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The Existence of Order-Preserving Coupling for Two-Dimentional Diffusion Processes*

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Abstract

This paper focus on order-preservation of paths for two diffusion processes. The existence of order-preserving coupling for two-dimentional nondegenerated diffusion processes is proved, furthermore, an order-preserving coupling operator is given.

Keywords: Diffusion process, Coupling, Order-preserving coupling.

AMS Subject Classification: 60J25, 60J35.

§1. Intoduction

Stochastic monotonicity plays an important role in the study of Markov processes. As for multidimentional diffusion processes, the study of order-preservation for semi-group (or distribution) is already very completed (see [1] and [2]); in [3], order-preservation for path (or order-preserving coupling) is well studied also. [3] presents some sufficient conditions and necessary ones for couplings of a multidimentional diffusion process to preserve the natural partial order on R^d, the next problem is: when does a order-preserving coupling exist?

Let $a(x)$ be a $d\times d$-order matrix and $b(x)\in R^d$ for each $x\in R^d$. We write $L\sim(a,b)$ if

$$L_x=\frac{1}{2}\sum_{1\le i,j\le d}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d}b_i(x)\frac{\partial}{\partial x_i},\qquad x\in R^d. \tag{1.1}$$

Let $L^k\sim(a^k,b^k)$, $k=1,2$. Assume that $a^k_{ij},b^k_i\in C(R^d)$ and the martingale problem for L^k is well-posed, $k=1,2$. Denote by P^k_t the semi-group of L^k, $k=1,2$. Write $P^1_t\le P^2_t$ if

$$P^1_tf(x)\le P^2_tf(y) \tag{1.2}$$

holds for all $t\ge 0$, $y\ge x$ and monotone function $f\in C(R^d)$. Here $\ge$ is the natural partial order and "f is monotonic" means "$f(x)\le f(y)$ if $x\le y$".

From [1], we know that $P^1_t\le P^2_t$ iff the following two conditions hold:

(1) *For any i and j, $a^1_{ij}=a^2_{ij}=:a_{ij}$ and $a_{ij}(x)$ depends only on x_i and x_j.*

(2) *For any i, $b^1_i(x)\le b^2_i(y)$ for $x\le y$ with $x_i=y_i$.*

On the other hand, let $\widehat{L}\sim(\widehat{a},\widehat{b})$, where

$$\widehat{a}(x,y)=\begin{pmatrix}a^1(x) & c(x,y)\\ c(x,y)' & a^2(y)\end{pmatrix},\qquad \widehat{b}(x,y)=\begin{pmatrix}b^1(x)\\ b^2(y)\end{pmatrix}. \tag{1.3}$$

for some $c(x,y)$ with $c_{ij}\in C(R^d\times R^d)$ such that $\widehat{a}(x,y)$ is nonnegative definite, where $c(x,y)'$ is the transpose of $c(x,y)$. We call $\widehat{L}$ a coupling operator of L^1 and L^2 (see[4]). $\{P^{x,y}:x,y\in R^d\}$ is said to be a coupling process (for simplicity, coupling) if it solves the martingale problem of a coupling operator.

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Definition 1 *A coupling $\{P^{x,y}\}$ is said to preserve order, if*

$$P^{x,y}(x_t \le y_t : \forall t \ge 0) = 1, \qquad x \le y, \;\; x, y \in Z^d. \tag{1.4}$$

Obviously, the existence of order-preserving coupling implies $P_t^1 \le P_t^2$, hence, (1) and (2) are necessary for a coupling preserving order. So, we will always assume that (1) and (2) hold. Next, when $L^1 = L^2$, let the marginal processes move together whenever they meet. In the case $d = 1$, all couplings preserve order since the two marginal processes must meet before the order is broken; in other case, i.e., $d \ge 2$, whether the marginal processes preserve order or not is a question. In [3], for the case $d \ge 2$, not required $L^1 = L^2$, WANG and XU proved the following sufficient condition for a coupling to preserve order (Theorem 1.1, (I) of [3]).

(3) *Let $\{P^{x,y}\}$ be a coupling with operator $\widehat{L}$. Suppose that (1) and (2) hold and for each $i \le d$, one of b_i^1 and b_i^2 is locally Lipschitz continuous. $\{P^{x,y}\}$ preserves order if for each $m > 0$, there exists a increasing function $\rho_m \in C(R_+)$ such that $\rho_m(0) = 0$, $\int_0^1 \rho_m(u)^{-1} du = \infty$ and*

$$|a_{ii}(x) + a_{ii}(y) - 2c_{ii}(x,y)| \le \rho_m(|x_i - y_i|), \qquad i \le d, \;\; x, y \in [-m, m]^d. \tag{1.5}$$

From (3) we get the following main theorem, which holds under the conditions (1) and (2).

Main Theorem *When $d = 2$, if $a^1 = a^2 = a = \sigma^2 > 0$, σ, b^1 and b^2 are locally Lipschitz continuous, then there exists a coupling operator $\widehat{L} \sim (\widehat{a}, \widehat{b})$ as (1.3) such that for coupling $\{P^{x,y}\}$ with operator $\widehat{L}$, $\{P^{x,y}\}$ preserves order.*

§2. Proof of The Main Theorem

To prove the main theorem, first we construct a coupling operator $\widehat{L}$, which has marginal operators L^1 and L^2, and then prove that $\{P^{x,y}\}$, which solves the martingale problem for $\widehat{L}$, preserves order. In other words, we try to find a "good" enough $c(x,y) \in C(R^2 \times R^2)$. From now on, we assume $c(x,y)$ in (1.3) has the specified form: $c(x,y) = \sigma(x) H(x,y) \sigma(y)$, where $H(x,y)$ is a 2×2 matrix.

The following lemma tells us when the above specified $c(x,y)$ is qualified to form a coupling operator $\widehat{L}$.

Lemma 1 *For $d \ge 2$, $a = \sigma^2$ be a positive definite $d \times d$-order matrix, then $\widehat{a}$ be nonnegative definite iff $H(x,y)$ be contractive [i.e., for all $\alpha \in R^d$, $|H\alpha| \le |\alpha|$].*

Proof The proof is given by Chen Mufa(see [5]):

For all $\alpha, \beta \in R^d$, we have

$$(\alpha', \beta')\widehat{a}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha' a \alpha + \beta' a \beta + 2\langle H\sigma\alpha, \sigma\beta\rangle = |\sigma\alpha|^2 + |\sigma\beta|^2 + 2\langle H\sigma\alpha, \alpha\beta\rangle.$$

Thus, $\widehat{a}$ is nonnegative definite iff

$$|\alpha|^2 + |\beta|^2 + 2\langle H\alpha, \beta\rangle \ge 0, \qquad \alpha, \beta \in R^d.$$

Setting $\beta = -H\alpha$, it follows that $|H\alpha| \le |\alpha|$. This proves the necessity. The sufficiency is easy. □

Before the proof of the main theorem, we give some notations, let

$$\lambda_1(x,y) = \frac{|x_1 - y_1|}{|x_1 - y_1| + |x_2 - y_2|}, \qquad \lambda_2(x,y) = \frac{|x_2 - y_2|}{|x_1 - y_1| + |x_2 - y_2|}, \qquad x, y \in R^2, \;\; x \ne y; \tag{2.1}$$

$$K(x,y) = \arccos \frac{a_{12}(x)}{\sqrt{a_{11}(x) a_{22}(x)}} - \arccos \frac{a_{12}(y)}{\sqrt{a_{11}(y) a_{22}(y)}}, \qquad x, y \in R^2; \tag{2.2}$$

$$\begin{aligned} &\theta_1(x,y) = \lambda_1 K(x,y), && \theta_2(x,y) = \lambda_2 K(x,y), && x, y \in R^2, \;\; x \ne y; \\ &\theta_1(x,x) = 0, && \theta_2(x,x) = 0, && x \in R^2; \end{aligned} \tag{2.3}$$

$$\overline{\sigma}(x) = \begin{pmatrix} \overline{\sigma}_{11}(x) & \overline{\sigma}_{12}(x) \\ \overline{\sigma}_{21}(x) & \overline{\sigma}_{22}(x) \end{pmatrix} = \begin{pmatrix} \dfrac{\sigma_{11}(x)}{\sqrt{a_{11}(x)}} & \dfrac{\sigma_{12}(x)}{\sqrt{a_{11}(x)}} \\ \dfrac{\sigma_{21}(x)}{\sqrt{a_{22}(x)}} & \dfrac{\sigma_{22}(x)}{\sqrt{a_{22}(x)}} \end{pmatrix}, \qquad x \in R^2; \tag{2.4}$$

(note that σ is symmetric, but $\overline{\sigma}$ is not.) let

$$H_1(x,y) = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}, \qquad H_2(x,y) = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix}, \tag{2.5}$$

for $x, y \in R^2$ with $x \neq y$; let

$$H_i(x,x) = \lim_{y\to x} H_i(x,y) = I, \qquad x \in R^2, \quad i = 1,2. \tag{2.6}$$

The following lemmas give some remarks on the above notations.

Lemma 2 *Under the assumptions of the main theorem,* $\Psi(x) := \arccos\dfrac{a_{12}(x)}{\sqrt{a_{11}(x)a_{22}(x)}}$ *in (2.2),* $\overline{\sigma}$ *in (2.4) and* $[\overline{\sigma}'(y)]^{-1}$ *are all locally Lipschitz continuous.*

Proof For $\Psi(x)$, we know that $\Psi(x)$ is the spatial angle of vectors $\sigma_1(x)$ and $\sigma_2(x)$ [the two row vectors of matrix $\sigma(x)$], write $\Psi_i(x,y)$ as the spatial angle of vectors $\sigma_i(x)$ and $\sigma_i(y)$, $i = 1,2$. Given $x \in R^2$, let $y \in R^2$ $\|x - y\|$ small enough, we have

$$|\Psi(x) - \Psi(y)| \leq \Psi_1(x,y) + \Psi_2(x,y). \tag{2.7}$$

It is easy to see that

$$\overline{\lim_{y\to x}} \frac{\Psi_i(x,y)}{|\sigma_i(x) - \sigma_i(y)|} \leq \frac{1}{|\sigma_i(x)|} = \frac{1}{a_{ii}(x)}, \qquad i = 1,2. \tag{2.8}$$

For all $m > 0$, since σ is locally Lipschitz continuous, there exists $\overline{C}_m < \infty$ such that

$$|\sigma_i(x) - \sigma_i(y)| \leq \overline{C}_m \|x - y\|, \qquad i = 1,2; \quad x, y \in [-m-1, m+1]^2. \tag{2.9}$$

On the other hand, by the fact that $a > 0$ and a is continuous, we have

$$K_m := \sup\left\{\frac{1}{a_{11}(x)} + \frac{1}{a_{22}(x)} : x \in [-m,m]^2\right\} < \infty. \tag{2.10}$$

By (2.7)–(2.10), we have

$$\overline{\lim_{y\to x}} \frac{|\Psi(x) - \Psi(y)|}{\|x - y\|} \leq K_m \cdot \overline{C}_m \tag{2.11}$$

for all $x \in [-m,m]^2$.

Now for any fixed $x, y \in [-m,m]^2$, write $l_{x,y}$ as the line interval in $[-m,m]^2$ with endvertex x and y, $\forall\epsilon > 0$, $\forall w \in l_{x,y}$, by (2.11), let $\delta_w > 0$ satisfies: For all z with $\|z - w\| < \delta_w$,

$$|\Psi(z) - \Psi(w)| \leq (K_m \cdot \overline{C}_m + \epsilon)\|z - w\|. \tag{2.12}$$

Denote $B(w, \delta_w)$ as the open ball centered at w and with radius δ_w, then by the compactness of $l_{x,y}$, there exists $\{w_0 = x, w_1, w_2 \cdots w_{k-1}, w_k = y\} \subset l_{x,y}$ such that

a) $l_{w_{i-1},w_i} \cap l_{w_{j-1},w_j}$, $i \neq j$ either be empty or be singleton.

b) $B(w_{i-1}, \delta_{w_{i-1}}) \cap B(w_i, \delta_{w_i}) \neq \phi$ and

c) $\bigcup_{i=0}^{k} B(w_i, \delta_{w_i}) \supset l_{x,y}$;

take $z_i \in B(w_{i-1}, \delta_{w_{i-1}}) \cap B(w_i, \delta_{w_i}) \cap l_{w_{i-1}, w_i}$, $1 \le i \le k$, by (2.12) and the above conditions a). b). c). we have

$$\begin{aligned}
&|\Psi(x) - \Psi(y)| \\
\le&|\Psi(x) - \Psi(z_1)| + |\Psi(z_1) - \Psi(w_1)| + \cdots + |\Psi(w_{k-1}) - \Psi(z_k)| + |\Psi(z_k) - \Psi(y)| \\
\le&(K_m \cdot \overline{C}_m + \epsilon)(\|x - z_1\| + \|z_1 - w_1\| + \cdots + \|w_{k-1} - z_k\| + \|z_k - y\|) \\
=&(K_m \cdot \overline{C}_m + \epsilon)\|x - y\|.
\end{aligned}$$

Let $\epsilon \to 0$, we have

$$|\Psi(x) - \Psi(y)| \le K_m \cdot \overline{C}_m \cdot \|x - y\|, \tag{2.13}$$

for all $x, y \in [-m, m]^2$. So, $\Psi(x)$ is locally Lipschitz continuous. The situation for $\bar{\sigma}$ and $[\overline{\sigma}'(y)]^{-1}$ are similar, we omit the detailed proofs. □

Lemma 3 *$\theta_i(\cdot,\cdot)$ $i = 1, 2$, defined in (2.9) are locally Lipschitz continuous.*

Proof Note that $\lambda_i(\cdot,\cdot), i = 1, 2$ in (2.1) are not locally Lipschitz continuous, but we have, for any $x, y, z \in R^2$ with $x \ne y$, $y \ne z$

$$\begin{aligned}
&|\lambda_1(x, y) - \lambda_1(x, z)| \\
=&|\lambda_2(x, y) - \lambda_2(x, z)| \\
\le&\frac{|y_1 - z_1| + |y_2 - z_2|}{|x_1 - y_1| + |x_2 - y_2|}.
\end{aligned} \tag{2.14}$$

Now, for any $x, y, z \in [-m, m]^2$, we observe $|\theta_1(x, y) - \theta_1(x, z)|$. When $x = y$, then

$$|\theta_1(x, y) - \theta_1(x, z)| = \lambda_1(y, z)|\Psi(y) - \Psi(z)|,$$

by the equivalence of L_1 distance and L_2 distance and (2.13), there exists $0 < C(m) < \infty$ such that

$$|\theta_1(x, y) - \theta_1(x, z)| \le C(m)(|y_1 - z_1| + y_2 - z_2|);$$

when $x \ne y$, then by (2.13) and (2.14)

$$\begin{aligned}
&|\theta_1(x, y) - \theta_1(x, z)| \\
=&|\lambda_1(x, y)(\Psi(x) - \Psi(y)) - \lambda_1(x, z)(\Psi(x) - \Psi(z))| \\
\le&|\lambda_1(x, y) - \lambda_2(x, z)||\Psi(x) - \Psi(y)| + \lambda_1(x, z)|\Psi(y) - \Psi(z)| \\
\le&\frac{|y_1 - z_1| + |y_2 - z_2|}{|x_1 - y_1| + |x_2 - y_2|}C(m)(|x_1 - y_1| + |x_2 - y_2|) + C(m)(|y_1 - z_1| + |y_2 - z_2|) \\
=&2C(m)(|y_1 - z_1| + |y_2 - z_2|).
\end{aligned}$$

Thus we get the Lemma for $\theta_1(\cdot,\cdot)$, the situation for $\theta_2(\cdot,\cdot)$ is the same. □

Proof of the main theorem We prove the main theorem by four steps.

Step 1 We construct a $H \in C(R^2 \times R^2)$ and, for all $x, y \in R^2$, $H(x, y)$ is contractive.

For $\sigma(x) = \begin{pmatrix} \sigma_1(x) \\ \sigma_2(x) \end{pmatrix}$, $x \in R^2$, denote the spatial angle of vector $\sigma_1(x)$ and $\sigma_2(x)$ by $\Psi_{12}(x)$, without loss of generality, we assume $0 < \Psi_{12}(x) < \pi$ and $\sigma_1(x)$ lies in the anticlockwise direction of $\sigma_2(x)$. So, since $\sigma(x) > 0$, we have $\det\sigma(x) > 0$ and $\det\overline{\sigma}(x) = \sin \Psi_{12}(x)$. Write

$$\sigma_1^*(x, y) = H_1(x, y) \begin{pmatrix} \overline{\sigma}_{11}(x) \\ \overline{\sigma}_{12}(x) \end{pmatrix}, \qquad \sigma_2^*(x, y) = H_2(x, y) \begin{pmatrix} \overline{\sigma}_{21}(x) \\ \overline{\sigma}_{22}(x) \end{pmatrix}.$$

Note that the length $|\sigma_1^*(x,y)| = |\sigma_2^*(x,y)| = 1$. Denote by $\Psi_{12}^*(x,y)$ the spatial angle of $\sigma_1^*(x,y)$ and $\sigma_2^*(x,y)$, we have

$$\begin{aligned}
&\cos\Psi_{12}^*(x,y) = \langle \sigma_1^*(x,y), \sigma_2^*(x,y)\rangle \\
=&\left[H_1(x,y)\begin{pmatrix}\overline{\sigma}_{11}(x)\\ \overline{\sigma}_{12}(x)\end{pmatrix}\right]'\left[H_2(x,y)\begin{pmatrix}\overline{\sigma}_{21}(x)\\ \overline{\sigma}_{22}(x)\end{pmatrix}\right] \\
=&\cos(\theta_1+\theta_2)[\overline{\sigma}_{11}(x)\overline{\sigma}_{21}(x)+\overline{\sigma}_{12}(x)\overline{\sigma}_{22}(x)] + \sin(\theta_1+\theta_2)[\overline{\sigma}_{11}(x)\overline{\sigma}_{22}(x)-\overline{\sigma}_{12}(x)\overline{\sigma}_{21}(x)] \\
=&\cos K(x,y)\cos\Psi_{12}(x) + \sin K(x,y)\sin\Psi_{12}(x) \\
=&\cos(K(x,y)-\Psi_{12}(x)) \\
=&\cos\Psi_{12}(y).
\end{aligned} \tag{2.15}$$

On the other hand, we have

$$\begin{aligned}
&\sin\Psi_{12}^*(x,y) = \det(\sigma_1^*(x,y) \;\; \sigma_2^*(x,y)) \\
=&\left|H_1(x,y)\begin{pmatrix}\overline{\sigma}_{11}(x)\\ \overline{\sigma}_{12}(x)\end{pmatrix} \;\; H_2(x,y)\begin{pmatrix}\overline{\sigma}_{21}(x)\\ \overline{\sigma}_{22}(x)\end{pmatrix}\right| \\
=&\cos(\theta_1+\theta_2)[\overline{\sigma}_{11}(x)\overline{\sigma}_{22}(x)-\overline{\sigma}_{12}(x)\overline{\sigma}_{21}(x)] + \sin(\theta_1+\theta_2)[\overline{\sigma}_{11}(x)\overline{\sigma}_{21}(x)+\overline{\sigma}_{12}(x)\overline{\sigma}_{22}(x)] \\
=&\cos K(x,y)\sin\Psi_{12}(x) - \sin K(x,y)\cos\Psi_{12}(x) \\
=&\sin\Psi_{12}(y).
\end{aligned} \tag{2.16}$$

From (2.15) and (2.16), we know that there exists an orthogonal matrix $H(x,y)$ with $\det H(x,y) = 1$ such that $(\sigma_1^*(x,y) \;\; \sigma_2^*(x,y)) = H(x,y)\overline{\sigma}'(y)$, where $\overline{\sigma}'(y)$ is the transpose of $\overline{\sigma}(y)$. So,

$$H(x,y) = (\sigma_1^*(x,y) \;\; \sigma_2^*(x,y))[\overline{\sigma}'(y)]^{-1}, \tag{2.17}$$

obviously, $H \in C(R^2\times R^2)$ and, since H is orthogonal, H is contractive.

Take $c(x,y) = \sigma(x)H(x,y)\sigma(y)$, construct $\widehat{L} \sim (\widehat{a},\widehat{b})$ as (1.3), then by Lemma 1 we know $\widehat{L}$ is a coupling operator of L^1 and L^2.

Step 2 Now we begin to study the properties of our specified $c(x,y)$. If $x = y$, by (2.1)–(2.6) and (2.17), $H(x,y) = I$, so

$$a_{ii}(x) + a_{ii}(y) - 2c_{ii}(x,y) = 2a_{ii}(x) - 2(\sigma(x)\sigma(x))_{ii} = 2a_{ii}(x) - 2a_{ii}(x) = 0; \qquad i = 1,2. \tag{2.18}$$

If $x \neq y$ and $x_i = y_i$ ($i = 1$ or 2), without loss of generality, assume $x_1 = y_1, x_2 \neq y_2$, then, by (2.1), $\theta_1 = 0$ and by the specified structure of $H(x,y)$ and $c(x,y)$, we have

$$H(x,y) = \left(\begin{pmatrix}\overline{\sigma}_{11}(x)\\ \overline{\sigma}_{12}(x)\end{pmatrix} \quad H_2(x,y)\begin{pmatrix}\overline{\sigma}_{21}(x)\\ \overline{\sigma}_{22}(x)\end{pmatrix}\right)[\overline{\sigma}'(y)]^{-1}, \tag{2.19}$$

and then

$$\begin{aligned}
&c(x,y) = \sigma(x)H(x,y)\sigma(y) \\
=&\sigma(x)\left(\begin{pmatrix}\overline{\sigma}_{11}(x)\\ \overline{\sigma}_{12}(x)\end{pmatrix} \quad H_2(x,y)\begin{pmatrix}\overline{\sigma}_{21}(x)\\ \overline{\sigma}_{22}(x)\end{pmatrix}\right)\begin{pmatrix}\sqrt{a_{11}(y)} & 0\\ 0 & \sqrt{a_{22}(y)}\end{pmatrix} \\
=&\begin{pmatrix}\sigma_{11}(x) & \sigma_{12}(x)\\ \sigma_{21}(x) & \sigma_{22}(x)\end{pmatrix}\begin{pmatrix}\sigma_{11}(x) & (\cos\theta_2\overline{\sigma}_{21}(x)+\sin\theta_2\overline{\sigma}_{22}(x))\sqrt{a_{22}(y)}\\ \sigma_{12}(x) & (-\sin\theta_2\overline{\sigma}_{21}(x)+\cos\theta_2\overline{\sigma}_{22}(x))\sqrt{a_{22}(y)}\end{pmatrix},
\end{aligned} \tag{2.20}$$

the computation in (2.20) used the fact: $\overline{\sigma}_{ij}(x) = \dfrac{\sigma_{ij}(x)}{\sqrt{a_{ii}(x)}}$, $i,j = 1,2$. By (2.20), together with the condition (1), we have

$$c_{11}(x,y) = \sigma_{11}^2(x) + \sigma_{12}^2(x) = a_{11}(x) = a_{11}(y). \tag{2.21}$$

So, by (2.18) and (2.21) we have

$$a_{ii}(x) + a_{ii}(y) - 2c_{ii}(x, y) = 0. \tag{2.22}$$

for all $x, y \in R^2$ with $x_i = y_i$, $i = 1$ or 2.

Step 3 By Lemma 2 and Lemma 3, it is easy to see that the matrix $c(\cdot, \cdot)$ we constructed in step 1 be locally Lipschitz continuous, so there exists $0 < C_m < \infty$ such that

$$|a_{ii}(y) - a_{ii}(z)| + 2|c_{ii}(x, y) - c_{ii}(x, z)| \leq C_m(|y_1 - z_1| + |y_2 - z_2|) \tag{2.23}$$

holds for $i = 1, 2$, $x, y, z \in [-m, m]^2$ uniformly.

Step 4 In this step, for $\widehat{L}$, we check the remained conditions of (3), and then finish the proof of the main theorem. Given $x, y \in [-m, m]^2$, let $z^1 = (x_1, y_2)$, $z^2 = (y_1, x_2)$, by (2.22)

$$a_{ii}(x) + a_{ii}(z^i) - 2c_{ii}(x, z^i) = 0, \qquad i = 1, 2;$$

then, by (2.23)

$$\begin{aligned} &|a_{ii}(x) + a_{ii}(y) - 2c_{ii}(x, y)| \\ =&|a_{ii}(x) + a_{ii}(y) - 2c_{ii}(x, y) - a_{ii}(x) - a_{ii}(z^i) + 2c_{ii}(x, z^i)| \\ \leq&|a_{ii}(y) - a_{ii}(z^i)| + 2|c_{ii}(x, y) - c_{ii}(x, z^i)| \\ \leq&C_m(|y_1 - z_1^i| + |y_2 - z_2^i|) \\ =&C_m|x_i - y_i|, \qquad i = 1, 2. \end{aligned} \tag{2.24}$$

Take $\rho_m(u) = C_m u$, we have $\rho_m(0) = 0$ and $\int_0^1 \rho_m^{-1}(u)\, du = \infty$, now (2.24) is just (1.5), by (3), the main theorem follows. □

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References

[1] Chen, M.F. and Wang, F.Y., On order-preservation and positive correlations for multidimensional diffusion processes, *Prob. Theory Rel. Fields*, **95**(1993), 421–428.

[2] Herbst, I. and Pitt, L., Diffusion equation techniques in stochastic monotonicity and positive correlations, *Prob. Theory Rel. Fields*, **87**(1991), 275–312.

[3] Wang, F.Y. and Xu, M.P., On order-preservation of couplings for multidimensional diffusion processes, *Chinese J. Appl. Prob. Stat.*, **13(2)**(1996), 142–148.

[4] Chen, M.F. and Li, S.F., Coupling methods for multidimensional diffusion processes, *Ann. of Prob.*, **17(1)**(1989), 151–177.

[5] Chen, M.F., Optimal markovian couplings and applications, *Acta Math. Sin. New Series*, **10(3)**(1994), 260–275.

二维扩散过程保序耦合的存在性

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本文研究扩散过程轨道的保序性, 对二维非退化扩散过程, 我们证明其保序耦合存在, 同时构造出一类保序算子.

关键词: 扩散过程, 耦合, 保序耦合.

学科分类号: O211.62.