big\}.$$

We use $\overline{\psi}_\delta(u) = 1 - \psi_\delta(u)$ to denote the survival probability (i.e. the probability that ruin never occurs).

## §3. Equations for $\overline{\psi}_\delta(u)$

In this section, we will try to obtain the integral equations for $\overline{\psi}_\delta(u)$. Using the renewal property of the surplus process, an integral equation and another exponential integral equation, satisfied by the survival probability, can be obtained. This is the common technique in risk theory. Although we cannot, in general, solve the integral equations, some asymptotic results maybe be obtained by using the integral equation. As consequences of main results, several integral equations satisfied by survival probability $\overline{\psi}_0(u)$ in a risk process with no interest force, where claim inter-arrival times have an Erlang(2) distribution, are obtained. At last, we present one two-order differential equation satisfied by the Laplace-Stieltjes transform of survival probability. The main results of this section are stated in the following Theorem 1, Theorem 2 and Theorem 3.

**Theorem 1**

$$\begin{aligned}\overline{\psi}_\delta(u) &= \frac{1}{(p+\delta u)^2}\int_0^u (3\delta p-\delta^2 u-2\beta\delta u+4\delta^2 v+4\beta\delta v+2\beta p)\overline{\psi}_\delta(v)\mathrm{d}v+\frac{p^2}{(p+\delta u)^2}\overline{\psi}_\delta(0)\\&\quad+\frac{\beta^2\mu-\delta p-2p\beta+f(\delta)}{(p+\delta u)^2}u-\frac{\beta^2}{(p+\delta u)^2}\int_0^u\int_0^v\overline{\psi}_\delta(v-y)(1-F(y))\mathrm{d}y\mathrm{d}v,\end{aligned}\tag{3.1}$$

where $f(\delta)$ is defined by (3.6) below.

**Proof** Given the first claim time $T_1=t$ and the first claim amount $Y_1=y$, the reserve just after the first claim is $ue^{\delta t}+p(e^{\delta t}-1)/\delta-y$. The conditional probability that the company will survive is $\overline{\psi}_\delta[ue^{\delta t}+p(e^{\delta t}-1)/\delta-y]$. Let $\overline{\psi}_\delta(u)=0$ when $u<0$. Thus we obtain

$$\begin{aligned}\overline{\psi}_\delta(u) &= \int_0^\infty \beta^2 te^{-\beta t}\int_0^\infty\overline{\psi}_\delta\Big(ue^{\delta t}+p\frac{e^{\delta t-1}}{\delta}-y\Big)\mathrm{d}F(y)\mathrm{d}t\\&= \int_0^\infty\beta^2 te^{-\beta t}\int_0^{ue^{\delta t}+pe^{\delta t-1}/\delta}\overline{\psi}_\delta(ue^{\delta t}+pe^{\delta t-1}/\delta-y)\mathrm{d}F(y)\mathrm{d}t.\end{aligned}$$

By using the substitution $s=ue^{\delta t}+pe^{\delta t-1}/\delta$, we have that

$$\begin{aligned}\overline{\psi}_\delta(u) &= \int_u^\infty\beta^2\Big(\frac{1}{\delta}\ln\frac{s\delta+p}{u\delta+p}\Big)e^{-(\beta/\delta)\ln[(s\delta+p)/(u\delta+p)]}\frac{1}{s\delta+p}\int_0^s\overline{\psi}_\delta(s-y)\mathrm{d}F(y)\mathrm{d}s\\&= \frac{\beta^2}{\delta}(u\delta+p)^{\beta/\delta}\int_u^\infty(s\delta+p)^{-1-\beta/\delta}\ln\frac{s\delta+p}{u\delta+p}\int_0^s\overline{\psi}_\delta(s-y)\mathrm{d}F(y)\mathrm{d}s.\end{aligned}\tag{3.2}$$

By taking derivative of the above expression with respect to $u$, and rearranging the terms, we get that

$$(u\delta+p)\frac{\mathrm{d}\overline{\psi}_\delta(u)}{\mathrm{d}u}=\beta\overline{\psi}_\delta(u)-\beta^2(u\delta+p)^{\beta/\delta}\int_u^\infty(p+s\delta)^{-1-\beta/\delta}\int_0^s\overline{\psi}_\delta(s-y)\mathrm{d}F(y)\mathrm{d}s.\tag{3.3}$$

Taking derivative on both sides of (3.3) yields

$$(p+\delta u)^2\frac{\mathrm{d}^2\overline{\psi}_\delta(u)}{\mathrm{d}u^2}+(\delta-2\beta)(p+\delta u)\frac{\mathrm{d}\overline{\psi}_\delta(u)}{\mathrm{d}u}+\beta^2\overline{\psi}_\delta(u)=\beta^2\int_0^u\overline{\psi}_\delta(u-y)\mathrm{d}F(y).\tag{3.4}$$

Clearly, integrating both sides of (3.4) from 0 to $u$ means

$$\begin{aligned}&(p+\delta u)^2\frac{\mathrm{d}\overline{\psi}_\delta(u)}{\mathrm{d}u}-p^2\overline{\psi}'_\delta(0)+(\delta^2+2\beta\delta)\int_0^u\overline{\psi}_\delta(v)\mathrm{d}v\\&-(\delta^2 u+2\beta\delta u+\delta p+2\beta p)\overline{\psi}_\delta(u)+(\delta p+2p\beta)\overline{\psi}_\delta(0)\\=\;&-\beta^2\int_0^u\overline{\psi}_\delta(u-y)(1-F(y))\mathrm{d}y.\end{aligned}\tag{3.5}$$

After some rearrangement of the terms, we obtain that

$$\begin{aligned}&(p+\delta u)^2\frac{\mathrm{d}\overline{\psi}_\delta(u)}{\mathrm{d}u}-(\delta^2 u+2\beta\delta u)\overline{\psi}_\delta(u)+(\delta^2+2\beta\delta)\int_0^u\overline{\psi}_\delta(v)\mathrm{d}v\\=\;&(\delta p+2\beta p)\overline{\psi}_\delta(u)+p^2\overline{\psi}'_\delta(0)-(\delta p+2p\beta)\overline{\psi}_\delta(0)-\beta^2\int_0^u\overline{\psi}_\delta(u-y)(1-F(y))\mathrm{d}y.\end{aligned}$$

Letting $u \to \infty$, the right hand side of the last equality tends to

$$(\delta p + 2\beta p) + p^2 \overline{\psi}'_\delta(0) - (\delta p + 2p\beta)\overline{\psi}_\delta(0) - \beta^2 \mu,$$

and the left hand side goes to a constant, denoted by $f(\delta)$, i.e.,

$$f(\delta) = \lim_{u\to\infty} \left\{ (p+\delta u)^2 \frac{\mathrm{d}\overline{\psi}_\delta(u)}{\mathrm{d}u} - (\delta^2 u + 2\beta\delta u)\overline{\psi}_\delta(u) + (\delta^2 + 2\beta\delta)\int_0^u \overline{\psi}_\delta(v)\mathrm{d}v \right\}. \tag{3.6}$$

It is obvious that $f(0) = 0$. So it follows that

$$\overline{\psi}'_\delta(0) = \frac{\beta^2\mu - (\delta p + 2p\beta)(1 - \overline{\psi}_\delta(0)) + f(\delta)}{p^2}. \tag{3.7}$$

Plugging this expression of $\overline{\psi}'_\delta(0)$ by (3.7) into (3.5), we have that

$$\begin{aligned}
&(p+\delta u)^2 \overline{\psi}'_\delta(u) - (\delta^2 u + 2\beta\delta u + \delta p + 2\beta p)\overline{\psi}_\delta(u) \\
= \ &(\beta^2\mu - \delta p - 2\beta p + f(\delta)) - \beta^2 \int_0^u \overline{\psi}_\delta(u-y)(1-F(y))\mathrm{d}y - (\delta^2 + 2\beta\delta)\int_0^u \overline{\psi}_\delta(v)\mathrm{d}v. \qquad (3.8)
\end{aligned}$$

Finally, by integrating both sides of (3.8) from 0 to $u$, we know that

$$\begin{aligned}
\overline{\psi}_\delta(u) \ = \ &\frac{1}{(p+\delta u)^2}\int_0^u (3\delta p - \delta^2 u - 2\beta\delta u + 4\delta^2 v + 4\beta\delta v + 2\beta p)\overline{\psi}_\delta(v)\mathrm{d}v + \frac{p^2}{(p+\delta u)^2}\overline{\psi}_\delta(0) \\
&+ \frac{\beta^2\mu - \delta p - 2p\beta + f(\delta)}{(p+\delta u)^2} u - \frac{\beta^2}{(p+\delta u)^2}\int_0^u \int_0^v \overline{\psi}_\delta(v-y)(1-F(y))\mathrm{d}y\mathrm{d}v. \qquad (3.9)
\end{aligned}$$

(3.9) is called the hidden integral equation that $\overline{\psi}_\delta(u)$ satisfies. #

**Remark 1** If we let $\delta = 0$, then it follows from (3.3), (3.4) and (3.7) that

$$\begin{aligned}
&p\frac{\mathrm{d}\overline{\psi}_0(u)}{\mathrm{d}u} = \beta\overline{\psi}_0(u) - \frac{\beta^2}{p}\int_u^\infty e^{-\beta(s-u)/p}\int_0^s \overline{\psi}_0(s-y)\mathrm{d}F(y)\mathrm{d}s, \\
&p^2\frac{\mathrm{d}^2\overline{\psi}_0(u)}{\mathrm{d}u^2} - 2\beta p\frac{\mathrm{d}\overline{\psi}_0(u)}{\mathrm{d}u} + \beta^2\overline{\psi}_0(u) = \beta^2\int_0^u \overline{\psi}_0(u-y)\mathrm{d}F(y), \\
&\overline{\psi}'_0(0) = \frac{\beta^2\mu - 2p\beta(1-\overline{\psi}_0(0))}{p^2}.
\end{aligned}$$

The last three equations were obtained by Dickson (1998) and Dickson & Hipp (1998).

**Corollary 1** If we consider the same risk model as in Dickson & Hipp (1998), then it follows from (3.9) that

$$\overline{\psi}_0(u) = \overline{\psi}_0(0) + \frac{\beta^2\mu u - 2p\beta u}{p^2} + \frac{2\beta}{p}\int_0^u \overline{\psi}_0(v)\mathrm{d}v - \frac{\beta^2}{p^2}\int_0^u\int_0^v \overline{\psi}_0(v-y)(1-F(y))\mathrm{d}y\mathrm{d}v, \tag{3.10}$$

which is the integral equation satisfied by survival probability $\overline{\psi}_0(u)$ in a risk process with no interest force, where claim inter-arrival times have an Erlang(2) distribution.

The (3.11) below is called the exponential hidden integral equation that survival probability $\overline{\psi}_\delta(u)$ satisfies.

**Theorem 2** The survival probability $\overline{\psi}_\delta(u)$ also satisfies the following exponential equation, i.e.,

$$\begin{aligned}
e^{-\chi(u)}\overline{\psi}_\delta(u) \ = \ &\overline{\psi}_\delta(0) - \int_0^u \frac{\beta^2}{(p+\delta v)^2}e^{-\chi(v)}\int_0^v \overline{\psi}_\delta(v-y)(1-F(y))\mathrm{d}y\mathrm{d}v \\
&- \int_0^u \frac{\delta^2 + 2\beta\delta}{(p+\delta v)^2}e^{-\chi(v)}\int_0^v \overline{\psi}_\delta(y)\mathrm{d}y\mathrm{d}v + \int_0^u \frac{\beta^2\mu - \delta p - 2p\beta + f(\delta)}{(p+\delta v)^2}e^{-\chi(v)}\mathrm{d}v, \qquad (3.11)
\end{aligned}$$

where

$$\chi(u) = \int_0^u \frac{\delta^2 w + 2\beta\delta w + \delta p + 2\beta p}{(p+\delta w)^2}\mathrm{d}w,$$

and $f(\delta)$ is defined by (3.6).

**Proof** If we rearrange (3.8), we can obtain

$$\begin{aligned}
&\overline{\psi}'_\delta(u) - \frac{\delta^2 u + 2\beta\delta u + \delta p + 2\beta p}{(p+\delta u)^2}\overline{\psi}_\delta(u) \\
= &\ \frac{\beta^2\mu - \delta p - 2p\beta + f(\delta)}{(p+\delta u)^2} - \frac{\delta^2 + 2\beta\delta}{(p+\delta u)^2}\int_0^u \overline{\psi}_\delta(v)\mathrm{d}v - \frac{\beta^2}{(p+\delta u)^2}\int_0^u \overline{\psi}_\delta(u-y)(1-F(y))\mathrm{d}y. \qquad (3.12)
\end{aligned}$$

Multiplying by $e^{-\int_0^u (\delta^2 v + 2\beta\delta v + \delta p + 2\beta p)/(p+\delta v)^2 \mathrm{d}v}$ on both sides of (3.12), then we can integrate both sides from 0 to $u$, and hence it follows that

$$\begin{aligned}
&\overline{\psi}_\delta(u) e^{-\int_0^u (\delta^2 v + 2\beta\delta v + \delta p + 2\beta p)/(p+\delta v)^2 \mathrm{d}v} - \overline{\psi}_\delta(0) \\
= &\ (\beta^2\mu - \delta p - 2\beta p + f(\delta)) \int_0^u \frac{1}{(p+\delta v)^2} e^{-\int_0^v (\delta^2 w + 2\beta\delta w + \delta p + 2\beta p)/(p+\delta w)^2 \mathrm{d}w} \mathrm{d}v \\
&-(\delta^2 + 2\beta\delta)\int_0^u \frac{1}{(p+\delta v)^2}\Big(\int_0^v \overline{\psi}_\delta(w)\mathrm{d}w\Big) e^{-\int_0^v (\delta^2 w + 2\beta\delta w + \delta p + 2\beta p)/(p+\delta w)^2 \mathrm{d}w} \mathrm{d}v \\
&-\beta^2 \int_0^u \frac{1}{(p+\delta v)^2}\Big(\int_0^v \overline{\psi}_\delta(v-y)(1-F(y))\mathrm{d}y\Big) e^{-\int_0^v (\delta^2 w + 2\beta\delta w + \delta p + 2\beta p)/(p+\delta w)^2 \mathrm{d}w} \mathrm{d}v. \qquad (3.13)
\end{aligned}$$

Let

$$\chi(u) = \int_0^u \frac{\delta^2 w + 2\beta\delta w + \delta p + 2\beta p}{(p+\delta w)^2}\mathrm{d}w.$$

So (3.11) comes immediately from (3.13) above. #

**Corollary 2** If we consider the same risk model as in Dickson & Hipp (1998), then it follows from (3.11) that

$$e^{-2\beta u/p}\overline{\psi}_0(u) = \overline{\psi}_0(0) + \Big(\frac{\beta\mu}{2p} - 1\Big)(1 - e^{-2\beta u/p}) - \frac{\beta^2}{p^2}\int_0^u e^{-2\beta v/p}\int_0^v \overline{\psi}_0(v-y)(1-F(y))\mathrm{d}y\mathrm{d}v, \qquad (3.14)$$

which is called the exponential integral equation satisfied by survival probability $\overline{\psi}_0(u)$ in a risk process with no interest force, where claim inter-arrival times have an Erlang(2) distribution.

We now have obtained the integral equation satisfied by $\overline{\psi}_\delta(u)$. As an application of Theorem 1, we get the following two-order differential equation satisfied by $\widehat{\phi}_\delta(s)$, the Laplace-Stieltjes transform of $\overline{\psi}_\delta(u)$, given by $\widehat{\phi}_\delta(s) = \int_0^\infty e^{-su}\mathrm{d}\overline{\psi}_\delta(u)$. To this end, let $\widehat{F}_1(s) = \int_0^\infty e^{-su}\mathrm{d}F_1(u)$ be the Laplace-Stieltjes transform of $F_1(u)$, where $F_1$ is the equilibrium distribution of $F$, given by $F_1(x) = (1/\mu)\int_0^x (1-F(y))\mathrm{d}y$.

**Theorem 3** The Laplace-Stieltjes transform of $\overline{\psi}_\delta(u)$ satisfies the following two-order differential equation, i.e.,

$$\begin{aligned}
&\delta^2\widehat{\phi}''_\delta(s) + \Big(\frac{\delta^2 + 2\beta\delta}{s} - 2p\delta\Big)\widehat{\phi}'_\delta(s) + \Big(p^2 + \frac{\beta^2\mu\widehat{F}_1(s)}{s} - \frac{\delta p + 2\beta p}{s}\Big)\widehat{\phi}_\delta(s) \\
= &\ \Big(\frac{\delta p + 2\beta p}{s} + p^2 - \frac{\beta^2\mu\widehat{F}_1(s)}{s}\Big)\overline{\psi}_\delta(0) + \frac{\beta^2\mu - \delta p - 2p\beta + f(\delta)}{s}. \qquad (3.15)
\end{aligned}$$

**Proof** Firstly, we note that

$$\begin{aligned}
\int_0^u\int_0^v \overline{\psi}_\delta(v-y)(1-F(y))\mathrm{d}y\mathrm{d}v &= \int_0^u\Big(\int_0^v \overline{\psi}_\delta(y)(1-F(v-y))\mathrm{d}y\Big)\mathrm{d}v \\
&= \int_0^u\Big(\int_y^u (1-F(v-y))\mathrm{d}v\Big)\overline{\psi}_\delta(y)\mathrm{d}y \\
&= \int_0^u\Big(\int_0^{u-y} (1-F(v))\mathrm{d}v\Big)\overline{\psi}_\delta(y)\mathrm{d}y \\
&= \mu\int_0^u F_1(u-y)\overline{\psi}_\delta(y)\mathrm{d}y \\
&= \mu\int_0^u \overline{\psi}_\delta(u-y)F_1(y)\mathrm{d}y,
\end{aligned}$$

so, it follows from (3.1) that

$$\begin{aligned}(p+\delta u)^2\overline{\psi}_\delta(u) &= p^2\overline{\psi}_\delta(0)+(3\delta p+2\beta p-\delta^2 u-2\beta\delta u)\int_0^u\overline{\psi}_\delta(v)\mathrm{d}v\\ &\quad+(\beta^2\mu-\delta p-2\beta p+f(\delta))u-\beta^2\mu\int_0^u\overline{\psi}_\delta(u-y)F_1(y)\mathrm{d}y\\ &\quad+(4\delta^2+4\beta\delta)\int_0^u v\overline{\psi}_\delta(v)\mathrm{d}v. \qquad (3.16)\end{aligned}$$

Taking Laplace-Stieltjes transforms on both sides of (3.16) yields

$$\begin{aligned}&p^2\int_0^\infty e^{-su}\mathrm{d}\overline{\psi}_\delta(u)+\delta^2\int_0^\infty e^{-su}\mathrm{d}(u^2\overline{\psi}_\delta(u))+2\delta p\int_0^\infty e^{-su}\mathrm{d}(u\overline{\psi}_\delta(u))\\ =\ &p^2\overline{\psi}_\delta(0)+(3\delta p+2\beta p)\int_0^\infty e^{-su}\overline{\psi}_\delta(u)\mathrm{d}u-(\delta^2+2\beta\delta)\int_0^\infty e^{-su}\mathrm{d}\Big(u\int_0^u\overline{\psi}_\delta(v)\mathrm{d}v\Big)\\ &+\frac{(\beta^2\mu-\delta p-2\beta p+f(\delta))}{s}-\beta^2\mu\int_0^\infty e^{-su}\mathrm{d}\Big(\int_0^u\overline{\psi}_\delta(u-y)F_1(y)\mathrm{d}y\Big)\\ &+(4\delta^2+4\beta\delta)\int_0^\infty e^{-su}u\overline{\psi}_\delta(u)\mathrm{d}u. \qquad (3.17)\end{aligned}$$

It is easy to derive the following identities by using the formula of integration by parts, i.e.,

$$\begin{aligned}\int_0^\infty e^{-su}\overline{\psi}_\delta(u)\mathrm{d}u &= -\frac{1}{s}\int_0^\infty\overline{\psi}_\delta(u)\mathrm{d}(e^{-su})=\frac{1}{s}\overline{\psi}_\delta(0)+\frac{1}{s}\widehat{\phi}_\delta(s),\\ \int_0^\infty e^{-su}\mathrm{d}(u\overline{\psi}_\delta(u)) &= \int_0^\infty e^{-su}\overline{\psi}_\delta(u)\mathrm{d}u+\int_0^\infty e^{-su}u\mathrm{d}(\overline{\psi}_\delta(u))\\ &= -\widehat{\phi}'_\delta(s)+\frac{1}{s}\overline{\psi}_\delta(0)+\frac{1}{s}\widehat{\phi}_\delta(s)\\ &= \frac{1}{s}\overline{\psi}_\delta(0)+\frac{1}{s}\widehat{\phi}_\delta(s)-\widehat{\phi}'_\delta(s),\\ \int_0^\infty ue^{-su}\overline{\psi}_\delta(u)\mathrm{d}u &= \frac{1}{s}\Big(\frac{1}{s}\overline{\psi}_\delta(0)+\frac{1}{s}\widehat{\phi}_\delta(s)\Big)-\frac{1}{s}\widehat{\phi}'_\delta(s),\\ \int_0^\infty e^{-su}\mathrm{d}(u^2\overline{\psi}_\delta(u)) &= 2\int_0^\infty ue^{-su}\overline{\psi}_\delta(u)\mathrm{d}u+\int_0^\infty e^{-su}u^2\mathrm{d}(\overline{\psi}_\delta(u))\\ &= \widehat{\phi}''_\delta(s)-\frac{2}{s}\int_0^\infty u\overline{\psi}_\delta(u)\mathrm{d}(e^{-su})\\ &= \widehat{\phi}''_\delta(s)+\frac{2}{s}\Big(\frac{1}{s}\overline{\psi}_\delta(0)+\frac{1}{s}\widehat{\phi}_\delta(s)\Big)-\frac{2}{s}\widehat{\phi}'_\delta(s),\\ \int_0^\infty e^{-su}\mathrm{d}\Big(u\int_0^u\overline{\psi}_\delta(v)\mathrm{d}v\Big) &= \int_0^\infty e^{-su}\int_0^u\overline{\psi}_\delta(v)\mathrm{d}v\mathrm{d}u+\int_0^\infty e^{-su}u\overline{\psi}_\delta(u)\mathrm{d}u\\ &= \frac{1}{s}\int_0^\infty e^{-su}\overline{\psi}_\delta(u)\mathrm{d}u+\int_0^\infty e^{-su}u\overline{\psi}_\delta(u)\mathrm{d}u\\ &= \frac{2}{s}\Big(\frac{1}{s}\overline{\psi}_\delta(0)+\frac{1}{s}\widehat{\phi}_\delta(s)\Big)-\frac{1}{s}\widehat{\phi}'_\delta(s),\\ \int_0^\infty e^{-su}F_1(u)\mathrm{d}u &= -\frac{1}{s}\int_0^\infty F_1(u)\mathrm{d}(e^{-su})=\frac{1}{s}\widehat{F}_1(s),\\ \int_0^\infty e^{-su}\mathrm{d}\Big(\int_0^u\overline{\psi}_\delta(u-y)F_1(y)\mathrm{d}y\Big) &= \widehat{F}_1(s)\Big[\frac{1}{s}\overline{\psi}_\delta(0)+\frac{1}{s}\widehat{\phi}_\delta(s)\Big].\end{aligned}$$

Hence, (3.15) follows immediately from the above mentioned identities and (3.17). #

## References

[1] Abate, J., Whitt, W., Fourier series method for inverting transforms of probability distributions, *Queuing Systems*, **10**(1992), 5–88.

[2] Asmussen, S., Approximations for the probability of ruin within finite time, *Scandinavian Actuarial Journal*, **64**(1984), 31–57.

[3] Asmussen, S., Conjugate process and the simulation of ruin problems, *Stoch. Proc. Appl.*, **20**(1985), 213–229.

[4] Boogaert, P. and Crijns, V., Upper bound on ruin probabilities in case of negative loadings and positive interest rates, *Insurance: Mathematics and Economics*, **6**(1987), 221–232.

[5] Dickson, D.C.M., On a class of renewal risk processes, *North American Actuarial Journal*, **1**(1998), 60–68.

[6] Dickson, D.C.M. and Hipp, C., Ruin probabilities for Erlang(2) risk process, *Insurance: Mathematics and Economics*, **22**(1998), 251–262.

[7] Dickson, D.C.M. and Hipp, C., Ruin problems for Phase-Type(2) risk processes, *Scandinavian Actuarial Journal*, **2**(2000), 147–167.

[8] Malinovskii, V.K., Non-Poissonian claims' arrivals and calculation of the probability of ruin, *Insurance: Mathematics & Economics*, **22**(1998), 123–138.

[9] Paulsen, J., Ruin theory with compounding assets – a survey, *Insurance: Mathematics and Economics*, **22**(1998), 3–16.

[10] Paulsen, J. and Gjessing, H.K., Ruin theory with stochastic return on investments, *Advances in Applied Probability*, **29**(1997), 965–985.

[11] Resnick, S.I., *Adventures in Stochastic Processes*, Birkhäuser, Boston, 1992.

[12] Sundt, B. and Teugels, J.L., Ruin estimates under interest force, *Insurance: Mathematics and Economics*, **16**(1995), 7–22.

[13] Thorin, O., Probability of ruin, *Scandinavian Actuarial Journal*, **No.1**(1982), 65–102.

[14] Willmot, G.E. and Lin, X., Exact and approximate properties of the distribution of surplus before and after ruin, *Insurance: Mathematics and Economics*, **23**(1997), 91–110.

[15] Yang, H., Non-exponential bounds for ruin probability with interest effect included, *Scandinavian Actuarial Journal*, **1**(1998), 66–79.

[16] Yang, H.L. and Zhang, L.H., On the distribution of surplus immediately after ruin under interest force, *Insurance: Mathematics and Economics*, **29**(2001a), 247–255.

[17] Yang, H.L. and Zhang, L.H., On the distribution of surplus immediately before ruin under interest force, *Statistics and Probability Letters*, **55**(2001b), 329–338.

[18] Yang, H.L. and Zhang, L.H., The joint distribution of surplus immediately before ruin and the deficit at ruin under interest force, *North American Actuarial Journal*, **5(3)**(2001c), 92–103.

# 带息力的 Erlang(2) 风险过程下的一类积分方程

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本文考虑了带息力的 Erlang(2) 风险模型，利用 Sundt 和 Teugels(1995), Yang 和 Zhang(2001a, 2001b 和 2001c) 文中的技巧，得到了生存概率所满足的积分方程和指数型的积分方程，然后研究了生存概率的 Laplace-Stieltjes 变换所满足的二阶微分方程.

**关 键 词**: Erlang(2) 过程，生存概率，利息力， Laplace-Stieltjes 变换.

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