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An Asymptotic Estimation for the Distribution of the Minimum Value in Partial Sum Sequence of Stationary Ergodic Markov Chain and Its Application*

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Abstract

This paper gives an asymptotic estimation for the distribution of the minimum value in partial sum sequence of a stationary ergodic Markov chain with finite state space and the estimation to describe the asymptotic behaviour of extinction probability in the Athreya-Karlin BPRE. (i.e. branching process with random environments.)

Keywords: Perron-Frobenius theorem; Athreya-Karlin BPRE.
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§1. Introduction

Let $\{\xi_n\}, n = 1, 2, \dots$ be anyone sequence of random variables. Define $\{X_n\}, n = 0, 1, 2, \dots$, the sequence of partial sums of $\{\xi_n\}$, by

$$X_0 = 0, \qquad X_n = \sum_{i=1}^{n} \xi_i, \qquad (n = 1, 2, \dots).$$

In many practical problems, we need to consider the asymptotic estimation for the distribution of the minimum value of $\{X_n\}$. When $\{\xi_n\}$ is an iid sequence, the corresponding $\{X_n\}$ usually is called a random walk as well known. This particular case has been studied and investigated rather penetratingly. By the means of the renewal theory, one can obtain the results that we are interested in here (see Feller (1971)). However, when $\{\xi_n\}$ is not iid, the situation bacomes very complicated in general, about which little is known currently.

To extend the knowledge in the subject, we suppose that $\{\xi_n\}$ is a stationary ergodic Markov chain with finite state space $\mathcal{E} = \{e_i | i = 1, 2, \dots, m\}$. Denote the equilibrium distribution and the transition matrix of the chain by $p_i = \mathrm{P}\{\xi_n = e_i\}$ and $P = (p_{ij})_{m\times m} = (\mathrm{P}\{\xi_{n+1} = e_j | \xi_n = e_i\})_{m\times m}$ respectively, where $i, j = 1, 2, \dots, m; n = 1, 2, \dots$. Furthermore, write $M = \min_{n\geq 0}\{X_n\}$ and let $\asymp$ be the symbol introduced by Bingham et.(1988) with the sense:

$$f(t) \asymp g(t), t \to +\infty \Longleftrightarrow 0 < \liminf_{t\to+\infty} \frac{f(t)}{g(t)} \leq \limsup_{t\to+\infty} \frac{f(t)}{g(t)} < +\infty.$$

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In this paper, we present an asymptotic tail estimation for the distribution of M, $P\{M \le -t\}$, i.e. we have

Theorem If the above $\{\xi_n\}$ also satisfies the additional conditions below:

i) $E[\xi_1] > 0$;

ii) there exists at least one state $e_i < 0$ in $\mathcal{E}$;

iii) in $\mathcal{E}$ there exists a k-cycle for some state $e_j : e_j, e_{j_1}, \ldots, e_{j_{k-1}}$ restricted by

$$e_j + e_{j_1} + e_{j_2} + \cdots + e_{j_{k-1}} < 0 \quad \text{and} \quad p_{jj_1} \cdot p_{j_1 j_2} \cdot \ldots p_{j_{k-1} j} > 0,$$

then there must exists an unique constant $\theta > 0$, such that

$$P\{M \le -t\} \asymp \exp(-\theta t), \qquad t \to +\infty.$$

In section 2 we shall prove the above theorem and in section 3 we shall describe an application to Athreya-Karlin BPRE[1].

§2. Proof of the Theorem

To prove the theorem, first let us show several lemmas. Note that the conditions of the theorem are the prerequisites of these lemmas. Only for saving space, we do not repeat them along with all the meanings of symbols mentioned in section 1.

Lemma 1 Denote $Q(\theta) = (q_{ij}(\theta))_{m\times m}$, where $q_{ij}(\theta) = p_{ij}\exp(-\theta e_j)$. Then there must exist a constant $\theta > 0$ and a corresponding column vector $C(\theta) = (c_1(\theta), \ldots, c_m(\theta))^\tau > 0$ (where τ is the transposition operation) such that $Q(\theta)C(\theta) = C(\theta)$.

Proof From the ergodicity of $\{\xi_n\}$, we can easily see that the $Q(\theta)$ is an irreducible nonnegative matrix for any $\theta > 0$. Then according to Perron-Frobenius Theorem [7], for each $\theta > 0$, there exist a Perron-Frobenius eigenvalue $\lambda(\theta) > 0$ and a corresponding strictly positive right eigenvector $C(\theta)$, which is unique to constant multiples, such that $Q(\theta)C(\theta) = \lambda(\theta)C(\theta)$. Obviously, following we only need to show that there exists some $\theta > 0$ such that $\lambda(\theta) = 1$.

Noting that $Q(0) = P$ and $PI = \mathbf{1} = (1, \ldots, 1)^\tau_{m\times 1}$, we know $\lambda(0) = 1$ and $C(0) = \mathbf{1}$. Denote

$$[Q(\theta)]^k = (q_{ij}^{(k)}(\theta))_{m\times m} \quad \text{and} \quad e_l = (0, \ldots, 0, 1, 0, \ldots, 0)^\tau_{m\times 1},$$

where all components of e_l are zero but the l-th is one.

Then $[Q(\theta)]^k e_l = (q_{1l}^{(k)}(\theta), \ldots, q_{ml}^{(k)}(\theta))^\tau_{m\times l} \ge (0, \ldots, 0)^\tau_{m\times 1}$ and the l-th component in left-hand side is

$$q_{ll}^{(k)}(\theta) = \sum_{i_1, \ldots, i_{k-1}=1}^{m} p_{li_1} \cdot p_{i_1 i_2} \cdot \cdots \cdot p_{i_{k-1} l} \exp(-\theta(e_l + e_{i_1} + \cdots + e_{i_{k-1}})).$$

particularly taking $l = j$, we have

$$q_{jj}^{(k)}(\theta) \ge c\exp(h\theta),$$

where $c = p_{jj_1} \cdot p_{jj_2} \cdot \ldots p_{j_{k-1} j} > 0$ and $h = -(e_j + e_{j_1} + \ldots e_{j_{k-1}}) > 0$ both are completely determined by $\{\xi_n\}$. It follows that

$$[Q(\theta)]^k e_j \geq c\exp(h\theta)e_j.$$

By Perron-Frobenius theorem, corresponding the same $\lambda(\theta)$ above , there also exists a strictly positive left eigenvector $V(\theta)_{1\times m}$, which is unique to constant multiples, too. Hence

$$[\lambda(\theta)]^k V(\theta)e_j = V(\theta)[Q(\theta)]^k e_j \geq c\exp(h\theta)V(\theta)e_j,$$

as $V(\theta)e_j$ is a positive number, which implies $[\lambda(\theta)]^k \geq c\exp(h\theta)$ and then $\lambda(\theta) \to \infty$, as $\theta \to \infty$..

Let $F(\theta,\lambda) = \det(Q(\theta) - \lambda I)$, where I is the $m \times m$ unit matrix. Applying the implicit function theorem on the equation $F(\theta,\lambda) = 0$, we may show that the Perron-Frobenius eigenvalue $\lambda(\theta)$ is continuous and differentiable on $\theta \geq 0$.

Similarly, from the set of equations

$$\sum_{j=1}^{m} p_{ij}\exp(-\theta e_j)c_j(\theta) = \lambda(\theta)c_i(\theta), \qquad i = 1,2,\cdots,m, \tag{$*$}$$

which is equivalent to $Q(\theta)C(\theta) = \lambda(\theta)C(\theta), c_i(\theta), i = 1,2,\cdots,m$ are also continuous and differentiable.

Taking derivative both sides of $(*)$ with respect to θ for $\theta = 0$ and noting that $\lambda(0) = \mathbf{1}$ and $C(0) = I$, we have

$$-\sum_{j=1}^{m} p_{ij}e_j + \sum_{j=1}^{m} p_{ij}c_j'(0) = \lambda'(0) + c_i'(0), i = 1,2,\cdots,m.$$

Now multiple by p_i and sum over $i = 1,2,\cdots,m$ and substitute $\sum_{i=1}^{m} p_i = 1, \sum_{i=1}^{m} p_{ij}p_i = p_j$, then

$$\lambda'(0) = -\sum_{j=1}^{m} p_j e_j = -\mathsf{E}[\xi_1] < 0.$$

Combining the above results, we see that there does exist some constant $\theta > 0$ such that$\lambda(\theta) = 1$.

By the stationary of $\{\xi_n\}, n = 1,2,\cdots$, we may shift $\{\xi_n\}$ forward one subscript unit (i.e. time scale) such that it becomes $\{\xi_n\}, n = 0,1,2,\cdots$. After fixing some $\theta > 0$ introduced in Lemma 1, then we have

Lemma 2 There must exist a constant $k_1 > 0$ such that

$$\mathsf{P}\{M \leq -t|\xi_0 = e_i\} \geq k_1 c_i(\theta)\exp(-\theta t), i = 1,2,\cdots,m,$$

for all $t \geq 0$.

Proof Since the $\theta > 0$ here is fixed. we can write $c_i(\theta)$ by $c(e_i)$ without any ambiguity.

As $t = 0$, the conclusion is trivally true due to $M \leq 0$, in which $k_1 = \exp(\theta\min_{1\leq i\leq m}\{e_i\})/\max_{1\leq i\leq m}\{c(e_i)\}$ is chosen particularly, we only need consider the case $t > 0$ hereafter.

Let $Y_n = \exp(-\theta X_n)c(\xi_n), F_n = \sigma(\xi_0,\xi_1,\cdots,\xi_n)$. it is easy to show that $\{Y_n\}, n = 0,1,2,\cdots$ is a martingale with respect to $\{F_n\}$. For constant $t,u > 0$, we define a Markov time by $T = \min\{n|X_n \leq -t \text{ or } X_n \geq u\}$. By the ergodic theorem and the assumption above we known that

$$\frac{X_n}{n} \xrightarrow{a.s.} \mathsf{E}[\xi_1] > 0, \qquad (n \to +\infty).$$

Hence $X_n \xrightarrow{a.s.} +\infty$ and then $P\{T < +\infty\} = 1$.

When $n < T$, we have

$$X_{n+1} < u + \max_{1\leq i\leq m}\{e_i\} \triangleq v > 0,$$

$$X_{n+1} > -t + \min_{1\leq i\leq m}\{e_i\} \triangleq -w < 0.$$

Therefore $\mathsf{P}\{-w < X_T < v\} = 1$ which implies $\mathsf{E}|Y_T| < +\infty$, from $0 \leq Y_n < \exp(\theta t)\max_{1\leq i\leq m}\{c(e_i)\} < +\infty$ for all $n < T$, we get $\mathsf{E}[Y_n I_{\{T>n\}}] \to 0, (n \to \infty)$, where $I_{\{T>n\}}$ is the indicator function of set $\{T > n\}$. Then by the optional stopping theorem (see e.g. [6]), we know that under the condition $\xi_0 = e_i$, it must be hold that

$$\mathsf{E}[Y_T|\xi_0 = e_i] = \mathsf{E}[Y_0|\xi_0 = e_i] = c(e_i).$$

Note that $\mathsf{E}[Y_T|\xi_0 = e_i] = \mathsf{E}[Y_T I_{\{X_T\leq -t\}}|\xi_0 = e_i] + \mathsf{E}[Y_T I_{\{X_T\geq u\}}|\xi_0 = e_i]$ and when $X_T \leq -t, X_T \geq -t + \min_{1\leq i\leq m}\{e_i\}$. We have

$$Y_T = \exp(-\theta X_T)c(\xi_T) \leq \exp(-\theta(-t + \min_{1\leq i\leq m}\{e_i\}))\max_{1\leq i\leq m}\{c(e_i)\},$$

and may deduce that

$$\mathsf{E}[Y_T I_{\{X_T\leq -t\}}|\xi_0 = e_i] \leq \exp(-\theta(-t + \min_{1\leq i\leq m}\{e_i\}))\max_{1\leq i\leq m}\{c(e_i)\}\mathsf{P}\{X_T \leq -t|\xi_0 = e_i\}.$$

Recalling $k_1 = \exp(\theta\min_{1\leq i\leq m}\{e_i\})/\max_{1\leq i\leq m}\{c(e_i)\} > 0$, we have

$$\mathsf{P}\{X_T \leq -t|\xi_0 = e_i\} \geq k_1\exp(-\theta t)(c(e_i) - \mathsf{E}[Y_T I_{\{X_T\geq u\}}|\xi_0 = e_i]).$$

Noting that $M \leq X_T, c(e_i) = c_i(\theta)$ and $0 \leq \mathsf{E}[Y_T I_{\{X_T\geq u\}}|\xi_0 = e_i] \leq \exp(-\theta u)\max_{1\leq i\leq m}\{c(e_i)\}$ $\mathsf{P}\{X_T \geq u|\xi_0 = e_i\} \to 0$ as $u \to \infty$, we obtain

$$\mathsf{P}\{M \leq -t|\xi_0 = e_i\} \geq k_1 c_i(\theta)\exp(-\theta t) \qquad i = 1, 2, \cdots, m,$$

as requied.

Lemma 3 Let $\theta > 0$ and ξ_0 be the same as in Lemma 2. Then there must exist another constant $k_2 > 0$ such that

$$\mathsf{P}\{M \leq -t|\xi_0 = e_i\} \leq k_2 c_i(\theta)\exp(-\theta t) \qquad i = 1, 2, \cdots, m,$$

for all $t \geq 0$

Proof Let Y_n, T and $c(e_i)$ be defined as before. Denote $k_2 = 1/\min_{1\leq i\leq m}\{c(e_i)\}$. then as $t = 0$ the conclusion is trivially true due to $M \leq 0$, too, we may suppose $t > 0$ below.

From the proof of Lemma 2, we have seen that

$$c(e_i) = \mathsf{E}[Y_T I_{\{X_T\leq -t\}}|\xi_0 = e_i] + \mathsf{E}[Y_T I_{\{X_T\geq u\}}|\xi_0 = e_i], \qquad i = 1, 2, \cdots, m.$$

Noting that $\mathsf{E}[Y_T I_{\{X_T\leq -t\}}|\xi_0 = e_i] \geq \exp(\theta t)\min_{1\leq i\leq m}\{c(e_i)\}$ and $\mathsf{E}[Y_T I_{\{X_T\geq u\}}|\xi_0 = e_i\} \geq 0$, we have

$$\mathsf{P}\{X_T \leq -t|\xi_0 = e_i\} \leq k_2 c(e_i)\exp(-\theta t), \qquad i = 1, 2, \cdots, m,$$

whenever $u > 0$. For any fixed $t > 0$, it is possible to regard the T in the formula above as $T = \min\{n|X_n \le -t\}$. On the other hand, for the fixed $t > 0$, denote $T(u) = \min\{n|X_n \le -t \text{ or } X_n \ge u\}$ $(u > 0)$, then $T = \lim_{u\to\infty} T(u)$. Taking $u = 1, 2, \cdots$, we have $\{M \le -t|\xi_0 = e_i\} \overset{a.s}{=} \bigcup_{n=1}^{\infty}\{X_{T(n)} \le -t|\xi_0 = e_i\}$. Note that $c(e_i) = c_i(\theta)$ and $\{X_{T(n)} \le -t|\xi_0 = e_i\} \subset \{X_{T(n+1)} \le -t|\xi_0 = e_i\}, n = 1, 2, \cdots$. By the continuity of probability, we can obtain

$$\mathsf{P}\{M \le -t|\xi_0 = e_i\} \le k_2 c_i(\theta)\exp(-\theta t), \qquad i = 1, 2, \cdots, m,$$

as required.

Remark k_1, k_2 are dependent on θ but independent of t.

Proof of theorem : By Lemma 2 we have

$$\begin{aligned}\mathsf{P}\{M \le -t\} &= \sum_{i=1}^{m} \mathsf{P}\{M \le -t|\xi_0 = e_i\}P\{\xi_0 = e_i\}\\ &\ge \sum_{i=1}^{m} k_1 c_i(\theta)\exp(-\theta t)p_i = c_1\exp(-\theta t),\end{aligned}$$

where $c_1 = k_1\sum_{i=1}^{m} c_i(\theta)p_i > 0$. Similarly, by Lemma 3, we have $\mathsf{P}\{M \le -t\} \le c_2\exp(-\theta t)$, where $c_2 = k_2\sum_{i=1}^{m} c_i(\theta)p_i > 0$. It follows that

$$0 < c_1 \le \frac{\mathsf{P}\{M \le -t\}}{\exp(-\theta t)} \le c_2 < +\infty,$$

which implies that $P\{M \le -t\} \asymp \exp(-\theta t), (t \to \infty)$.

With regard to the uniqueness of $\theta > 0$, we only need to show that the $\theta > 0$ defined in Lemma 1 is unique, which we can prove by reduction to absurdity. On the contrary, we suppose that such $\theta > 0$ in Lemma 1 is more than one. Noting that the c_1, c_2 above are dependent on θ. We take θ_1 and $\theta_2, \theta_1 < \theta_2$ say, in Lemma 2 and Lemma 3 respectively. Then there should exist constants $c_1(\theta_1), c_2(\theta_2) > 0$ such that

$$c_1(\theta_1)\exp(-\theta_1 t) \le \mathsf{P}\{M \le -t\} \le c_2(\theta_2)\exp(-\theta_2 t), \qquad (t \ge 0).$$

However in the other hand, since

$$c_2(\theta_2)\exp(-\theta_2 t)/c_1(\theta_1)\exp(-\theta_1 t) = \frac{c_2(\theta_2)}{c_1(\theta_1)}\exp((\theta_1 - \theta_2)t) \to 0, \quad (t \to +\infty),$$

we get $c_2(\theta_2)\exp(-\theta_2 t) < c_1(\theta_1)\exp(-\theta_1 t)$ for all sufficiently large t. Therefore the contradition implies that such $\theta > 0$ here must be unique.

§3. An Application to the Athreya- Karlin BPRE.

Now we suppose that $\{Z_n\}, n = 0, 1, 2, \cdots$ is a simple type of Athreya-Karlin BPRE., in which the environment $\{\zeta_n\}, n = 0, 1, 2, \cdots$, is a stationary ergodic Markov chain with finite state space $\mathcal{E} = \{e_i|i = 1, 2, \cdots, m\}$. Denote the corresponding equilibrium distribution and the transition matrix by

$$p_i = \mathsf{P}\{\zeta_n = e_i\} \quad \text{and} \quad \boldsymbol{P} = (p_{ij})_{m\times m} = (\mathsf{P}\{\zeta_{n+1} = e_j|\zeta_n = e_i\})_{m\times m},$$

respectively, where $i,j = 1,2,\cdots,m; n = 0,1,2,\cdots$. Furthermore, we assume that there exists a k_{-} cycle: $e_j, e_{j_1}, \cdots, e_{j_{k-1}}$ in $\mathcal{E}$ such that

$$p_{jj_1} \cdot p_{j_1 j_2} \cdots p_{j_{k-1}j} > 0 \quad \text{and} \quad r(e_j) + r(e_{j_1}) + \cdots + r(e_{j_{k-1}}) < 0,$$

where $r(e_i) = \log \varphi'_{e_i}(1)$ and φ_{e_i} is the reproducing probability generating function when ζ_n turns out to be e_i. If our consideration is under the sufficient condition for supercritical BPRE.:

$$0 < \mathsf{E}[\log \varphi'_{\zeta_n}(1)] < +\infty; \qquad \mathsf{E}[-\log(1 - \varphi_{\zeta_n}(0))] < +\infty,$$

and the φ_{ζ_n} has good analysis properties: $\varphi''_{e_i}(1) < +\infty$ for all $e_i \in \mathcal{E}$, then we can present following conclusion.

Conclusion There exists an unique constant $\theta > 0$ such that as k, the starting population size, tends to infinity, the corresponding extinction probability, $q_k = \mathsf{P}\{Z_n \to 0, (n \to +\infty) | Z_0 = k\}$ satisfies $q_k \asymp k^{-\theta}, (k \to +\infty)$.

To show the conclusion, first we extend $\{\zeta_n\}, n = 0,1,2,\cdots$, into a double-ended process $\{\zeta_n\}, n = \cdots,-2,-1,0,1,2,\cdots$ by the standard way (see e.g. [3]) keeping the stationarity and ergodicity like it before. Let

$$X_n = \lim_{m \to +\infty} \varphi_{\zeta_n}(\varphi_{\zeta_{n+1}}(\cdots(\varphi_{\zeta_{n+m}}(0))\cdots)), n = \cdots,-2,-1,0,1,2,\cdots.$$

By reversing the time scale define $X_n^* = X_{-n}$ and $\zeta_n^* = \zeta_{-(n+1)}$. We obtain a "dual process" $\{X_n^*\}$ satisfying $X_{n+1}^* = \varphi^*_{\zeta_n}(X_n^*)$, for all n, and (X_n^*, ζ_n^*) becomes a bivariate stationary ergodic process. From [1] we know that $q_k = \mathsf{E}[X_n^{*k}]$ for all n. Define

$$Y_n = -\log(1 - X_n^*); \qquad \xi_n = -\log\left[\frac{1 - \varphi_{\zeta^*_{n-1}}(X^*_{n-1})}{1 - X^*_{n-1}}\right].$$

Then $Y_n = Y_{n-1} + \xi_n$. Denote

$$\bar{\xi}_n = -\log \varphi'_{\zeta^*_{n-1}}(1); \qquad \hat{\xi}_n = -\log\left[\frac{1 - \varphi_{\zeta^*_{n-1}}(x_0)}{1 - x_0}\right],$$

where x_0 is any fixed constant in [0, 1). Clearly $\{\bar{\xi}_n\}$ and $\{\hat{\xi}_n\}$ both are stationary ergodic Markov chain with finite state space. Now we construct $\{W_n\}$ and $\{U_n\}$ by the recursive relations:

$$\begin{cases} W_0 = 0, \\ W_{n+1} = \max\{0, W_n + \bar{\xi}_n\}; \end{cases} \qquad \begin{cases} U_0 = \max\{y_0, Y_0\}, \\ U_{n+1} = \max\{y_0, U_n + \hat{\xi}_{n+1}\}, \end{cases}$$

respectively, where $y_0 = -\log(1 - x_0)$. By the ergodic theorem and induction we can easily deduce that as x_0 is sufficiently close to 1, $W_n \le Y_n \le U_n$, for all n.

Let $\eta_n = -\log \varphi'_{\zeta_n}(1)$. Then $\eta_{-n} = \bar{\xi}_n$ and $\{\eta_n\}$ also is stationary ergodic. By our assumption we have $\mathsf{E}[\eta_n] = \mathsf{E}[-\log \varphi'_{\xi_n}(1)] < 0$. From ergodic theorem we can see that $\sum_{i=1}^n \eta_i \xrightarrow{a.s.} -\infty$, as $n \to +\infty$. Therefore, if we write

$$\overline{W}_0 = 0, \quad \overline{W}_n = \max\{0, \eta_1 + \eta_2, \cdots, \eta_1 + \eta_2 + \cdots + \eta_n\},$$

then there must exist an integer N such that

$$\overline{W}_N = \overline{W}_{N+1} = \overline{W}_{N+2} = \cdots \triangleq W \quad (a.s.).$$

That is $\overline{W}_n \xrightarrow{a.s.} W$ as $n \to +\infty$. Noting that

$$\begin{aligned} W_n &= \max\{0, \overline{\xi}_n, \overline{\xi}_n + \overline{\xi}_{n-1}, \cdots, \overline{\xi}_n + \overline{\xi}_{n-1} + \cdots + \overline{\xi}_1\} \\ &= \max\{0, \eta_{-n}, \eta_{-n} + \eta_{-(n-1)}, \cdots, \eta_{-n} + \eta_{-(n-1)} + \cdots \eta_{-1}\} \\ &\overset{D}{=} \max\{0, \eta_1, \eta_1 + \eta_2, \cdots, \eta_1 + \eta_2 + \cdots \eta_n\} = \overline{W}_n, \end{aligned}$$

where "$\overset{D}{=}$" denotes equality in distribution. Hence we get

$$W_n \xrightarrow{D} W \qquad \text{as} \quad n \to +\infty.$$

On the other hand, if we let $\overline{\eta}_n = -\log\Big[\dfrac{1 - \varphi_{\zeta_n}(x_0)}{1 - x_0}\Big]$ and

$$\begin{aligned} &\overline{U}_0 = \max\{y_0, Y_0\}, \\ &\overline{U}_n = \max\{y_0, y_0 + \overline{\eta}_1, y_0 + \overline{\eta}_1 + \overline{\eta}_2, \cdots, y_0 + \overline{\eta}_1 + \cdots + \overline{\eta}_{n-1}, U_0 + \overline{\eta}_1 + \cdots \overline{\eta}_n\}, \end{aligned}$$

then $\overline{\eta}_{-n} = \widehat{\xi}_n$ and $\{\overline{\eta}_n\}$ also is stationary ergodic. Noting that

$$\lim_{x_0 \uparrow 1}\Big[-\log\Big[\frac{1 - \varphi_{\zeta_n}(x_0)}{1 - x_0}\Big]\Big] = -\log \varphi'_{\zeta_n}(1),$$

when we choose x_0 close enough to 1 it must be true that

$$\sum_{i=1}^{n} \overline{\eta}_1 \xrightarrow{a.s.} -\infty, \qquad \text{as} \quad n \to +\infty,$$

and there must exist an integer N^* such that

$$\overline{U}_{N^*} = \overline{U}_{N^*+1} = \overline{U}_{N^*+2} = \cdots \overset{\Delta}{=} U \qquad (a.s.).$$

That is $\overline{U}_n \xrightarrow{a.s.} U$ as $n \to +\infty$. Noting that

$$\begin{aligned} U_n &= \max\{y_0, y_0 + \widehat{\xi}_n, \cdots, y_0 + \widehat{\xi}_n + \cdots + \widehat{\xi}_2, U_0 + \widehat{\xi}_n + \cdots + \widehat{\xi}_1\} \\ &= \max\{y_0, y_0 + \overline{\eta}_{-n}, \cdots, y_0 + \overline{\eta}_{-n} + \cdots + \eta_{-2}, U_0 + \overline{\eta}_{-n} + \cdots + \overline{\eta}_{-1}\} \\ &\overset{D}{=} \max\{y_0, y_0 + \overline{\eta}_1, \cdots, y_0 + \overline{\eta}_1 + \cdots + \eta_{n-1}, U_0 + \overline{\eta}_1 + \cdots + \overline{\eta}_n\} = \overline{U}_n. \end{aligned}$$

Hence we get $U_n \xrightarrow{D} U$ as $n \to +\infty$.

Furthermore, we write

$$\overline{W} = \min\{0, -\eta_1, -\eta_1 - \eta_2, \cdots\}; \quad \overline{U} = \min\{-y_0, -y_0 - \overline{\eta}_1, -y_0 - \overline{\eta}_1 - \overline{\eta}_2, \cdots\}.$$

Then $\mathrm{P}\{W > t\} = \mathrm{P}\{\overline{W} < -t\}, \mathrm{P}\{U > t\} = \mathrm{P}\{\overline{U} < -t\}$.

Denote martices by

$$Q(\theta) = (p_{ij}\exp(-\theta r_j))_{m\times m}; \qquad \overline{Q}(\theta) = (p_{ij}\exp(-\theta \overline{r}_j(x_0)))_{m\times m},$$

where $r_j = \log \varphi'_{e_j}(1), \overline{r}_j = \log\Big[\dfrac{1 - \varphi_{e_j}(x_0)}{1 - x_0}\Big]$. Noting that $\overline{r}_j(x_0) < r_j$ we have $\overline{r}_j(x_0) + \overline{r}_{j_1}(x_0) + \cdots + \overline{r}_{j_{k-1}}(x_0) < 0$. Applying the theorem in section 1 to $\overline{W}$ and $\overline{U}$ respectively we can derive that there exist constant $\theta, \overline{\theta}(x_0) > 0$ and $0 < \overline{c}_1, \overline{c}_2(x_0) < +\infty$ such that

$$1 - \overline{c}_2(x_0)\exp(-\overline{\theta}(x_0)t) \le \mathrm{P}\{Y_n \le t\} \le 1 - \overline{c}_1\exp(-\theta t), \qquad t \ge 0,$$

where x_0 is sufficiently close to 1.

Note that $\bar{r}_j(x_0) \uparrow r_j$ as $x_0 \uparrow 1$. By the implicit function theorem and Perron-Frobenius theorem we get $\bar{\theta}(x_0) \uparrow \theta$ as $x_0 \uparrow 1$ under the condition $\varphi''_{e_i}(1) < +\infty$ for all $e_i \in \mathcal{E}$. Since $\mathsf{P}\{X_n^* > x\} = 1 - P\{Y_n \le \log(1-x)\}$ from an Abelian theorem, we may obtain

$$q_k = k \int_0^1 x^{k-1}(1 - \mathsf{P}\{Y_n \le \log(1-x)\})dx.$$

Denote $c_1 = \bar{c}_1 \Gamma(\theta+1), c_2 = \bar{c}_2(x_0)\Gamma(\bar{\theta}(x_0)+1)$ and note that

$$B(k,\alpha) \sim \frac{\Gamma(\alpha)}{k^\alpha}, \qquad (k \to +\infty).$$

We have $c_1 k^{-\theta} \le q_k \le c_2(x_0) k^{-\bar{\theta}(x_0)}$, for all sufficiently large k. Let $x_0 \uparrow 1$ and write $c_2 = \limsup_{x_0 \uparrow 1} c_2(x_0)$. Then $0 < c_1 \le q_k / k^{-\theta} \le c_2 < +\infty$, for all sufficiently large k, which implies that $q_k \asymp k^{-\theta}, (k \to +\infty)$.

Obviously, this conclusion is an extension result of Grey & Lu (1993) in Classification A.

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平稳遍历马氏链部分和序列最小值分布的渐近估计及其应用

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本文给出了有限状态平稳遍历Markov链部分和序列最小值分布的一个渐近估计式并利用它对一类Athreya-Karlin BPRE．灭种概率的渐近行为作出估计．

关键词: Perron-Frobenius 定理, Athreya-Karlin BPRE．

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