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Exact Asymptotics in Complete Moment Convergence for Record Times and the Associated Counting Process *

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Abstract: Let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables with absolutely continuous distribution function. Denote the record times and the associated counting process of $\{X_n, n \geqslant 1\}$ by $\{L(n), n \geqslant 1\}$ and $\{\mu(n), n \geqslant 1\}$, respectively. In this paper, we obtain the exact asymptotics in complete moment convergence of $\{L(n), n \geqslant 1\}$ and $\{\mu(n), n \geqslant 1\}$.

Keywords: counting process; record times; exact asymptotics; complete moment convergence

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§1. Introduction

Let $\{X, X_n; n \in \mathbb{N}\}$ be a sequence of i.i.d. random variables with continuous distribution function. And define the partial sum $S_n = \sum\limits_{k=1}^{n} X_k$, $n \in \mathbb{N}$. It is well-known that, given $0 < p < 2$ and $r \geqslant p$,

$$\sum_{n=1}^{\infty} n^{r/p-2} \mathsf{P}(|S_n| \geqslant \epsilon n^{1/p}) < \infty, \qquad \epsilon > 0, \tag{1}$$

if and only if $\mathsf{E}|X|^r < \infty$, and when $r \geqslant 1$, $\mathsf{E}[X] = 0$. For $r = 2$ and $p = 1$, Hsu and Robbins [1] first proved the sufficiency. Later, Erdős [2,3] obtained the necessity. For the special case $r = p = 1$, we refer to [4]. Baum and Katz [5] obtained the general case.

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Note that the sum in (1) tends to infinity as $\epsilon \searrow 0$. Hence finding the precise rate at which this occurs becomes an interesting topic. In fact, it has been studied extensively. For example, Heyde [6] proved that

$$\lim_{\epsilon \searrow 0} \epsilon^2 \sum_{n=1}^{\infty} \mathsf{P}(|S_n| \geqslant \epsilon n) = \mathsf{E}[X^2],$$

if and only if $\mathsf{E}[X] = 0$ and $\mathsf{E}[X^2] < \infty$. For more information on this topic, we refer to [7–12] and so on.

On the other hand, based on (1), another interesting topic is to study the complete moment convergence. Let $p \geqslant 1$, $\alpha > 1/2$, $p\alpha > 1$ and $\mathsf{E}\{|X|^p + |X| \ln(1 + |X|)\} < \infty$. Under the assumption that $\mathsf{E}[X] = 0$, Chow [13] obtained that for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^{(p-1)\alpha-2} \mathsf{E}\Big\{ \max_{1 \leqslant j \leqslant n} |S_j| - \epsilon n^{\alpha} \Big\}_+ < \infty,$$

where $\{x\}_+ = \max\{x, 0\}$.

In this paper, the main aim is to study the exact asymptotics in complete moment convergence for the record times and the associated counting process. Hence, we first introduce the corresponding definitions.

Let $\{X, X_n;\ n \in \mathbb{N}\}$ be a sequence of i.i.d. random variables with absolutely continuous distribution function. Set $L(1) = 1$, and recursively,

$$L(n) = \min\{k > L(n-1) : X_k > X_{L(n-1)}\}, \qquad n \geqslant 2.$$

We call $\{L(n), n \geqslant 1\}$ as the record times of $\{X_n, n \geqslant 1\}$. The associated counting process $\{\mu(n), n \geqslant 1\}$ is defined by

$$\mu(n) = \max\{k : L(k) \leqslant n\}.$$

Many results about $\{\mu(n)\}$ have been established. For example, see [14–16] and so on. Here, we should point out the recent results obtained by [17]. The results read as follows.

Theorem 1 (i) Let $r > 0$, then

$$\lim_{\epsilon \searrow \sqrt{2r}} \sqrt{\epsilon^2 - 2r} \sum_{n=9}^{\infty} \frac{(\ln\ln n)^{r-1}}{n \ln n} \mathsf{P}\big\{|\mu(n) - \ln n| > \epsilon\sqrt{\ln n \ln\ln\ln n}\big\} = \sqrt{\frac{2}{r}}.$$

(ii) When $r = 0$, we have

$$\lim_{\epsilon \searrow 0} \frac{1}{-\ln \epsilon} \sum_{n=3}^{\infty} \frac{1}{n \ln n \ln\ln n} \mathsf{P}\big\{|\mu(n) - \ln n| > \epsilon\sqrt{\ln n \ln\ln n}\big\} = 2$$

and

$$\lim_{\epsilon \searrow 0} \epsilon^2 \sum_{n=9}^{\infty} \frac{1}{n \ln n \ln\ln n} \mathsf{P}\big\{|\mu(n) - \ln n| > \epsilon\sqrt{\ln n \ln\ln\ln n}\big\} = 1.$$

Motivated by the above results, we aim to investigate the exact asymptotics in complete moment convergence for $\{L(n), n \in \mathbb{N}\}$ and $\{\mu(n), n \in \mathbb{N}\}$ in this work. The rest of this paper is organized as follows. In Section 2, we list some basic facts about the record times and the associated counting process. Furthermore, we obtain the exact asymptotics in complete moment convergence for the associated counting process. Section 3 is devoted to getting the exact asymptotics for the record times.

Throughout this paper, we use C to denote an positive constant whose value may vary from line to the next. The notation $f(x) \sim g(x)$ means that $f(x)/g(x) \to 1$ as $x \to \infty$.

§2. Exact Asymptotics for the Counting Process

In this section, we study the exact asymptotics for the counting process. In order to reach our aim, we first recall some basic facts about the record times and the counting process. Let

$$I_k = \begin{cases} 1, & \text{if } X_k \text{ is a record,} \\ 0, & \text{otherwise,} \end{cases}$$

then $\mu(n) = \sum\limits_{k=1}^{n} I_k$, $n \geqslant 1$. It follows from [15] that, as $n \to \infty$,

(i) $m_n = \mathsf{E}\mu(n) = \sum\limits_{k=1}^{n} 1/k = \ln n + \gamma + \mathrm{o}(1)$,

(ii) $\mathsf{Var}\,\mu(n) = \sum\limits_{k=1}^{n} (1 - 1/k)/k = \ln n + \gamma - \pi^2/6 + \mathrm{o}(1)$,

(iii) $\mu(n)/\ln n \to 1$ a.s.,

(iv) $[\mu(n) - \ln n]/\sqrt{\ln n} \xrightarrow{d} \mathrm{N}(0,1)$,

(v) $\ln L(n)/n \to 1$ a.s.,

(vi) $(\ln L(n) - n)/\sqrt{n} \xrightarrow{d} \mathrm{N}(0,1)$,

where $\gamma = 0.577\cdots$, is the Euler's constant.

Now, we state the main results of this section. We have

Theorem 2 Given $r > 0$,

$$\lim_{\epsilon \searrow \sqrt{2r}} \frac{1}{-\ln(\epsilon^2 - 2r)} \sum_{n=9}^{\infty} \frac{(\ln\ln n)^{r-1}}{n(\ln n)^{3/2}} \mathsf{E}\big\{|\mu(n) - \ln n| - \epsilon\sqrt{\ln n \ln\ln\ln n}\big\}_+ = \frac{1}{r\sqrt{2\pi}}.$$

Theorem 3 When $r = 0$,

$$\lim_{\epsilon\searrow 0} \frac{1}{-\ln\epsilon} \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^{3/2}\ln\ln n} \mathsf{E}\big\{|\mu(n) - \ln n| - \epsilon\sqrt{\ln n \ln\ln n}\big\}_+ = 2\mathsf{E}|N|, \tag{2}$$

and

$$\lim_{\epsilon\searrow 0} \epsilon^2 \sum_{n=9}^{\infty} \frac{1}{n(\ln n)^{3/2}\ln\ln n} \mathsf{E}\big\{|\mu(n) - \ln n| - \epsilon\sqrt{\ln n \ln\ln\ln n}\big\}_+ = \frac{1}{3}\mathsf{E}|N|^3. \tag{3}$$

Remark 4 Zang and Fu [18] established the exact asymptotics in complete moment convergence for the counting process. However, they dealt with the case of large deviation, i.e., $\ln n$, and in this paper, we investigate the case of moderate deviation, i.e., $\sqrt{\ln n \ln\ln n}$, in (2), and the case of small deviation, i.e., $\sqrt{\ln n \ln\ln\ln n}$, in Theorem 2 and the equaiton (3).

Remark 5 When we checked the proof of this article, we found by chance that some similar results were obtained by [19] and [20]. They used the truncated method to estimate the remainder terms. However, in the proofs of our results, we mainly rely on the estimate of Δ_n, which is given by Lemma 9.

In order to prove Theorem 2, we need some technical results. We have the following propositions.

Proposition 6 Given $r > 0$, we have

$$\lim_{\epsilon\searrow\sqrt{2r}} \frac{1}{-\ln(\epsilon^2 - 2r)} \sum_{n=9}^{\infty} \frac{(\ln\ln n)^{r-1}}{n\ln n} \mathsf{E}\big\{|N| - \epsilon\sqrt{\ln\ln\ln n}\big\}_+ = \frac{1}{r\sqrt{2\pi}}.$$

Before we prove it, we need a technical lemma, which comes from [21].

Lemma 7 For large enough x,

$$\Psi(x) = 2\mathsf{P}(N \geqslant x) \sim \frac{2}{\sqrt{2\pi}x}\mathrm{e}^{-x^2/2},$$

where $\Psi(x) = \mathsf{P}(|N| \geqslant x)$ with N being the standard normal random variable.

Next, we prove the Proposition 6.

Proof of Proposition 6 For convenience, let

$$\Gamma = \lim_{\epsilon\searrow\sqrt{2r}} \frac{1}{-\ln(\epsilon^2 - 2r)} \sum_{n=9}^{\infty} \frac{(\ln\ln n)^{r-1}}{n\ln n} \mathsf{E}\big\{|N| - \epsilon\sqrt{\ln\ln\ln n}\big\}_+.$$

Note that

$$\Gamma = \lim_{\epsilon\searrow\sqrt{2r}} \frac{1}{-\ln(\epsilon^2 - 2r)} \int_9^{\infty} \frac{(\ln\ln x)^{r-1}}{x\ln x} \int_{\epsilon\sqrt{\ln\ln\ln x}}^{\infty} \Psi(y)\mathrm{d}y\mathrm{d}x. \tag{4}$$

Let $t = \epsilon\sqrt{\ln\ln\ln x}$. Then (4) is equivalent to

$$\begin{aligned}\Gamma &= \lim_{\epsilon\searrow\sqrt{2r}} \frac{1}{-\ln(\epsilon^2-2r)} \int_{\epsilon\sqrt{\ln\ln\ln 9}}^{\infty} \frac{2t}{\epsilon^2} \mathrm{e}^{rt^2/\epsilon^2} \int_t^{\infty} \Psi(y)\mathrm{d}y\mathrm{d}t \\ &= \lim_{\epsilon\searrow\sqrt{2r}} \frac{1}{-\ln(\epsilon^2-2r)} \int_{\epsilon\sqrt{\ln\ln\ln 9}}^{\infty} \Psi(y) \int_{\epsilon\sqrt{\ln\ln\ln 9}}^{y} \frac{2t}{\epsilon^2} \mathrm{e}^{rt^2/\epsilon^2} \mathrm{d}t\mathrm{d}y \\ &= \lim_{\epsilon\searrow\sqrt{2r}} \frac{1}{-r\ln(\epsilon^2-2r)} \int_{\epsilon\sqrt{\ln\ln\ln 9}}^{\infty} \Psi(y)(\mathrm{e}^{ry^2/\epsilon^2} - \mathrm{e}^{r\ln\ln\ln 9})\mathrm{d}y \\ &= \lim_{\epsilon\searrow\sqrt{2r}} \frac{1}{-r\ln(\epsilon^2-2r)} \int_{\epsilon\sqrt{\ln\ln\ln 9}}^{\infty} \Psi(y)\mathrm{e}^{ry^2/\epsilon^2}\mathrm{d}y. \end{aligned} \tag{5}$$

(5) and Lemma 7 imply that

$$\Gamma = \lim_{\epsilon\searrow\sqrt{2r}} \frac{1}{-r\ln(\epsilon^2-2r)} \int_{\epsilon\sqrt{\ln\ln\ln 9}}^{\infty} \frac{2}{\sqrt{2\pi}y} \mathrm{e}^{-(\epsilon^2-2r)y^2/(2\epsilon^2)}\mathrm{d}y.$$

Let $s = (\epsilon^2-2r)y^2/(\epsilon^2\ln\ln\ln 9)$. Then

$$\Gamma = \lim_{\epsilon\searrow\sqrt{2r}} \frac{1}{-r\sqrt{2\pi}\ln(\epsilon^2-2r)} \int_{\epsilon^2-2r}^{\infty} \frac{1}{s}\mathrm{e}^{-s\ln\ln\ln 9/2}\mathrm{d}s = \frac{1}{r\sqrt{2\pi}}.$$

We complete the proof of Proposition 6. □

Proposition 8 Let

$$A(n) = \left|\mathsf{E}\left\{|\mu(n)-\ln n| - \epsilon\sqrt{\ln n\ln\ln\ln n}\right\}_+ - \sqrt{\ln n}\,\mathsf{E}\left\{|N| - \epsilon\sqrt{\ln\ln\ln n}\right\}_+\right|.$$

Then, for any $r > 0$,

$$\lim_{\epsilon\searrow\sqrt{2r}} \frac{1}{-\ln(\epsilon^2-2r)} \sum_{n=9}^{\infty} \frac{(\ln\ln n)^{r-1}}{n(\ln n)^{3/2}} A(n) = 0.$$

In order to prove it, we also need a technical lemma. Before we state it, we first introduce the following notation. For any $x \geqslant 0$, let

$$\Delta_n = \sup_x \left|\mathsf{P}(|\mu_n - \ln n| > x) - \mathsf{P}\left(|N| > \frac{x}{\sqrt{\ln n}}\right)\right|. \tag{6}$$

Following [22], we have

Lemma 9

$$\Delta_n \leqslant \frac{3.8}{\sqrt{\ln n}}, \qquad \forall\, n \geqslant 2,$$

where Δ_n is defined by (6).

Next, we prove the Proposition 8.

Proof of Proposition 8 In order to prove the proposition, we only need to prove that, for any $r > 0$,

$$\sum_{n=9}^{\infty} \frac{(\ln\ln n)^{r-1}}{n(\ln n)^{3/2}} A(n) < \infty.$$

For convenience, let

$$\Lambda = \sum_{n=9}^{\infty} \frac{(\ln\ln n)^{r-1}}{n(\ln n)^{3/2}} A(n).$$

Note that

$$\Lambda = \sum_{n=9}^{\infty} \frac{(\ln\ln n)^{r-1}}{n(\ln n)^{3/2}} \Big| \int_{\epsilon\sqrt{\ln n \ln\ln\ln n}}^{\infty} \mathsf{P}(|\mu(n) - \ln n| \geqslant x) \mathrm{d}x - \sqrt{\ln n} \int_{\epsilon\sqrt{\ln\ln\ln n}}^{\infty} \mathsf{P}(|N| \geqslant t) \mathrm{d}t \Big|.$$

By using the change of variable $u = x/\sqrt{\ln n}$, we have

$$\begin{aligned}
\Lambda &= \sum_{n=9}^{\infty} \frac{(\ln\ln n)^{r-1}}{n\ln n} \Big| \int_{0}^{\infty} \Big[\mathsf{P}\Big(\Big| \frac{\mu(n) - \ln n}{\sqrt{\ln n}} \Big| \geqslant u + \epsilon\sqrt{\ln\ln\ln n} \Big) \\
&\quad - \mathsf{P}\big(|N| \geqslant u + \epsilon\sqrt{\ln\ln\ln n} \big) \Big] \mathrm{d}u \Big| \\
&\leqslant \sum_{n=9}^{\infty} \frac{(\ln\ln n)^{r-1}}{n\ln n} \int_{0}^{\infty} \Big| \mathsf{P}\Big(\Big| \frac{\mu(n) - \ln n}{\sqrt{\ln n}} \Big| \geqslant u + \epsilon\sqrt{\ln\ln\ln n} \Big) \\
&\quad - \mathsf{P}\big(|N| \geqslant u + \epsilon\sqrt{\ln\ln\ln n} \big) \Big| \mathrm{d}u \\
&=: \sum_{n=9}^{\infty} \frac{(\ln\ln n)^{r-1}}{n\ln n} (I_1 + I_2),
\end{aligned}$$

where

$$I_1 = \int_{0}^{1/\sqrt{\Delta_n}} \Big| \mathsf{P}\Big(\Big| \frac{\mu(n) - \ln n}{\sqrt{\ln n}} \Big| \geqslant u + \epsilon\sqrt{\ln\ln\ln n} \Big) - \mathsf{P}\big(|N| \geqslant u + \epsilon\sqrt{\ln\ln\ln n} \big) \Big| \mathrm{d}u$$

and

$$I_2 = \int_{1/\sqrt{\Delta_n}}^{\infty} \Big| \mathsf{P}\Big(\Big| \frac{\mu(n) - \ln n}{\sqrt{\ln n}} \Big| \geqslant u + \epsilon\sqrt{\ln\ln\ln n} \Big) - \mathsf{P}\big(|N| \geqslant u + \epsilon\sqrt{\ln\ln\ln n} \big) \Big| \mathrm{d}u.$$

By Lemma 9, we have

$$I_1 \leqslant \sqrt{\Delta_n} \leqslant \frac{C}{(\ln n)^{1/4}}. \tag{7}$$

Note that

$$\frac{\mu(n) - \ln n}{\sqrt{\ln n}} \xrightarrow{d} \mathrm{N}(0,1).$$

Thus, Lemma 7 implies that, as $u \to +\infty$,

$$\mathsf{P}\Big(\Big| \frac{\mu(n) - \ln n}{\sqrt{\ln n}} \Big| \geqslant u \Big) \leqslant \frac{C}{u} \mathrm{e}^{-u^2/2} \tag{8}$$

and

$$\mathsf{P}(|N| \geqslant u) \leqslant \frac{C}{u}\mathrm{e}^{-u^2/2}. \tag{9}$$

On the other hand, we have

$$\int_y^\infty \frac{1}{u}\mathrm{e}^{-u^2/2}\mathrm{d}u \leqslant \frac{1}{y}, \qquad \text{as } y \to +\infty. \tag{10}$$

Combining (8), (9) and (10), we get

$$I_2 \leqslant \int_{1/\sqrt{\Delta_n}}^\infty \frac{C}{u}\mathrm{e}^{-u^2/2}\mathrm{d}u \leqslant C\sqrt{\Delta_n} \leqslant \frac{C}{(\ln n)^{1/4}}. \tag{11}$$

Thus, by (7) and (11), we have

$$\Lambda \leqslant \sum_{n=9}^\infty \frac{(\ln\ln n)^{r-1}}{n(\ln n)^{5/4}} < \infty. \tag{12}$$

We complete the proof of Proposition 8. □

Remark 10 By (12), Λ is also finite when $r \leqslant 0$.

Now we stand at a point where we can prove the theorem 2.

Proof of Theorem 2 Theorem 2 follows from Propositions 6 and 8, and the triangle inequality directly. □

Next, we prove Theorem 3. Similar to the proof of Theorem 2, in order to prove it, we also need the following technical propositions.

Proposition 11

$$\lim_{\epsilon\searrow 0} \frac{1}{-\ln\epsilon} \sum_{n=3}^\infty \frac{1}{n\ln n\ln\ln n}\mathsf{E}\big\{|N| - \epsilon\sqrt{\ln\ln n}\big\}_+ = 2\mathsf{E}|N|, \tag{13}$$

and

$$\lim_{\epsilon\searrow 0} \epsilon^2 \sum_{n=9}^\infty \frac{1}{n\ln n\ln\ln n}\mathsf{E}\big\{|N| - \epsilon\sqrt{\ln\ln\ln n}\big\}_+ = \frac{1}{3}\mathsf{E}|N|^3. \tag{14}$$

Proof The proof of Proposition 11 is similar to that of Proposition 6. In fact, by some calculations, we have

$$\begin{aligned}
&\lim_{\epsilon\searrow 0} \frac{1}{-\ln\epsilon} \sum_{n=3}^\infty \frac{1}{n\ln n\ln\ln n}\mathsf{E}\big\{|N| - \epsilon\sqrt{\ln\ln n}\big\}_+ \\
&= \lim_{\epsilon\searrow 0} \frac{1}{-\ln\epsilon} \int_3^\infty \frac{1}{x\ln x\ln\ln x} \int_{\epsilon\sqrt{\ln\ln x}}^\infty \Psi(y)\mathrm{d}y\mathrm{d}x \\
&= \lim_{\epsilon\searrow 0} \frac{2}{-\ln\epsilon} \int_{\epsilon\sqrt{\ln\ln 3}}^\infty (\ln y - \ln\epsilon - \ln\sqrt{\ln\ln 3})\Psi(y)\mathrm{d}y \\
&= 2\mathsf{E}|N|,
\end{aligned}$$

and

$$\begin{aligned}&\lim_{\epsilon\searrow 0}\epsilon^2\sum_{n=9}^{\infty}\frac{1}{n\ln n\ln\ln n}\mathsf{E}\big\{|N|-\epsilon\sqrt{\ln\ln\ln n}\big\}_+\\ =&\lim_{\epsilon\searrow 0}\epsilon^2\int_9^{\infty}\frac{1}{x\ln x\ln\ln x}\int_{\epsilon\sqrt{\ln\ln\ln x}}^{\infty}\Psi(y)\mathrm{d}y\mathrm{d}x\\ =&\lim_{\epsilon\searrow 0}\int_{\epsilon\sqrt{\ln\ln\ln 9}}^{\infty}y^2\Psi(y)\mathrm{d}y=\frac{1}{3}\mathsf{E}|N|^3,\end{aligned}$$

where we used the fact that

$$\lim_{\epsilon\searrow 0}\frac{2}{-\ln\epsilon}\int_{\epsilon\sqrt{\ln\ln 3}}^{\infty}\ln y\Psi(y)\mathrm{d}y=0.$$

We complete the proof of Proposition 11. □

Proposition 12 Let

$$B(n)=\big|\mathsf{E}\big\{|\mu(n)-\ln n|-\epsilon\sqrt{\ln n\ln\ln n}\big\}_+-\sqrt{\ln n}\,\mathsf{E}\big\{|N|-\epsilon\sqrt{\ln\ln n}\big\}_+\big|.$$

Then

$$\lim_{\epsilon\searrow 0}\frac{1}{-\ln\epsilon}\sum_{n=3}^{\infty}\frac{1}{n(\ln n)^{3/2}\ln\ln n}B(n)=0.$$

Proposition 12 can be proved in the same way as the proof of Proposition 8. We omit the details here.

Next, we prove Theorem 3.

Proof of Theorem 3 It follows from (13), Proposition 12 and the triangle inequality that (2) holds. On the other hand, by (14), Remark 10 and the triangle inequality, we obtain (3). The proof of Theorem 3 is finished. □

§3. Exact Asymptotics for the Record Times

In this section, we study the precise asymptotics for the record times $\{L(n):n\in\mathbb{N}\}$. The following results for record times can be proved in the same way as the proof of Theorem 2 with the estimate given by Theorem 4 in [22].

Theorem 13 Given $\delta>-1$ and $p>0$, we have

$$\lim_{\epsilon\searrow 0}\epsilon^{p(\delta+1)}\sum_{n=3}^{\infty}n^{\delta}\mathsf{E}\big\{|\ln L(n)-n|-\epsilon n^{1/p+1/2}\big\}_+=\frac{\mathsf{E}|N|^{p\delta+p+1}}{p(\delta+1)(p\delta+p+1)}.$$

Theorem 14 Given $\delta>-1$, we have

$$\lim_{\epsilon\searrow 0}\epsilon^{2(\delta+1)}\sum_{n=3}^{\infty}\frac{(\ln n)^{\delta}}{n^{3/2}}\mathsf{E}\big\{|\ln L(n)-n|-\epsilon\sqrt{n\ln n}\big\}_+=\frac{\mathsf{E}|N|^{2\delta+3}}{(\delta+1)(2\delta+3)}.$$

Theorem 15 Given $\delta > 0$, we have

$$\lim_{\epsilon \searrow \sqrt{2\delta}} \frac{1}{-\ln(\epsilon^2 - 2\delta)} \sum_{n=3}^{\infty} \frac{(\ln n)^{\delta-1}}{n^{3/2}} \mathsf{E}\big\{|\ln L(n) - n| - \epsilon\sqrt{n \ln\ln n}\big\}_+ = \frac{1}{\delta\sqrt{2\pi}}.$$

Remark 16 Theorem 13 deals with the exact asymptotics in complete moment convergence of large deviation for $\{L(n), n \geqslant 1\}$, while Theorems 14 and 15 consider the moderate deviation and small deviation, respectively.

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References

[1] Hsu P L, Robbins H. Complete convergence and the law of large numbers [J]. *Proc. Nat. Acad. Sci. U.S.A.*, 1947, **33(2)**: 25–31.

[2] Erdős P. On a theorem of Hsu and Robbins [J]. *Ann. Math. Statist.*, 1949, **20(2)**: 286–291.

[3] Erdős P. Remark on my paper "On a theorem of Hsu and Robbins" [J]. *Ann. Math. Statist.*, 1950, **21(1)**: 138.

[4] Spitzer F. A combinatorial lemma and its application to probability theory [J]. *Trans. Amer. Math. Soc.*, 1956, **82(2)**: 323–339.

[5] Baum L E, Katz M. Convergence rates in the law of large numbers [J]. *Trans. Amer. Math. Soc.*, 1965, **120(1)**: 108–123.

[6] Heyde C C. A supplement to the strong law of large numbers [J]. *J. Appl. Probab.*, 1975, **12(1)**: 173–175.

[7] Chen R. A remark on the tail probability of a distribution [J]. *J. Multivariate Anal.*, 1978, **8(2)**: 328–333.

[8] Gut A, Spătaru A. Precise asymptotics in the Baum-Katz and Davis laws of large numbers [J]. *J. Math. Anal. Appl.*, 2000, **248(1)**: 233–246.

[9] Gut A, Spătaru A. Precise asymptotics in the law of the iterated logarithm [J]. *Ann. Probab.*, 2000, **28(4)**: 1870–1883.

[10] Li D L, Spătaru A. Asymptotics related to a series of T.L. Lai [J]. *Statist. Probab. Lett.*, 2012, **82(8)**: 1538–1548.

[11] Spătaru A. Precise asymptotics in Spitzer's law of large numbers [J]. *J. Theoret. Probab.*, 1999, **12(3)**: 811–819.

[12] Spătaru A. Precise asymptotics for a series of T.L. Lai [J]. *Proc. Amer. Math. Soc.*, 2004, **132(11)**: 3387–3395.

[13] Chow Y S. On the rate of moment convergence of sample sums and extremes [J]. *Bull. Inst. Math. Acad. Sinica*, 1988, **16(3)**: 177–201.

[14] Nevzorov V B. *Records: Mathematical Theory* (*Translations of Mathematical Monographs, Vol. 194*) [M]. Providence, RI: American Mathematical Society, 2001.

[15] Rényi A. On the extreme elements of observations [J]. *MTA III, Oszt. Közl*, 1962, **12**: 105–121. Also in: *Collected Works, Vol. III* [M]. Budapest: Akadémiai Kiadó, 1976: 50–66.

[16] Resnick S I. *Extreme Values, Regular Variation, and Point Processes* [J]. New York: Springer-Verlag, 1987.

[17] Gut A. Precise asymptotics for record times and the associated counting process [J]. *Stochastic Process. Appl.*, 2002, **101(2)**: 233–239.

[18] Zang Q P, Fu K A. Precise asymptotics in complete moment convergence of the associated counting process [J]. *J. Math. Anal. Appl.*, 2009, **359(1)**: 76–80.

[19] Ji Y Q, Cao Y S. The quadrature complete convergence for record time and the associated counting process [J]. *Chinese J. Appl. Probab. Statist.*, 2008, **24(3)**: 265–273. (in Chinese)

[20] Shen Y F. Precise asymptotics in the complete moment convergence of record times and the associated counting process [J]. *J. Zhejiang Univ. Sci. Ed.*, 2010, **37(1)**: 12–16. (in Chinese)

[21] Billingsley P. *Convergence of Probability Measures* [M]. 2nd ed. New York: Wiley, 1999.

[22] Gut A. Convergence rates for record times and the associated counting process [J]. *Stochastic Process. Appl.*, 1990, **36(1)**: 135–151.

记录时及相应计数过程关于矩精确完全收敛的渐进性质

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摘　要： 我们考虑一列独立同分布且分布函数绝对连续的随机变量序列及其记录时和相应的计数过程, 得到了记录时和相应计数过程的关于矩精确完全收敛的渐进性质.

关键词： 计数过程; 记录时; 精确渐进性; 矩完全收敛

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